数理解析研究所講究録
914 巻 1995 年 85-97

Nonlinear Stability of Travelling Waves for One-dimensional Viscoelastic Materials Without Non-Convex Nonlinearity

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ABSTRACT

The aim of this paper is to study the stability of travelling wave solutions with shock profiles for one-dimensional viscoelastic materials with the non-degenerate and the degenerate shock conditions by means of an elementary weighted energy method. The stress function is not necessarily assumed to be convex or concave, and the third derivative of this stress function is also not necessarily assumed to be non-negative or non-positive. The travelling waves are proved to be stable for suitably small initial disturbance and shock strengths, which improves recent stability results. The key points of our proofs are to choose the suitable weight function and weighted Sobolev spaces of the solutions.

1. Introduction

In this paper we study the asymptotic stability of travelling wave solutions with shock profiles for one-dimensional viscoelastic materials with non-convex nonlinearity in the form

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t - \sigma(v)_x = \mu u_{xx}, \tag{1.2}$$

with the initial data

$$(v,u)|_{t=0} = (v_0,u_0)(x) \to (v_\pm,u_\pm) \qquad \text{as} \quad x \to \pm\infty, \tag{1.3}$$

which arises in the theory of viscoelastic materials. Here, $x \in R^1$ and $t \geq 0$, v is the strain, u the velocity, $\mu > 0$ the viscous constant, $\sigma(v)$ is the smooth stress function satisfying

$$\sigma'(v) > 0 \quad \text{for all} \quad v \quad \text{under consideration}, \tag{1.4}$$

$$\sigma''(v) \lesseqgtr 0 \quad \text{for} \quad v \lesseqgtr 0 \quad \text{under consideration}, \tag{1.5}$$

so that $\sigma(v)$ is neither convex nor concave, and has a point of inflection at $v = 0$. We find that the system (1.1),(1.2) with $\mu = 0$ is strictly hyperbolic, with the characteristic roots

$$\lambda = \pm\lambda(v), \qquad \text{where} \quad \lambda(v) = \sqrt{\sigma'(v)}$$

and with the corresponding right eigenvectors

$$r_\pm(v) = \begin{pmatrix} 1 \\ \mp\lambda(v) \end{pmatrix}.$$

Moreover, we see that both characteristic fields are neither genuinely nonlinear nor linearly degenerate in the neighborhood of $v = 0$. In fact, the quantity

$$\nabla\lambda(v) \cdot r_\pm(v) = \lambda'(v) = \sigma''(v)/2\sqrt{\sigma'(v)},$$

changes its sign at $v = 0$, where ∇ denottes the gradient with respect to (v,u).

The travelling wave solutions are solutions of the form

$$(v,u)(t,x) = (V,U)(\xi), \quad \xi = x - st, \tag{1.6}$$

$$(V,U)(\xi) \to (v_\pm,u_\pm), \qquad \xi \to \pm\infty, \tag{1.7}$$

where s is the shock speed and $(v_\pm,u_\pm)$ are constant stats at $\pm\infty$. Let the system (1.1),(1.2) admit the existence of travelling wave solutions, then both $(v_\pm,u_\pm)$ and s satisfy the Rankine-Hugoniot condition

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) - (\sigma(v_+) - \sigma(v_-)) = 0, \end{cases} \tag{1.8}$$

and the generalized shock condition

$$\frac{1}{s}h(v) \equiv \frac{1}{s}[-s^2(v-v_\pm)+\sigma(v)-\sigma(v_\pm)]\begin{cases} <0, & \text{if} \quad v_+<v<v_- \\ >0, & \text{if} \quad v_-<v<v_+. \end{cases} \tag{1.9}$$

We note that the condition (1.9) with (1.4) and (1.5) implies

$$\lambda(v_+) \le s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_-) < s \le -\lambda(v_+), \tag{1.10}$$

and that, especially when $\sigma''(v) > 0$, the condition (1.9) is equivalent to

$$\lambda(v_+) < s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_-) < s < -\lambda(v_+), \tag{1.11}$$

which is well-known as Lax's shock condition(Lax[5]). We call the degenerate or non-degenerate shock condition following $s = \lambda(v_+)$ or (1.11), respectively. Through this paper, without loss of generality, let us suppose $\sigma(0) = 0$.

