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Asymptotic Equivalence of A Reaction-Diffusion System to the Corresponding System of Ordinary Differential Equations

(ある反応拡散方程式系と対応する常微分方程式系との漸近的同値性)

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1. Introduction

In this report Large-time behavior of a global solution to a reaction-diffusion system with homogeneous Neumann boundary conditions is studied. It is proved that the solution behaves like the solution to the corresponding system of ordinary differential equations as time goes to infinity. This report is based on the paper [5] by Hoshino and Kawashima.

We consider the following initial-boundary value problem which stems from a model for a simple irreversible chemical reaction:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - u^m v^n, & t > 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - u^m v^n, & t > 0,\ x \in \Omega, \end{cases} \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega, \tag{1.2}$$

$$(u, v)(0, x) = (u_0, v_0)(x), \qquad x \in \Omega. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative to $\partial\Omega$, the coefficients d_1, d_2 and the exponents m, n are fixed real constants satisfying

$$d_1, d_2 > 0, \qquad m, n \geq 1 \tag{1.4}$$

and the initial data $(u_0, v_0)(x)$ are bounded and nonnegative on Ω, that is,

$$(u_0, v_0) \in L^\infty(\Omega)^2 \ \text{ and } \ u_0(x), v_0(x) \geq 0 \ \text{ for } x \in \Omega. \tag{1.5}$$

In this report we may assume without loss of generality that $\overline{u}_0 \geq \overline{v}_0 > 0$, where

$$\overline{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0(x)dx, \qquad \overline{v}_0 = \frac{1}{|\Omega|} \int_\Omega v_0(x)dx. \tag{1.6}$$

If (1.1) is replaced with

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} = d_1 \Delta \tilde{u} - K_1 \tilde{u}^m \tilde{v}^n, & t > 0, \ x \in \Omega, \\ \dfrac{\partial \tilde{v}}{\partial t} = d_2 \Delta \tilde{v} - K_2 \tilde{u}^m \tilde{v}^n, & t > 0, \ x \in \Omega \end{cases}$$

and we put $k_1 = K_1^{-(n-1)/(m+n-1)} K_2^{n/(m+n-1)}$ and $k_2 = K_1^{m/(m+n-1)} K_2^{-(m-1)/(m+n-1)}$, then $(u, v) = (k_1 \tilde{u}, k_2 \tilde{v})$ satisfies (1.1); so that we deal with (1.1). Note that we have $K_1/K_2 = m/n$ in the case where we consider the chemical reaction $mX + nY \to \ell Z$ and $\tilde{u}$ (resp. $\tilde{v}$) stands for the concentration of X (resp. Y).

The initial-boundary value problem (1.1) - (1.3) was studied in [6] and it was proved there that a unique solution $(u, v)(t, x)$ exists globally in time, this solution uniformly converges to the equilibrium state $(u_\infty, 0) = (\overline{u}_0 - \overline{v}_0, 0)$ as $t \to \infty$ (note that this equilibrium state becomes $(0, 0)$ when $\overline{u}_0 = \overline{v}_0$), when $\overline{u}_0 > \overline{v}_0$ and $n = 1$, $(u, v)(t, x)$ approaches $(\overline{u}_0 - \overline{v}_0, 0)$ with exponential rate and when $\overline{u}_0 > \overline{v}_0$ and $n > 1$, there exist positive constants T and K such that

$$\|(u, v)(t) - (\overline{u}_0 - \overline{v}_0, 0)\|_{L^\infty(\Omega)^2} \le K(1 + t - T)^{-\alpha}, \qquad t \ge T,$$

where $\alpha = 1/(n-1)$.

In [5], Hoshino and Kawashima studied more detailed large-time behavior of the solution $(u, v)(t, x)$ mentioned above. Concerning rate of convergence, they have shown the following: When $\overline{u}_0 = \overline{v}_0$, $(u, v)(t, x)$ converges to $(0, 0)$ at the rate $t^{-\beta}, \beta = 1/(m+n-1)$, as $t \to \infty$ (Theorem 1).

