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COMPLEX DYNAMICAL SYSTEMS : ALGEBRAIC CURVES IN THE CUBIC MAPS

Kiyoko NISHIZAWA
Asako NOJIRI
DEP.MATH. FAC.SCI. JOSAI UNIV.
DEP.MATH. SHIRAYURI GAKUENN 1-1
Keyakidai
Sakado
Saitama 350-02 JAPAN e-mail:
kiyoko@euclides.josai.ac.jp

Abstract. We define center curves in the moduli space, consisting of all affine conjugacy classes of cubic maps and analyse the dynamics of maps along these curves.

25.1 Moduli space of the complex cubic polynomials.

We consider the family of cubic maps $x \mapsto g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ $(c_3 \neq 0, c_i \in \mathbf{C})$. For such a cubic map g , we have two normal forms ; $x^3 - 3Ax \pm \sqrt{B}$, $A, B \in \mathbf{C}$. Therefore, the complex affine conjugacy class of g can be represented by (A, B). The **moduli space**, denoted by $\mathcal{M}$, consisting of all affine conjugacy classes of cubic maps, can be identified with the coordinate space $\mathbf{C}^2 = \{(A, B)\}$ ([5]).

25.1.1 Center in a hyperbolic component.

A complex cubic map f, or the corresponding point $(A, B) \in \mathcal{M}$, belongs to the **connectedness locus** if the orbits of both critical points p_i such that $f'(p_i) = 0$, $i = 1, 2$, are bounded. And f is **hyperbolic** if both of these critical orbits converge towards attracting periodic orbits. The set of all hyperbolic points in the moduli space $\mathcal{M}$ forms an open set.

Each connected component of this open set is called a **hyperbolic component**. By M.Rees([9]), each hyperbolic component contains a unique post-critically finite complex cubic map. So following A. Douady and J. Hubbard ([2]), this map is called a **center map** or**Thurston map** and the coordinates (A, B) of f will be called a **center** in the moduli space. Following M.Rees([9])and J.Milnor([5]), the centers are roughly classified into four diffeent types, as follows. (In the following t, p, q denote integers.) A center is of the type $\mathcal{A}_p$ if two critical points p_1, p_2 of the center map coinside and has the period p : $f^p(p_1) = p_1$. (In fact, only possible values for p in this case are $1, 2$.) A center is of the type $\mathcal{B}_{p+q}$ if $f^p(p_1) = p_2$ and $f^q(p_2) = p_1$; of the type $\mathcal{C}_{(t)q}$ if $f^t(p_1) = p_2$ and $f^q(p_2) = p_2$; of the type $\mathcal{D}_{p,q}$ if $f^p(p_1) = p_1$ and $f^q(p_2) = p_2$.

These exhaust all types of centers. It is clear that there are only a finite number of centers of a given type.

Example: There exist a unique center of type $\mathcal{A}_i$. The corresponding parameter is $(0, 0)$ for $i = 1$ and $(0, -1)$ for $i = 2$. There exist three centers of type $\mathcal{C}_{(3)1}$. The corresponding parameters are

$(A, B) = (-.75040, -.18820)$, $(-.74949, -.18679)$, $(-.0924912, -.0614376)$.

25.2 Center curves in the moduli space.

The **center curves** CDp, BCp, which are algebraic curves, can be defined according to the above four renormalization-type. We show how the equations of these curves are obtained by induction on p ([7] and [8]).

Theorem 2.1 : Defining equation of a center curve *For a given p, there exist an algebraic curve CDp containing all centers of the type $\mathcal{C}_{(k)p}$ and $\mathcal{D}_{k,p}$, and another algebraic curve BCp containing all centers of the type $\mathcal{B}_{p+k}$ and $\mathcal{C}_{(p)k}$. For example, we obtain precisely the following curves;*

$$\begin{aligned}
CD1 &: \quad B = 4A(A+\tfrac{1}{2})^2,\\
BC1 &: \quad B = 4A(A-\tfrac{1}{2})^2,\\
CD2 &: \quad B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5\\
&\qquad\qquad -12A^4 + 16A^3 - 4A + 1 = 0,\\
BC2 &: \quad B^3 - 12A^3B^2 - 6AB^2 + 2B^2 + 48A^6B + 24A^3B + 21A^2B\\
&\qquad -6AB + B - 64A^9 + 96A^7 - 20A^5 - 12A^3 - A = 0,
\end{aligned}$$

Proof: Let $f(x) = x^3 - 3Ax + \sqrt{B}$, with critical points $\pm\sqrt{A}$.