Notations

H^l_w $(l \ge 0)$ denotes the weighted Sobolev space of L^2_w-functions f on R whose derivatives $\partial^j_x f, j = 1, \cdots, l$, are also L^2_w-functions, where $w(x) > 0$ is a called weight function, with the norm

$$|f|_{l,w} = \Big(\sum_{j=0}^{l} |\partial^j_x f|^2_w\Big)^{1/2}.$$

C^l_w $(l \ge 0)$ denotes the weighted l-times continuously differentiable space with the weight function $w(x) > 0$, whose functions $f(x)$ satisfy $w(x)\partial^j_x f \in C^0$, $j = 0, 1, \cdots, l$, with the norm

$$\|f\|_{C^l_w} = \sup_{x\in R} \sum_{j=0}^{l} w(x)|\partial^j_x f|.$$

Denoting

$$\langle x\rangle_+ = \begin{cases} \sqrt{1+x^2}, & \text{if} \quad x \ge 0 \\ 1, & \text{if} \quad x < 0, \end{cases}$$

we will make use of the space $L^2_{\langle x\rangle_+}$ and $H^l_{\langle x\rangle_+}(l = 1,2)$. We also denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g \le f \le Cg$ in a neighborhood of a. When $C^{-1} \le w(x) \le C$ for $x \in R$, we note that $L^2 = H^0 = L^2_w = H^0_w$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$.
Let us define

$$G(v) \equiv h(v)\sigma''(v) - h'(v)\sigma'(v), \qquad v \in [0, v_*],$$

where v_* is a unique point in $(0, v_-)$ suth that $s^2 = \sigma'(v_*)$, and pay our attention to the points $v_i \in (0, v_*)$ which are defined as the followings

$$\begin{cases} v_1 & = \sup\{v|G(v) \ge 0 \text{ on } [0,v]\}, \\ v_2 & = \sup\{v|G(v) \le 0 \text{ on } [v_1,v]\}, \\ v_{2i-1} & = \sup\{v|G(v) \ge 0 \text{ on } [v_{2i-2},v]\}, \\ v_{2i} & = \sup\{v|G(v) \le 0 \text{ on } [v_{2i-1},v]\}. \end{cases}$$

Without loss generality, say say n points (n should be an odd number), $v_i \in (0, v_*), i = 1, \cdots, n$, we denote

$$I_0 \equiv (v_+, 0], \quad , \quad I_{n+2} \equiv [v_*, v_-]$$

$$I_{2j-1} \equiv [v_{2j-2}, v_{2j-1}], \quad I_{2j} \equiv [v_{2j-1}, v_{2j}], j = 1, 2, \cdots, \frac{n+1}{2}.$$

2. Stability Theorems

In this section, we shall prove the stability of travelling wave solutions with shock profiles for Cauchy problem (1.1)-(1.3) without the condition $\sigma'''(v) > 0$.

Now, without loss generality, we restrict our attention to the case

$$s > 0 \quad \text{and} \quad v_+ < 0 < v_-, \quad i.e., \quad \mu s V_\xi = h(V) < 0. \tag{2.1}$$

Let $(V, U)(x - st)$ be a pair of travelling wave solutions connecting $(v_\pm, u_\pm)$, we assume the integrability of $(v_0 - V, u_0 - U)(x)$ over R and express that integral in the form

$$\int_{-\infty}^{\infty} (v_0 - V, u_0 - U)(x)dx = 0. \tag{2.2}$$

Let us define (ϕ_0, ψ_0) by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (v_0 - V, u_0 - U)(y)dy. \qquad (2.3)$$