We see that the polynomial rates of convergence stated above are just the same as those for the solution $(U, V)(t)$ to the corresponding system of ordinary differential equations

$$\frac{dU}{dt} = -U^m V^n, \quad \frac{dV}{dt} = -U^m V^n, \qquad t > 0 \tag{1.7}$$

with the averaged initial data

$$(U, V)(0) = (\overline{u}_0, \overline{v}_0). \tag{1.8}$$

This suggests the possibility that our solution $(u, v)(t, x)$ might behave like the solution $(U, V)(t)$ to the problem (1.7), (1.8) as $t \to \infty$. They have proved in [5] that the $(U, V)(t)$

becomes an asymptotic solution for $t \to \infty$ to the problem (1.1) - (1.3). In fact, when $\overline{u}_0 > \overline{v}_0$ and $n > 1$,

$$(1.9) \qquad (u,v)(t,x) = (U,V)(t) + O(t^{-\alpha-1}), \qquad \alpha = 1/(n-1),$$

as $t \to \infty$, while in the case where $\overline{u}_0 = \overline{v}_0$,

$$(1.10) \qquad (u,v)(t,x) = (U,V)(t) + O(t^{-\beta-1}), \qquad \beta = 1/(m+n-1),$$

as $t \to \infty$ (Theorem 2).

Moreover, they have also shown that the following asymptotic relations hold true: When $\overline{u}_0 > \overline{v}_0$ and $n > 1$,

$$(1.11) \qquad (u,v)(t,x) = (\overline{u},\overline{v})(t) + O(t^{\mu-\alpha}e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ as $t \to \infty$ and when $\overline{u}_0 = \overline{v}_0$,

$$(1.12) \qquad (u,v)(t,x) = (\overline{u},\overline{v}(t) + O(t^{\nu-\beta}e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ as $t \to \infty$, where $d_0 = \min\{d_1, d_2\}$, λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$, $\alpha = 1/(n-1)$, $\beta = 1/(m+n-1)$, $\mu = k/(n-1)$ with $k = 1 + ((\sqrt{2}-1)/2)n$ and $\nu = \ell/(m+n-1)$ with $\ell = 1 + (\sqrt{2(m^2+n^2)} - (m+n))/2$. Exponential decay estimates similar to above ones are obtained in e.g., [2] and [8].

Our main results stated above are essentially based on a simple energy method which makes use of the Poincaré inequality. Namely, we derive fundamental $L^2(\Omega)$-estimates by an energy method and then prove $L^\infty(\Omega)$-estimates by applying what we call L^p-L^q estimate for the semigroup associated with the problem (1.1) - (1.3).

The plan of this report is as follows. In Section 2 we give precise statements of our main theorems. In Section 3 we discuss rates of convergence of the solution toward the equilibrium. We prove the large-time approximation results (1.9) - (1.12) in the final section.

We will only give outlines of our discussion in this report. For the details, refer the reader to [5]. Furthermore, Hoshino [4] has shown that the argument in [5] is valid for a different reaction-diffusion system with homogeneous Neumann boundary conditions.

2. Main Results

In this section, we state our main results on the initial-boundary value problem of the reaction-diffusion system (1.1) - (1.3). Throughout the report we assume (1.4), (1.5) and $\overline{u}_0 \geq \overline{v}_0 > 0$, where $\overline{u}_0$ and $\overline{v}_0$ are given in (1.6). The following results are proved in [5]: Under the assumptions stated above, the initial-boundary value problem (1.1) - (1.3) has a unique global solution $(u, v)(t, x)$ which is smooth for $t > 0$. This solution verifies the estimates $0 \leq u(t, x) \leq \|u_0\|_{L^\infty(\Omega)}$ and $0 \leq v(t, x) \leq \|v_0\|_{L^\infty(\Omega)}$ for $t \geq 0, x \in \Omega$ and converges to the equilibrium $(\overline{u}_0 - \overline{v}_0, 0)$ uniformly in $x \in \Omega$ as $t \to \infty$, that is,

$$\lim_{t\to\infty} \|(u, v)(t) - (\overline{u}_0 - \overline{v}_0, 0)\|_{L^\infty(\Omega)^2} = 0.$$

In order that we state our results, we define

$$u_\infty = \overline{u}_0 - \overline{v}_0 = \frac{1}{|\Omega|}\int_\Omega (u_0 - v_0)(x)dx$$

and

$$\overline{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(t, x)dx, \qquad \overline{v}(t) = \frac{1}{|\Omega|}\int_\Omega v(t, x)dx. \tag{2.1}$$

Integrate (1.1) over $(0, t) \times \Omega$ and apply the Green formula to the resulting expression. Then, it follows from (1.2) and (1.3) that $\overline{u}(t) - \overline{v}(t) = \overline{u}_0 - \overline{v}_0$ for $t \geq 0$. In particular, we have: When $\overline{u}_0 > \overline{v}_0$,

$$\overline{u}(t) = u_\infty + \overline{v}(t), \qquad t \geq 0, \tag{2.2}$$

and when $\overline{u}_0 = \overline{v}_0$,

$$\overline{u}(t) = \overline{v}(t), \qquad t \geq 0. \tag{2.3}$$

Our first theorem, which is on rates of the above convergence as $t \to \infty$, can now be stated as follows.