The equation of curve BC1: $B = A(2A-1)^2$ is obtained by the following equations:

$$\begin{aligned}
f(\sqrt{A}) - (-\sqrt{A}) &= (-2A+1)\sqrt{A} + \sqrt{B} = 0\\
f(-\sqrt{A}) - \sqrt{A} &= (2A-1)\sqrt{A} + \sqrt{B} = 0.
\end{aligned}$$

The equation of curve CD1: $B = A(2A+1)^2$ is obtained by the following equations:

$$\begin{aligned}
f(\sqrt{A}) - \sqrt{A} &= (-2A-1)\sqrt{A} + \sqrt{B} = 0,\\
f(-\sqrt{A}) - (-\sqrt{A}) &= (2A+1)\sqrt{A} + \sqrt{B} = 0.
\end{aligned}$$

The equation of curve BC2 is obtained as follows:

$$f^2(\sqrt{A}) - (-\sqrt{A}) = 0, \quad f^2(-\sqrt{A}) - \sqrt{A} = 0$$

Therefore,

$$\begin{aligned}
\text{BC2}: \quad & B^3 - 12A^3B^2 - 6AB^2 + 2B^2 + 48A^6B + 24A^3B + 21A^2B\\
& -6AB + B - 64A^9 + 96A^7 - 20A^5 - 12A^3 - A = 0.
\end{aligned}$$

The equation of curve CD2 is obtained by the equations:

$$f^2(\sqrt{A}) - \sqrt{A}, \quad f^2(-\sqrt{A}) - (-\sqrt{A}) = 0$$

Thus

$$B(12A^3 - 3A + 1 + B)^2 - A(-8A^4 + 6A^2 - 1 - 6AB)^2 = 0$$

Fixed points can be also considered as periodic points of period 2. So, this curve contains CD1. Dividing the left-hand side of the last equation by the defining polynomial of CD1, we get the equation of CD2 as follows:

$$\begin{aligned}
\text{CD2}: B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5&\\
-12A^4 + 16A^3 - 4A + 1 = 0.&
\end{aligned}$$

Suppose now,

$$\begin{aligned} f^p(\sqrt{A}) &= P_p\sqrt{A} + Q_p\sqrt{B}, \\ f^n(-\sqrt{A}) &= -P_p\sqrt{A} + Q_p\sqrt{B}, \end{aligned}$$

where P_p, Q_p are polynomials of A, B. Then we have

$$\begin{aligned} P_p &= AP_{p-1}^3 + 3BP_{p-1}Q_{p-1}^2 - 3AP_{p-1}, \\ Q_p &= 3AP_{p-1}^2 Q_{p-1} + BQ_{p-1}^3 - 3AQ_{p-1} + 1. \end{aligned}$$

The equation of curve BCp: $(P_p+1)^2 A - Q_p^2 B = 0$ is obtained as follows:

$$\begin{aligned} f^p(\sqrt{A}) - (-\sqrt{A}) &= (P_p+1)\sqrt{A} + Q_p\sqrt{B} = 0, \\ f^p(-\sqrt{A}) - \sqrt{A} &= (-P_p-1)\sqrt{A} + Q_p\sqrt{B} = 0. \end{aligned}$$

The equation of curve CDp is obtained as follows:

$$\begin{aligned} f^p(\sqrt{A}) - \sqrt{A} &= (P_p-1)\sqrt{A} + Q_p\sqrt{B} = 0, \\ f^p(-\sqrt{A}) - (-\sqrt{A}) &= (-P_p+1)\sqrt{A} + Q_p\sqrt{B} = 0. \end{aligned}$$

Let

$$\tilde{\phi}_p(A,B) := (P_p-1)^2 A - Q_p^2 B.$$

If $\phi_q(A,B) = 0$ is the defining equaiton of CDq, then we have

$$\tilde{\phi}_p(A,B) = \prod_{q \mid p} \phi_q(A,B).$$

Therefore if $\{q_1, \cdots, q_n\}$ is the set of all divisors of p except p, then

$$\text{CD}p : \phi_p(A,B) = \tilde{\phi}_p(A,B) / \prod_{i=1}^{n} \phi_{q_i}(A,B) = 0.$$