Our main theorems are the followings

Theorem 2.1(*Non-degenerate case: $\lambda(v_+) < s < \lambda(v_-)$.*) *Suppose that (1.4), (1.5), (1.8) and (2.1) hold, and assume that $|(v_+ - v_-, u_+ - u_-)| \ll 1$. When $(v_0, u_0)(x)$ and $(V, U)(\xi)$ satisfy (2.2), and suppose that $(\phi_0, \psi_0) \in H^2$, then there exists a positive constant δ_1 such that if $\|(\phi_0, \psi_0)\|_2 < \delta_1$, then (1.1)-(1.3) has a unique global solution $(v, u)(t, x)$ satisfying*

$$v - V \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^1),$$

$$u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2).$$

Furthermore, the solution verifies

$$\sup_{x \in R} |(v, u)(t, x) - (V, U)(x - st)| \to 0 \quad as \quad t \to \infty. \qquad (2.4)$$

Theorem 2.2(*Degenerate case: $\lambda(v_+) = s < \lambda(v_-)$*). *Suppose that (1.4), (1.5), (1.8) and (2.1) hold, and assume that $|(v_+ - v_-, u_+ - u_-)| \ll 1$ and that there exists $\overline{\delta}$ $(0 < \overline{\delta} < 1)$ such that*

$$\sigma'(0)v_+ - \sigma(v_+) < \overline{\delta} v_+(\sigma'(0) - \sigma'(v_+)) \quad as \quad v_+ \to 0_-. \qquad (2.5)$$

When $(v_0, u_0)(x)$ and $(V, U)(\xi)$ satisfy (2.2), then the followings hold:

(i) Suppose that $(\phi_0, \psi_0) \in H^2_{\langle x \rangle_+}$, then there exists a positive constant δ_2 such that if $|(\phi_0, \psi_0)|_{2, \langle x \rangle_+} < \delta_2$, then (1.1)-(1.3) has a unique global solution $(v, u)(t, x)$ satisfying

$$v - V \in C^0([0, \infty); H^1_{\langle x \rangle_+}) \cap L^2([0, \infty); H^1_{\langle x \rangle_+^{\frac{1}{2}}})$$

$$u - U \in C^0([0, \infty); H^1_{\langle x \rangle_+}) \cap L^2([0, \infty); H^2_{\langle x \rangle_+}).$$

Furthermore, the solution verifies (2.4).

(ii) Suppose that $(\phi_0, \psi_0) \in H^2 \cap L^2_{\langle x\rangle_+}$ *and* $\phi_{0,x} \in L^2_{\langle x\rangle_+^{\frac{3}{4}}}$. *Then there exists a constant* $\delta_3 > 0$ *such that if* $\|(\phi_0, \psi_0)\|_2 + |(\phi_0, \psi_0)|_{\langle x\rangle_+} + |\phi_{0,x}|_{\langle x\rangle_+^{\frac{3}{4}}} < \delta_3$, *then there is a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v - V \in C^0([0,\infty); H^1 \cap L^2_{\langle x\rangle_+}) \cap L^2([0,\infty); H^1 \cap L^2_{\langle x\rangle_+^{\frac{3}{4}}})$$

$$u - U \in C^0([0,\infty); H^1 \cap L^2_{\langle x\rangle_+}) \cap L^2([0,\infty); H^2 \cap L^2_{\langle x\rangle_+}).$$

Furthermore, the solution verifies (2.4).

Remark *1.) We note that the stability results in [4,10] both* $\sigma'''(v) > 0$ *and smallness of shock strength* $|(v_+ - v_-, u_+ - u_-)|$ *are sufficient conditions. For the degenerate shock condition,* $\lambda(v_+) = s < \lambda(v_-)$, *since Kawashima-Matsumura[4]'s estimates cannot be applied to this case, Nishihara[10], at the first time, showed the stability result for this case provided that the integral of the initial disturbance over* $(-\infty, \xi]$, *say* $(\phi_0, \psi_0)(\xi)$, *have an polynomial decay* $O(|\xi|^{-\frac{1+\alpha}{2}})$ $(0 < \alpha < 1)$ *as* $\xi \to +\infty$. *This sufficint condition is stronger than one in this paper(i.e.,* $O(|\xi|^{-\frac{1}{2}})$ *as* $\xi \to +\infty$*). It seems to get hardly the stability results by the schemes in [4,10] without* $\sigma'''(v) > 0$. *Here we show an example of* $\sigma(v)$ *as the following*