Theorem 1. *There exists a positive constant T such that the solution $(u, v)(t, x)$ for (1.1) - (1.3) satisfies the following properties:*

(i) *When* $\overline{u}_0 > \overline{v}_0$ *and* $n > 1$,

$$\|(u,v)(t) - (u_\infty, 0)\|_{L^\infty(\Omega)^2} \leq K(1+t-T)^{-\alpha}, \tag{2.4}$$

$$\|(u-\overline{u}, v-\overline{v})(t)\|_{L^\infty(\Omega)^2} \leq K(1+t-T)^K e^{-d_0\lambda(t-T)} \tag{2.5}$$

for $t \geq T$, *where* $\alpha = 1/(n-1)$, $d_0 = \min\{d_1, d_2\}$, λ *is the smallest positive eigenvalue of* $-\Delta$ *with homogeneous Neumann boundary condition on* $\partial\Omega$ *and* K *denotes a constant depending on* $\|(u_0, v_0)\|_{L^\infty(\Omega)^2}$ *but not on* T.

(ii) *When* $\overline{u}_0 = \overline{v}_0$,

$$\|(u,v)(t)\|_{L^\infty(\Omega)^2} \leq K(1+t-T)^{-\beta}, \tag{2.6}$$

$$\|(u-\overline{u}, v-\overline{v})(t)\|_{L^\infty(\Omega)^2} \leq K(1+t-T)^K e^{-d_0\lambda(t-T)} \tag{2.7}$$

for $t \geq T$, *where* $\beta = 1/(m+n-1)$ *and* d_0, λ *and* K *are the same as in* (i).

Our second theorem gives a large-time approximation of the solution $(u,v)(t,x)$. Before stating the results, we summarize basic properties of the solution $(U,V)(t)$ to the problem (1.7), (1.8) for the corresponding system of ordinary differential equations. When $\overline{u}_0 > \overline{v}_0$ and $n > 1$, we have

$$U(t) = u_\infty + V(t), \qquad t \geq 0, \tag{2.8}$$

$$V(t) \sim t^{-\alpha} \quad \text{as} \quad t \to \infty, \tag{2.9}$$

where $\alpha = 1/(n-1)$. When $\overline{u}_0 = \overline{v}_0$, as the counterparts of (2.8) and (2.9), we have

$$U(t) = V(t), \qquad t \geq 0, \tag{2.10}$$

$$V(t) \sim t^{-\beta} \quad \text{as } t \to \infty, \tag{2.11}$$

where $\beta = 1/(m+n-1)$. In fact, in this case, $V(t) = U(t)$ satisfies

$$\frac{dV}{dt} = -V^{m+n}, \quad t > 0 \quad \text{and} \quad V(0) = \overline{v}_0 \tag{2.12}$$

and is given explicitly as

$$V(t) = \overline{v}_0\{1 + (m+n-1)\overline{v}_0{}^{m+n-1}t\}^{-1/(m+n-1)}. \tag{2.13}$$

Large-time behavior of the solution $(u,v)(t,x)$ can now be described in terms of the $(U,V)(t)$ as follows.

Theorem 2. *Let $(u,v)(t,x)$ be the solution for* (1.1) - (1.3) *and let $(U,V)(t)$ be the solution to the problem* (1.7), (1.8). *Then the following asymptotic relations hold:*

(i) *When $\overline{u}_0 > \overline{v}_0$ and $n > 1$,*

$$(u,v)(t,x) = (U,V)(t) + O(t^{-\alpha-1}), \tag{2.14}$$

$$(u,v)(t,x) = (\overline{u},\overline{v})(t) + O(t^{\mu-\alpha}e^{-d_0\lambda t}), \tag{2.15}$$

uniformly in $x \in \Omega$ as $t \to \infty$. Here $(U,V)(t)$ and $(\overline{u},\overline{v})(t)$ verify (2.8), (2.9) *and* (2.2); *α, d_0 and λ are the same as in Theorem* 1, *and $\mu = k/(n-1)$ with*

$$k = 1 + ((\sqrt{2}-1)/2)n. \tag{2.16}$$

(ii) When $\overline{u}_0 = \overline{v}_0$,

$$(u,v)(t,x) = (U,V)(t) + O(t^{-\beta-1}), \tag{2.17}$$

$$(u,v)(t,x) = (\overline{u},\overline{v})(t) + O(t^{\nu-\beta}e^{-d_0\lambda t}), \tag{2.18}$$