Remark: *The defining equations of Center curves BCp and CDp ,*$1 < p < 5$ *are obtained by RISA/ASIR (computer algebra system by FUJITSU CO.LTD.)*

25.3 Algebraic-geometric properties of center curves

We can embed $\mathbf{C}^2$ canonically in $\mathbf{P}^2(\mathbf{C}) : (A,B) \to (1 : A : B)$. Then an affine algebraic curve $V_0 = \{(A,B) \in \mathbf{C}^2 : h(A,B) = 0\}$ uniquely determines a projective algebraic curve $V = \{(C : A :$

$B) \in \mathbf{P}^2(\mathbf{C}) : H(C : A : B) = 0\}$ in $\mathbf{P}^2(\mathbf{C})$ such that $h(A, B) = H(1 : A : B)$ and $V \cap \mathbf{C}^2 = V_0$.

Definition. For a center curve V_0, the corresponding projective algebraic curve V is called the **projective center curve**. We denote by PBCp and PCDp, these curves corresponding to BCp and CDp respectively.

We give some algebraic-geometric properties of these curves.

Theorem 3.1 : The interseciton with the line at infinity ([8]). *Each projective center curve and the line at infinity,* $L_\infty : C = 0$, *intersect at the point* $(0 : 0 : 1)$ *only. This point* $(0 : 0 : 1)$ *is singular and its multiplicity can be calculated explicitly.*

Remark. *PCD1 and PBC1 are both cuspidal cubic. But for* $p \geq 2$, *the point* $(0 : 0 : 1)$ *is not a "simple cusp".*

25.3.1 Case $p = 1, 2$.

We get the following theorem about the irreducibility of each projective center curve, which is based on Kaltofen's algorithms on *RISA/ASIR* (computer algebra system by FUJITSU CO.LTD.) ([10] , [6]).

Theorem 3.2 : Irreducibility and Singurarity ([8]). *For projective center curves PCDi and PBCi* $(i = 1, 2)$,

•*PCD*1 *and PBC*1.

These two are irreducible curves of degree 3. *Hence, no other singular points exist.*

•*PCD*2.

It is an irreducible curve of degree 6. *It has one 4-fold point* $(0 : 0 : 1)$ *and one ordinary double point* $(0.25, -0.4375)$.

•*PBC*2.

It is an irreducible curve of degree 9. *It has one 6-fold point* $(0 : 0 : 1)$ *and four ordinary double points:*

$$(-0.1341351918179714, -1.37344484910264),$$
$$(-0.5531033117555605, -0.6288238268413773),$$
$$(0.3041906503790061 * i + 0.3436192517867655,$$
$$0.6886343379400248 - 0.04267412324347224 * i),$$
$$(0.3436192517867655 - 0.3041906503790061 * i,$$
$$0.04267412329900053 * i + 0.6886343379735695),$$

25.3.2 Genus of center curves.

Definition

([3]).

To calculate genus g of each projective center curve Γ, we determine the principal part at $(0:0:1)$ of the curves by using Newton Polygons and apply the Plücker's formula. I am grateful to Y. Komori([4]) for helpful suggestions on the genus.

Lemma 3.3 : Principal part of the center curves. *The principal part at* $(0:0:1)$ *of PCD1 and of PBC1 is* $(C^2-4A^3)^1$, *of PCD2 is* $(C^2-4A^3)^2$, *and of PBC2 is* $(C^2-4A^3)^3$.

Plücker's formula *Let* Γ *be an irreducible curve of degree* n. *Let* $Sing\Gamma = \{P_1, \cdots, P_k\}$ *be the set of singular points* P_i *of* Γ *and of its strict transform obtained by blowing up several times. Let* r_i *be the multiplicity of* P_i. *Then,*

$$g = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{k} \frac{r_i(r_i-1)}{2}.$$

Theorem 3.4 : Genus. *The curves PCD1 and PBC1 are rational. Hence the genus is 0. The genus of PCD2 is* 1. *The genus of PBC2 is* 3.

We would like to state the following conjectures for the projective center curves:

Conjectures • All projective center curves are irreducible.

- All singular points except $(0:0:1)$ are ordinary double points.
- Especially, for real graph of center curves, the sigular point exists only in $\mathcal{R}_1$.
- The pricipale part at $(0:0:1)$ of every projective center curve has a form $(C^2-4A^3)^k$.

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