$$\sigma(v) = bv + \int_0^v \int_0^x y^3 (sin\frac{1}{y} + 2) dy dx,$$

where

$$b > \max \left| \int_0^v x^3 (sin\frac{1}{x} + 2) dx \right|,$$

which satisfies $\sigma'(v) > 0$ *and* $\sigma''(v) \lesseqgtr 0$ *for* $v \lesseqgtr 0$, *and* $\sigma'''(v)$ *changes signs on* $[v_+, v_-]$. *Therefore, we improve the stability results in [4,10].*

2.) In Theorem 2.2, it means that when the initial datas are stronger, then the property of the solution is also better. The result in (ii) is better than one in (i), because we get the stability under the *weaker conditions in (ii), i.e, we don't restrict the higher derivate of the initial data in the weighted space.*

3.) If $\sigma'''(v) > 0$ for $v < 0$, namely, $\sigma'(v)$ is convex for $v < 0$, then $\overline{\delta}$ in (2.5) can be taken as $\overline{\delta} = \frac{1}{2}$.

In order to solve the stability, we make a reformulation for the problem (1.1)-(1.3) as in [3,4,6,7,10] in the form

$$(v,u)(t,x) = (V,U)(\xi) + (\phi_\xi, \psi_\xi)(t,\xi), \quad \xi = x - st \tag{2.6}$$

Then the problem (1.1)-(1.3) is reduced to

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - \sigma'(V)\phi_\xi - \mu\psi_{\xi\xi} = F \\ (\phi,\psi)(0,\xi) = (\phi_0,\psi_0)(\xi) \end{cases} \tag{2.7}$$

with

$$F = \sigma(V + \phi_\xi) - \sigma(V) - \sigma'(V)\phi_\xi.$$

We define the solution spaces of (4.8) as

$$X_0(0,T) = \{(\phi,\psi) \in C^0([0,\infty); H^2), \phi_\xi \in L^2([0,\infty); H^1),$$

$$\psi_\xi \in L^2([0,\infty); H^2)\},$$

$$X_1(0,T) = \{(\phi,\psi) \in C^0([0,\infty); H^2_{\langle\xi\rangle_+}), \phi_\xi \in L^2([0,\infty); H^1_{\langle\xi\rangle_+^{\frac{1}{2}}}),$$

$$\psi_\xi \in L^2([0,\infty); H^2_{\langle\xi\rangle_+})\},$$

$$X_2(0,T) = \{(\phi,\psi) \in C^0([0,\infty); H^2 \cap L^2_{\langle\xi\rangle_+}), \phi_\xi \in L^2([0,\infty);$$

$$H^1 \cap L^2_{\langle\xi\rangle_+^{\frac{3}{4}}}), \quad \psi_\xi \in L^2([0,\infty); H^2 \cap L^2_{\langle\xi\rangle_+})\},$$

with $0 < T \leq \infty$. By the embedding theorem, and let

$$N_0(t) = \sup_{0\leq\tau\leq t} \|(\phi,\psi)(\tau)\|_2,$$

$$N_1(t) = \sup_{0\leq\tau\leq t} |(\phi,\psi)(\tau)|_{2,\langle\xi\rangle_+},$$

$$N_2(t) = \sup_{0\le\tau\le t}(\|(\phi,\psi)(\tau)\|_2 + |(\phi,\psi)(\tau)|_{\langle\xi\rangle_+} + |\phi_\xi(\tau)|_{\langle\xi\rangle_+^{\frac{3}{4}}}),$$

we have

$$\begin{cases} \sup_{\xi\in R}|(\phi,\psi)(t,\xi)| \le CN_0(t), \\ \sup_{\xi\in R}|(\phi,\psi)(t,\xi)| \le C\sup_{\xi\in R}|\langle\xi\rangle_+^{1/2}(\phi,\psi)(t,\xi)| \le CN_1(t), \\ \sup_{\xi\in R}|\psi(t,\xi)| \le C\sup_{\xi\in R}|\langle\xi\rangle_+^{3/4}\psi(t,\xi)| \le CN_2(t), \\ \sup_{\xi\in R}|(\phi,\psi)(t,\xi)| \le CN_2(t). \end{cases}$$