uniformly in $x \in \Omega$ as $t \to \infty$. Here $(U,V)(t)$ and $(\overline{u},\overline{v})(t)$ verify (2.10), (2.11) and (2.3); β, d_0 and λ are the same as in Theorem 1, and $\nu = \ell/(m+n-1)$ with

$$\ell = 1 + (\sqrt{2(m^2+n^2)} - (m+n))/2. \tag{2.19}$$

Remark. (i) Since $k > 1$ in (2.16), we have $\mu > \alpha$ so that (2.15) actually involves the polynomial growth order $t^{\mu-\alpha}$. Similarly, we have $\ell \geq 1$ in (2.19) and hence $\nu \geq \beta$. However, the equality $\nu = \beta$ holds if and only if $m = n$, and we can simplify (2.18) as

$$(u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(e^{-d_0 \lambda t})$$

in this case. This would be the optimal estimate.

(ii) If we put $m = 0$ formally in (2.19), then the resulting ℓ coincides with the k in (2.16).

3. Rate of Convergence

In this section, we give an outline of proof of Theorem 1. We only discuss the case where $\overline{u}_0 = \overline{v}_0$ and show the estimates (2.6) and (2.7) in a series of lemmas. For (2.4) (resp. (2.5)) refer to [6] (resp. [5]).

First, we prove the $L^2(\Omega)$-decay estimate by a simple energy method which makes use of the following Poincaré inequality: For $w \in W^{1,2}(\Omega)$ satisfying $\partial w/\partial \nu = 0$ on $\partial\Omega$,

$$\lambda \|w - \overline{w}\|^2_{L^2(\Omega)} \leq \|\nabla w\|^2_{L^2(\Omega)}, \tag{3.1}$$

where λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$, and $\overline{w}$ is the mean value of $w(x)$ over Ω, that is,

$$\overline{w} = \frac{1}{|\Omega|} \int_\Omega w(x) dx. \tag{3.2}$$

Lemma 3.1. *Let $\overline{u}_0 = \overline{v}_0$. Then there exists a positive constant T (which is determined by (3.4) below) such that for $1 \leq p \leq 2$,*

$$\|(u, v)(t)\|_{L^p(\Omega)^2} \leq K(1 + t - T)^{-\beta}, \qquad t \geq T, \tag{3.3}$$

where $\beta = 1/(m + n - 1)$ and K is a constant depending on $\|(u_0, v_0)\|_{L^\infty(\Omega)^2}$ but not on T.

Outline of proof. It suffices to prove (3.3) for $p = 2$. Note that we can choose a positive constant T so large that

$$\|(u, v)(t)\|^{m+n-1}_{L^\infty(\Omega)^2} \leq d_0 \lambda / C \qquad \text{for } t \geq T. \tag{3.4}$$

By the simple energy method, the Poincaré inequality (3.1) and the Hölder inequality, we can get

$$\frac{1}{2}\frac{d}{dt}\|(u,v)(t)\|^2_{L^2(\Omega)^2} + C\|(u,v)(t)\|^{m+n+1}_{L^2(\Omega)^2} \le 0, \qquad t > T,$$

which leads us to (3.3) with $p = 2$. $\Box$

Next we show the $L^\infty(\Omega)^2$-decay estimate (2.6). We will use the L^p-L^q estimate for the semigroup associated with the heat equation. Let us denote by A the operator $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$ and let $\{e^{-tA}\}_{t\ge 0}$ be the corresponding semigroup. For given $w(x)$, we define

$$P_0 w = \frac{1}{|\Omega|}\int_\Omega w(x)dx, \tag{3.5}$$

$$(P_+ w)(x) = w(x) - P_0 w. \tag{3.6}$$

Note that $P_0 w$ is just the mean value $\overline{w}$ defined in (3.2). It is well known that P_0 and P_+ are the projections onto the eigenspaces of A corresponding to the principal eigenvalue $\lambda = 0$ and to the totality of positive eigenvalues, respectively. It is also known that the semigroup e^{-tA} satisfies the following L^p-L^q estimate: For $1 \le q \le p \le \infty$,

$$\|e^{-tA}P_+ w\|_{L^p(\Omega)} \le Cm(t)^{-(N/2)(1/q-1/p)}e^{-\lambda t}\|P_+ w\|_{L^q(\Omega)}, \tag{3.7}$$

where $m(t) = \min\{1, t\}$, λ is the smallest positive eigenvalue of A (the same as the λ in (3.1)) and C is some positive constant. For the details of (3.14), see [1], [3] and [7].

By means of the L^p-L^q estimate stated above, we can prove the $L^\infty(\Omega)^2$-decay estimate (2.6).

Lemma 3.2. *Let $\overline{u}_0 = \overline{v}_0$. Then the $L^\infty(\Omega)$-decay estimate* (2.6) *holds true for $t \ge T$, where T is the constant in Lemma* 3.1 *and is determined by* (3.4).

Outline of proof. We make use of the decomposition $w = P_0 w + P_+ w$. The P_0-part of the solution is estimated by using $|P_0 w| \le |\Omega|^{-1}\|w\|_{L^1(\Omega)}$ and (3.3) with $p = 1$ as

$$|P_0(u,v)(t)| \le K(1 + t - T)^{-\beta}, \qquad t \ge T,$$

where $\beta = 1/(m+n-1)$. Here and in what follows K denotes a constant depending on $\|(u_0, v_0)\|_{L^\infty(\Omega)^2}$ but not on T.

It remains to prove

$$(3.8) \qquad \|P_+(u,v)(t)\|_{L^\infty(\Omega)^2} \le K(1+t-T)^{-\beta}, \qquad t \ge T,$$

with the same β. To this end we use the semigroup e^{-td_1A} and transform the first equation of (1.1) into an integral equation. After applying the projection P_+ defined by (3.6), we obtain

$$(P_+u)(t) = e^{-(t-T)d_1A}(P_+u)(T) - \int_T^t e^{-(t-\tau)d_1A}P_+(u^m v^n)(\tau)d\tau, \qquad t \ge T.$$

We have a similar equation also for v, in which d_1 is replaced by d_2. If we apply the L^p-L^q estimate (3.7) to these integral equations, then we get

$$(3.9) \qquad \|P_+(u,v)(t)\|_{L^p(\Omega)^2} \le Ke^{-d_0\lambda(t-T)}$$
$$+K\int_T^t m(t-\tau)^{-(N/2)(1/q-1/p)}e^{-d_0\lambda(t-\tau)}\|P_+(u,v)(\tau)\|_{L^q(\Omega)^2}d\tau, \qquad t \ge T,$$

where $1 \le q \le p \le \infty$. Here we used the fact that $P_+(u^m v^n)$ satisfies

$$\|P_+(u^m v^n)(t)\|_{L^p(\Omega)}$$
$$\le C(\|u(t)\|^{m-1}_{L^\infty(\Omega)}\|v(t)\|^n_{L^\infty(\Omega)}\|P_+u(t)\|_{L^p(\Omega)} + \|u(t)\|^m_{L^\infty(\Omega)}\|v(t)\|^{n-1}_{L^\infty(\Omega)}\|P_+v(t)\|_{L^p(\Omega)})$$

or, in a more compact form,

$$(3.10) \qquad \|P_+(u^m v^n)(t)\|_{L^p(\Omega)} \le C\|(u,v)(t)\|^{m+n-1}_{L^\infty(\Omega)^2}\|P_+(u,v)(t)\|_{L^p(\Omega)^2},$$

where $1 \le p \le \infty$. We require

$$(3.11) \qquad (N/2)(1/q - 1/p) < 1.$$

We use (3.9) for suitable p and q to prove the desired estimate (3.8). When $N = 1, 2$ or 3, we put $p = \infty$ and $q = 2$ in (3.9). When $N \ge 4$, we define $\{p_j\}$ by $p_0 = 1$ and $1/p_j - 1/p_{j+1} = 1/N$, $j = 0, 1, 2, \cdots, N-1$. We see that $\{p_j\}$ is an increasing sequence such that $p_{N-1} < \infty$ and $p_N = \infty$. We now put $p = p_{j+1}$ and $q = p_j$ in (3.9) which satisfy (3.11), we can show (3.8). $\square$

Once the $L^\infty(\Omega)^2$-decay estimate (2.6) is known, one can prove the asymptotic relation (2.7) rather easily as follows.

Lemma 3.3. *Let $\overline{u}_0 = \overline{v}_0$. Then* (2.7) *holds true for $t \geq T$, where T is the constant in Lemma* 3.1.