Theorem 2.1 and Theorem 2.2 can be treated from the following theorem. So, it is our purpose to prove the following theorem.

Theorem 2.3 *(A) (Non-degenerate Case): In addition to the assumptions in Theorem 2.1. Then there exists a positive constant δ_4 such that if $\|(\phi_0,\psi_0)\|_2 < \delta_4$, then (2.7) has a unique global solution $(\phi,\psi) \in X_0(0,\infty)$ satisfying*

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\}d\tau \le C\|(\phi_0,\psi_0)\|_2^2 \quad (2.8)_0$$

for any $t \ge 0$. Moreover, the stability

$$\sup_{\xi\in R}|(\phi_\xi,\psi_\xi)(t,\xi)| \to 0 \quad as \quad t\to\infty. \quad (2.9)$$

holds.

(B) (Degenerate Case): In addition to the assumptions in Theorem 2.2.

(i) Then there exists a constant $\delta_5 > 0$ such that if $|(\phi_0,\psi_0)|_{2,\langle\xi\rangle_+} < \delta_5$, then (2.7) has a unique global solution $(\phi,\psi) \in X_1(0,\infty)$ satisfying

$$|(\phi,\psi)(t)|_{2,\langle\xi\rangle_+}^2 + \int_0^t \{|\phi_\xi(\tau)|_{1,\langle\xi\rangle_+^{1/2}}^2 + |\psi_\xi(\tau)|_{2,\langle\xi\rangle_+}^2\}d\tau$$

$$\le C|(\phi_0,\psi_0)|_{2,\langle\xi\rangle_+}^2 \quad (2.8)_1$$

for any $t \ge 0$. Moreover, the stability (2.9) holds.

(ii) Then there exists a positive constant δ_6 such that if $\|(\phi_0,\psi_0)\|_2 + |(\phi_0,\psi_0)|_{\langle\xi\rangle_+} + |\phi_{0,\xi}|_{\langle\xi\rangle_+^{\frac{3}{4}}} < \delta_6$, then (2.7) has a unique global solution $(\phi,\psi) \in X_2(0,\infty)$ satisfying

$$\begin{aligned}
&\|(\phi,\psi)(t)\|_2^2 + |(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + |\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{3}{4}}} \\
&+ \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} + \|\psi_\xi(\tau)\|_2^2 + |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}\} d\tau \\
&\le C(\|(\phi_0,\psi_0)\|_2^2 + |(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}),
\end{aligned} \tag{2.8$_2$}$$

for any $t \ge 0$. Moreover, the stability (2.9) also holds.

Theorem 2.3 is proved by a weighted energy method combining the local existence with *a priori* estimates.

Proposition 2.4 *(Local existence) For any $\delta_0 > 0$, there exists a positive constant T_0 depending on δ_0 such that*

(A) (Non-degenerate Case): If $(\phi_0,\psi_0) \in H^2$ and $\|(\phi_0,\psi_0)\|_2 \le \delta_0$, then the problem (2.7) has a unique solution $(\phi,\psi) \in X_0(0,T_0)$ satisfying $\|(\phi,\psi)(t)\|_2 \le 2\delta_0$ for $0 \le t \le T_0$.

(B) (Degenerate Case): (i) If $(\phi_0,\psi_0) \in H^2_{\langle\xi\rangle_+}$ and $|(\phi_0,\psi_0)|_{2,\langle\xi\rangle_+} \le \delta_0$, then the problem (2.7) has a unique solution $(\phi,\psi) \in X_1(0,T_0)$ satisfying $|(\phi,\psi)(t)|_{2,\langle\xi\rangle_+} \le 2\delta_0$ for $0 \le t \le T_0$.