Outline of proof. It follows from (3.10) and (2.6) that

$$\begin{aligned}\|P_+(u,v)(t)\|_{L^p(\Omega)^2} \leq\ & Ce^{-d_0\lambda(t-T)}\|P_+(u,v)(T)\|_{L^p(\Omega)^2} \\ & +K\int_T^t e^{-d_0\lambda(t-\tau)}(1+\tau-T)^{-1}\|P_+(u,v)(\tau)\|_{L^p(\Omega)^2}\,d\tau\end{aligned}$$

for $t \geq T$. Then Gronwall's lemma yields

$$\|P_+(u,v)(t)\|_{L^p(\Omega)^2} \leq C\|P_+(u,v)(T)\|_{L^p(\Omega)^2}(1+t-T)^K e^{-d_0\lambda(t-T)}, \qquad t \geq T. \qquad \square$$

4. Large-time approximation

The aim of this section is to prove Theorem 2. We only discuss the case where $\overline{u}_0 = \overline{v}_0$ also here. The case where $\overline{u}_0 > \overline{v}_0$ and $n > 1$ can be studied along the similar manner. We first give an outline of proof of the asymptotic relations (2.17) and (2.18) under the additional restriction that the initial perturbation from the mean value $(\overline{u}_0, \overline{v}_0)$ is small enough in Subsection 4.1. Then in Subsection 4.2 we remove this restriction imposed on the initial data and prove (2.17) and (2.18) in full generality. For the details of (2.18) and (2.19), refer to [5].

4.1. The case $\overline{u}_0 = \overline{v}_0$ with small initial data

Let us consider the case where $\overline{u}_0 = \overline{v}_0$. We want to estimate the solution $(u,v)(t,x)$ to the problem (1.1) - (1.3) in the form

$$(4.1) \qquad u(t,x) = V(t)(1+\phi(t,x)), \quad v(t,x) = V(t)(1+\psi(t,x)),$$

where $V(t)$ is the solution of (2.12) and is given explicitly as in (2.13). A straightforward computation shows that the $(\phi,\psi)(t,x)$ defined by (4.1) must satisfy

$$\begin{cases} \phi_t = d_1\Delta\phi - V(t)^{m+n-1}\{(m-1)\phi + n\psi + f\}, & t>0,\ x\in\Omega, \\ \psi_t = d_2\Delta\psi - V(t)^{m+n-1}\{m\phi + (n-1)\psi + f\}, & t>0,\ x\in\Omega, \end{cases} \tag{4.2}$$

$$\frac{\partial\phi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0, \qquad t>0,\ x\in\partial\Omega, \tag{4.3}$$

$$(\phi,\psi)(0,x) = (\phi_0,\psi_0)(x), \qquad x\in\Omega,$$

where

$$f = (1+\phi)^m(1+\psi)^n - (1+m\phi+n\psi),$$

$$\phi_0(x) = (u_0(x)-\overline{u}_0)/\overline{v}_0, \qquad \psi_0(x) = (v_0(x)-\overline{v}_0)/\overline{v}_0. \tag{4.4}$$

Note that we used $\overline{u}_0 = \overline{v}_0$ in the first relation of (4.4). Note also that $P_0\phi_0 = P_0\psi_0 = 0$, namely, the mean value of $(\phi_0,\psi_0)(x)$ vanishes.

We will prove the following theorem for $(\phi,\psi)(t,x)$ that implies (ii) of Theorem 2 in the case where the initial perturbation from the mean value is small enough.

Theorem 4.1. *Let $\overline{u}_0 = \overline{v}_0$. Then there exists a positive constant ε_0 such that if $\|(\phi_0,\psi_0)\|_{L^\infty(\Omega)^2} \le \varepsilon_0$, then the $(\phi,\psi)(t,x)$ defined by (4.1) satisfies*

$$\|(\phi,\psi)(t)\|_{L^\infty(\Omega)^2} \le C\|(\phi_0,\psi_0)\|_{L^\infty(\Omega)^2}(1+t)^{-1}, \tag{4.5}$$

$$\|P_+(\phi,\psi)(t)\|_{L^\infty(\Omega)^2} \le C\|(\phi_0,\psi_0)\|_{L^\infty(\Omega)^2}(1+t)^{\nu}e^{-d_0\lambda t} \tag{4.6}$$

for $t\ge 0$, where C is a constant and d_0, λ and ν are the same as in Theorem 2, that is, $d_0 = \min\{d_1,d_2\}$, λ is the smallest positive eigenvalue of the operator A defined in Section 3 and $\nu = \ell/(m+n-1)$ with ℓ in (2.19).

This theorem will be proved in a series of lemmas below. We use the following notations: For $1 \leq p \leq \infty$,

$$
\begin{aligned}
I_p &= \|(\phi_0, \psi_0)\|_{L^p(\Omega)^2}, \\
M_p(t) &= \sup_{0 \leq \tau \leq t} (1+\tau)\|(\phi, \psi)(\tau)\|_{L^p(\Omega)^2}, \\
M_\infty^0(t) &= \sup_{0 \leq \tau \leq t} (1+\tau)|P_0(\phi, \psi)(\tau)|, \\
M_p^+(t) &= \sup_{0 \leq \tau \leq t} (1+\tau)^{-\nu} e^{d_0 \lambda \tau}\|P_+(\phi, \psi)(\tau)\|_{L^p(\Omega)^2}.
\end{aligned}
$$

Obviously we have

$$I_q \leq C I_p, \quad M_q(t) \leq C M_p(t), \quad M_q^+(t) \leq C M_p^+(t)$$

for $1 \leq q \leq p \leq \infty$, where $C = |\Omega|^{1/q - 1/p}$.

We state some lemmas which are used in order to prove Theorem 4.1 without proofs. First, we estimate the P_0-part of the $(\phi, \psi)(t, x)$.

Lemma 4.2. *It holds true that*

$$M_\infty^0(t) \leq K(t) M_2(t)^2, \qquad t \geq 0, \tag{4.7}$$

where and in what follows $K(t)$ denotes a quantity depending only on $\sup_{0 \leq \tau \leq t} \|(\phi, \psi)(\tau)\|_{L^\infty(\Omega)^2}$.

This lemma can be shown with use of (1.1), (3.5) and the facts that $P_0 \phi_0 = P_0 \psi_0 = 0$ and $P_0 \phi(t) = P_0 \psi(t)$ for all $t \geq 0$.

Second, we estimate the P_+-part of the $(\phi, \psi)(t, x)$ by using an energy method.

Lemma 4.3. *For $1 \leq p \leq 2$,*

$$M_p^+(t) \leq C I_2 + K(t) M_\infty(t) M_2^+(t). \tag{4.16}$$

Third, we derive the $L^\infty(\Omega)^2$-estimate by applying the L^p-L^q estimate (3.7) to the integral equations which (ϕ, ψ) must satisfy.

Lemma 4.4. *The following estimate holds true:*

$$M_\infty^+(t) \le CI_\infty + K(t)M_\infty(t)M_\infty^+(t). \tag{4.8}$$

Outline of proof of Theorem 4.1. We can complete the proof of Theorem 4.1 as follows. By definition we see that

$$M_\infty(t) \le C(M_\infty^0(t) + M_\infty^+(t)) \tag{4.9}$$

for some constant C. Therefore, combining (4.7) and (4.8), we have

$$M_\infty^0(t) + M_\infty^+(t) \le CI_\infty + K(t)(M_\infty^0(t) + M_\infty^+(t))^2. \tag{4.10}$$

Recall that $K(t)$ depends only on $\sup_{0\le\tau\le t} \|(\phi,\psi)(\tau)\|_{L^\infty(\Omega)^2}$ and hence is considered as a function of $M_\infty(t)$ or of $M_\infty^0(t) + M_\infty^+(t)$ by (4.9). Therefore (4.10) can be regarded as an inequality for $M_\infty^0(t) + M_\infty^+(t)$ and is solved in the form

$$M_\infty^0(t) + M_\infty^+(t) \le CI_\infty, \qquad t \ge 0,$$

provided that I_∞ is suitably small, say $I_\infty \le \varepsilon_0$. Consequently, we have $M_\infty(t) \le CI_\infty$ and $M_\infty^+(t) \le CI_\infty$ for $t \ge 0$, which imply the desired estimates (4.5) and (4.6), respectively. This completes the proof of Theorem 4.1. □

4.2. The case $\overline{u}_0 = \overline{v}_0$ with large initial data

In the previous subsection, we have proved (ii) of Theorem 2 for initial data close to the mean value. The aim of this subsection is to remove that restriction imposed on the initial data. To this end we first observe the following asymptotic relation.

Lemma 4.5. *Let $\overline{u}_0 = \overline{v}_0$ and let $(u,v)(t,x)$ be the solution for (1.1) - (1.3). Then it holds true that*

$$\|(u-\overline{u}, v-\overline{v})(t)\|_{L^\infty(\Omega)^2}/\overline{v}(t) \longrightarrow 0 \quad \text{as } t \to \infty,$$

where $(\overline{u},\overline{v})(t)$ is the mean value of $(u,v)(t,x)$ and is defined by (2.1); note that $\overline{u}(t) = \overline{v}(t)$ for $t \ge 0$ by (2.3).