(ii) If $(\phi_0,\psi_0) \in H^2 \cap L^2_{\langle\xi\rangle_+}$, $\phi_{0,\xi} \in L^2_{\langle\xi\rangle_+^{\frac{3}{4}}}$, $\|(\phi_0,\psi_0)\|_2 + |(\phi_0,\psi_0)|_{\langle\xi\rangle_+} + |\phi_{0,\xi}|_{\langle\xi\rangle_+^{\frac{3}{4}}} \le \delta_0$, then the problem (2.7) has a unique solution $(\phi,\psi) \in X_2(0,T_0)$ satisfying $\|(\phi,\psi)(t)\| + |(\phi,\psi)(t)|_{\langle\xi\rangle_+} + |\phi_\xi|_{\langle\xi\rangle_+^{\frac{3}{4}}} \le 2\delta_0$ for $0 \le t \le T_0$.

Proposition 2.5 *(A priori estimate) (A) (Non-degenerate Case): Let $(\phi,\psi) \in X_0(0,T)$ be a solution for a positive T. Then there exists a positive constant δ_7 such that if $N_0(T) < \delta_7$, then (ϕ,ψ) satisfies the a priori estimate (2.8)$_0$ for $0 \le t \le T$.*

(B) (Degenerate Case): (i) Let $(\phi,\psi) \in X_1(0,T)$ be a solution for a positive T. Then there exists a positive constant δ_8 such that

if $N_1(T) < \delta_8$, then (ϕ, ψ) satisfies the a priori estimate $(2.8)_1$ for $0 \le t \le T$.

(ii) Let $(\phi, \psi) \in X_2(0, T)$ be a solution for a positive T. Then there exists a positive constant δ_9 such that if $N_2(T) < \delta_9$, then (ϕ, ψ) satisfies the a priori estimate $(2.8)_2$ for $0 \le t \le T$.

Proposition 2.4 can be proved in the standard way. So we omit the proof. To prove Proposition 2.5 is our global aim in the next section.

3. The Proofs of A Priori Estimates

In this section, we will give a skecth of the proofs for our stability theorems. At first, let's introduce our desired weight functions which pay a key role for our *a priori* estimates. Let a weight function be

$$w(v) = \begin{cases} w_0(v) & = \frac{v^2 - v_+^2}{h(v)}, \quad v \in I_0, \\ w_{2j-1}(v) & = k_{2j-1} \cdot \frac{-1}{h(v)}, \quad v \in I_{2j-1}, \\ w_{2j}(v) & = k_{2j} \cdot \frac{1}{\sigma'(v)}, \quad v \in I_{2j}, \\ w_{n+2}(v) & = k_{n+1} \cdot \frac{1}{\sigma'(v)}, \quad v \in I_{n+2}, \end{cases} \tag{3.1}$$

where $j = 1, \cdots, \frac{n+1}{2}$, $k_1 = v_+^2$, $k_2 = -k_1 \sigma'(v_1)/h(v_1)$, $k_{2j-1} = -k_{2j-2} h(v_{2j-2})/\sigma'(v_{2j-2})$, $k_{2j} = -k_{2j-1}\sigma'(v_{2j-1})/h(v_{2j-1})$, $j = 2, \cdots, \frac{n+1}{2}$. So $k_i > 0$ $(i = 1, 2, \cdots, n+1)$. We also denote $r(\xi)$ as another weight function in the form

$$r(\xi) = \begin{cases} 1 + \xi - \xi_0, & \text{as } \xi \ge \xi_0, \\ 1, & \text{as } \xi \le \xi_0, \end{cases} \tag{3.2}$$

where ξ_0 is defined as such number such that $V(\xi_0) = 0$ in the section 2. Then we know that $w(V) \in C^0(v_+, v_-]$, $w(V) \notin C^1(v_+, v_-]$, but $w_i(V) \in C^2(I_i)$, $i = 0, 1, \cdots, n+1, n+2$. $r(\xi)$ has the same property of $w(V)$. Moreover, we find

$$\text{non-degenerate case:} \quad w(V(\xi)) \sim Const., \quad L^2_w = L^2, \tag{3.3$_1$}$$

$$\text{degenerate case:} \quad w(V(\xi)) \sim r(\xi) \sim \langle \xi \rangle_+, \quad L^2_w = L^2_r = L^2_{\langle \xi \rangle_+}. \tag{3.3$_2$}$$

Let $(\phi,\psi) \in X_1(0,T)$ be a solution of (2.7). On the every interval R_i $(i = 0,1,\cdots,n+2)$, multiplying the first equation of (4.7) by $(w_i\sigma')(V)\phi$ and the second equation of (4.7) by $w_i(V)\psi$ and adding those equations, we have

$$\begin{aligned}
&\frac{1}{2}\{(w_i\sigma')(V)\phi^2 + w_i(V)\psi^2\}_t - \{(w_i\sigma')(V)\phi\psi + \mu w_i(V)\psi\psi_\xi\}_\xi \\
&- \frac{s}{2}\{(w_i\sigma')(V)\phi^2 + w_i(V)\psi^2\}_\xi + \mu w_i(V)\psi_\xi^2 + A_i(t,\xi) \\
&= F w_i(V)\psi, \qquad (3.4)
\end{aligned}$$

where

$$\begin{aligned}
A_i(t,\xi) &= \frac{s}{2}(w_i\sigma')'(V)V_\xi\phi^2 + \mu w_i'(V)V_\xi\psi\psi_\xi \\
&+ (w_i\sigma')'(V)V_\xi\phi\psi + \frac{s}{2}w_i'(V)V_\xi\psi^2, \quad i = 0,1,\cdots,n+2. \qquad (3.5)
\end{aligned}$$

Integrating (3.4) over R_i and adding thses integrated equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R ((w\sigma')(V)\phi^2 + w(V)\psi^2)d\xi + \mu\sum_{i=0}^{n+2}\int_{R_i} w(V)\psi_\xi^2 d\xi \\
&+ \sum_{i=0}^{n+2}\int_{R_i} A_i(t,\xi)d\xi = \int_R F w(V)\psi d\xi. \qquad (3.6)
\end{aligned}$$

By some detail estimates on (3.6), we can prove the following Key Lemma. .

Key Lemma 3.1 *It holds*

$$\begin{aligned}
&|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+} d\tau + \int_0^t\int_{R_0} |V_\xi|\psi(\tau,\xi)^2 d\xi d\tau \\
&\qquad \le C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + N_1(t)\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} d\tau). \qquad (3.7)
\end{aligned}$$

From equations (2.7), we have

$$\mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + \sigma'(V)\phi_\xi + s\psi_\xi - \psi_t = -F. \qquad (3.8)$$

Since $L^2_{w(V)} = L^2_{r(\xi)} = L^2_{\langle\xi\rangle_+}$, firstly, let's consider our problem in the weighted space $L^2_{r(\xi)}$. Multiplying (3.8) by $r(\xi)^{\frac{1}{2}}\phi_\xi$ on the intervals $[\xi_0, +\infty)$ and $(-\infty, \xi_0]$, respectively, then adding them and integrating resulted equation over $R \times [0, t]$, we obtain

Key Lemma 5.5 *It holds*

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} + (1 - CN_1(t)) \int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} d\tau \\ \leq C(|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}) \tag{3.9}$$

provided that $N_1(t)$ is small.

Due to both of two Key Lemmas, futherly, estimating the higher order derivations of (ϕ, ψ), we can prove (i) of Part (B) in Proposition 2.5.

Similarly, we can prove Part (A) and (ii) of Part (B) in Proposition 2.5.

Acknowledgement The authors would like to express their sincere thanks to Prof. Akitaka Matsumura for his helpful suggestions and discussions.

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