This lemma can be proved by the comparison theorem and (2.7).

By virtue of Lemma 4.5, we can apply Theorem 4.1 for large time and prove (ii) of Theorem 2 without any restriction on the size of the initial data.

Outline of proof of (ii) *of Theorem 2.* Let ε_0 be the positive constant in Theorem 4.1. By virtue of Lemma 4.5, we can choose a constant $T_0 \geq T$ (T in Lemma 3.1) so large that

$$\|(u-\overline{u}, v-\overline{v})(t)\|_{L^\infty(\Omega)^2}/\overline{v}(t) \leq \varepsilon_0 \qquad \text{for } t \geq T_0. \tag{4.11}$$

For this choice of T_0, we define $(\tilde{U}, \tilde{V})(t)$ for $t \geq T_0$ as the solution to the problem

$$\frac{d\tilde{U}}{dt} = -\tilde{U}^m\tilde{V}^n, \quad \frac{d\tilde{V}}{dt} = -\tilde{U}^m\tilde{V}^n, \qquad t > T_0,$$

$$(\tilde{U}, \tilde{V})(T_0) = (\overline{u}, \overline{v})(T_0).$$

Since $\overline{u}(T_0) = \overline{v}(T_0)$ by (2.3), we see that

$$\tilde{U}(t) = \tilde{V}(t), \qquad t \geq T_0,$$

$$\tilde{V}(t) \sim t^{-\beta} \qquad \text{as } t \to \infty, \tag{4.12}$$

where $\beta = 1/(m+n-1)$, and that $\tilde{V}(t)$ solves

$$\frac{d\tilde{V}}{dt} = -\tilde{V}^{m+n}, \quad t > T_0, \qquad \tilde{V}(T_0) = \overline{v}(T_0).$$

Now, as in the previous subsection, we estimate the solution $(u, v)(t, x)$ to the problem (1.1) - (1.3) in the form

$$u(t,x) = \tilde{V}(t)(1 + \tilde{\phi}(t,x)), \quad v(t,x) = \tilde{V}(t)(1 + \tilde{\psi}(t,x)).$$

The $(\tilde{\phi}, \tilde{\psi})(t, x)$ introduced here satisfies (4.2) and (4.3) (for $t > T_0$) with $V(t)$ replaced by $\tilde{V}(t)$. Moreover, we have $\|(\tilde{\phi}, \tilde{\psi})(T_0)\|_{L^\infty(\Omega)^2} \leq \varepsilon_0$ by (4.11). Therefore Theorem 4.1 can be applied to $(\tilde{\phi}, \tilde{\psi})(t, x)$ for $t \geq T_0$ and we obtain

$$\|(\tilde{\phi}, \tilde{\psi})(t)\|_{L^\infty(\Omega)^2} \leq C\|(\tilde{\phi}, \tilde{\psi})(T_0)\|_{L^\infty(\Omega)^2}(1 + t - T_0)^{-1}, \tag{4.13}$$

$$\|P_+(\tilde{\phi},\tilde{\psi})(t)\|_{L^\infty(\Omega)^2} \le C\|(\tilde{\phi},\tilde{\psi})(T_0)\|_{L^\infty(\Omega)^2}(1+t-T_0)^\nu e^{-d_0\lambda(t-T_0)} \tag{4.14}$$

for $t \ge T_0$. The estimates (4.12) and (4.14) applied to the expression

$$(u,v)(t,x) = (\overline{u},\overline{v})(t) + \tilde{V}(t)(P_+\tilde{\phi}, P_+\tilde{\psi})(t,x)$$

give the desired asymptotic relation (2.18). In order that we prove (2.17), we write

$$u(t,x) = U(t) + \tilde{V}(t)(1 - V(t)/\tilde{V}(t) + \tilde{\phi}(t,x)),$$

$$v(t,x) = V(t) + \tilde{V}(t)(1 - V(t)/\tilde{V}(t) + \tilde{\psi}(t,x)),$$

where $V(t)$ is the solution of (2.12) and where we used (2.10). Since we already verified (4.12) and (4.13) and we can show the following lemma that concerns with the difference between $V(t)$ and $\tilde{V}(t)$ for $t \to \infty$, we can obtain (2.17).

Lemma 4.6. When $\overline{u}_0 = \overline{v}_0$, we have

$$V(t)/\tilde{V}(t) - 1 = O(t^{-1}) \qquad \text{as } t \to \infty.$$

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