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ON THE GLOBAL EXISTENCE OF SOLUTIONS FOR THE DISCRETE BOLTZMANN EQUATION WITH LINEAR AND QUADRATIC TERMS

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In this paper, we study the discrete Boltzmann equation in one-dimensional space with linear and quadratic terms. This system, which is more general than the usual one by the intervention of linear terms, describes the gas motion of molecules which take only a finite number ($\sharp I < \infty$) of velocities $c_i (i \in I)$ under the interactions between particles represented by the quadratic terms and also under the reflection of molecules at the inner wall of an infinite thin tube, represented by the linear terms which we treated in the papers [9], [10], [11], [12], [13], [14].

$$\text{(B)} \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) , \\ \qquad u_i|_{t=0} = u_i^0(x) \ , \ i \in I, t \in \mathbf{R}_+, x \in \mathbf{R}^1 \end{cases}$$

where

$$Q_i(u) = \sum_{j,k,\ell \in I} (A_{ij}^{k\ell} u_k u_\ell - A_{k\ell}^{ij} u_i u_j) ,$$

$$L_i(u) = \sum_{k \in I} (\alpha_i^k u_k - \alpha_k^i u_i) .$$

The physical theory imposes to this system the natural conditions :

Conditions .—

$$(*) \qquad \begin{cases} A_{ij}^{k\ell} \geqq 0, \quad A_{ij}^{k\ell} = A_{ji}^{k\ell} = A_{ij}^{\ell k} , \\ A_{k\ell}^{ij} \neq 0 \quad \Rightarrow \quad i \neq j , \\ \alpha_i^k \geqq 0 \quad \text{and} \quad \alpha_i^i = 0 \qquad \text{for all} \quad i, k \in I . \end{cases}$$

This linear terms are more general than the ones which are obtained by considering solutions around constant stationary solutions (M_i) with $Q_i(M) = 0$, called constant Maxwellian. We suppose furthermore the condition of distinct velocities and the microreversibility condition for the quadratic terms.

Condition vd.—

$$i \neq j \rightarrow c_i \neq c_j$$

Condition μrQ.—

$$\sum_{k,\ell\in I} A^{ij}_{k\ell} = \sum_{k,\ell\in I} A^{k\ell}_{ij} \text{ for } \forall i,j \in I$$

We introduce the space $\mathcal{B}(\mathbf{R}\times[0,T])$ defined as follows : for $T<\infty$ fixed,

$$\mathcal{B}(\mathbf{R}\times[0,T]) = \{u(x,t) \ : u^\sharp_i(x,t) \in C(0,T;L^1_{\ell oc}(\mathbf{R}))$$
$$\text{such that} \sum_{i\in I}\int_K \sup_{t\in[0,T]} \left|u^\sharp_i(x,t)\right| dx < \infty, \text{ for } \forall K \subset\subset \mathbf{R}\},$$

$$\text{where } u^\sharp_i(x,t) = u_i(x+c_it,t) \ .$$

Definition.— *Suppose the condition (vd). Let be $u^0_i \in L^1_{\ell oc}(\mathbf{R})$ and $u \in \mathcal{B}(\mathbf{R}\times[0,T])$. We say that u is a solution of the system (B) with Cauchy data $(u^0_i)_i$ if and only if*

$$u^\sharp_i(x,t) = u^0_i(x) + \int_0^t \{Q^\sharp_i(u)(x,s) + L^\sharp_i(u)(x,s)\}ds \ .$$

Remark : The solutions defined as above are weaker than those in the distribution sense.

Main Theorem.— *Suppose the conditions (vd) and (μrQ). Let the Cauchy data be positive with locally finite entropy (not necessary bounded). Then there exists a global solution in time and the solution is unique and positive.*

Let's show the local existence in time for the small mass :

Theorem 2.— *Suppose the condition (vd). There exists $\delta>0$ such that, for $\left\|u^0\right\|_{L^1} \leqq \delta$, there exists an unique solution u in $\mathcal{B}(\mathbf{R}\times[0,\delta])$, and we have $\|u\|_{\mathcal{B}} \leqq 2\delta$. Furthermore, the mapping $u^0 \mapsto u$ is continuous from the ball with radius δ of L^1 to the ball with radius 2δ of $\mathcal{B}(\mathbf{R}\times[0,\delta])$. Finally, we have the finite velocity propagation and the conservation of the positivity.*

Proof. The equation can be written in the form $u - Ku = f$, where

$$(Ku)_i(x,t) = \int_0^t (Q_i(u)+L_i(u))\,(x-c_i(t-s),s)ds$$

and $(f)_i = u^0_i(x-c_it)$. For sufficiently small δ, K is Lipschitz continuous with Lipschitz constant $1/2$ from the ball with radius δ of L^1 to the ball with radius 2δ of $\mathcal{B}(\mathbf{R}\times[0,\delta])$. We have also $K(0)=0$ and $\|v\|_{\mathcal{B}} \leqq \delta' \Rightarrow \|Kv\|_{\mathcal{B}} \leqq \delta'/2$ for $\delta' \leqq \delta$. Then the inverse of $1-K$ exists in the form of the Neumann series $\sum_{n=0}^\infty K^n$ and it is Lipschitz continuous with Lipschitz constant 2. The mapping $u^0 \mapsto u$ is continuous from the ball with radius δ of L^1 to the ball with radius 2δ of $\mathcal{B}(\mathbf{R}\times[0,\delta])$.

Let's show the finite velocity propagation. Now we suppose that the Cauchy data are supported in $[-B,B]$ with $B<\infty$. Then it is easy to see that Ku is supported in

$\{(x,t)\ :x\in[-B-Ct,B+Ct]\}$ where $C=\max_i|c_i|$. Similarly, K^2u, then all the K^nu $(n\geqq 1)$, are supported in the same region. Because we have $u=\sum_{n=0}^{\infty}K^nf$, the support of u is included in $\{(x,t)\ :x\in[-B-Ct,B+Ct]\}$. Then we have the finite velocity propagation.

Now we suppose that u^0 is bounded. Let [0,T] with $T\leqq\delta$ be the widest interval where $\sup_{i,x}|u_i(x,\cdot)|$ is bounded. Then u is a solution in the distribution sense. It is easy to see $T>0$. We put $M=\sup_{t\in[0,T]}\sup_{i,x}|u_i(x,t)|<\infty$. By the equation, we have

$$\begin{aligned}\sup_x u_i^\sharp(x,t)&\leqq\left\|u^0\right\|_{L^\infty}+C_*M\|u\|_{\mathcal{B}}+C_*\|u\|_{\mathcal{B}}+MT\\&\leqq C_*\delta M+C_*\delta+\left\|u^0\right\|_{L^\infty}\end{aligned}$$

where the constant C_* depends only on the equation. For $\delta<(2C_*)^{-1}$, we have

$$M=\sup_{t\in[0,T]}\sup_{i,x}|u_i(x,t)|\leqq 2\left(1+\left\|u^0\right\|_{L^\infty}\right)\ .$$

This upperbound depends only on the Cauchy data. Therefore we have $T=\delta$. We omit a proof for the conservation of the positivity. (See [14]) ∎

Corollary 3.— *Suppose the condition (vd). Let u^0 be positive data in L^1 and h a number such that $\int_a^{a+h}u_i^0\leqq\delta$ for any $a\in\mathbf{R}$, then there exists an unique solution u in $\mathcal{B}(\mathbf{R}\times[0,\theta])$ with $\theta=\min\{\delta,h/C\}$ and $C=\max_i|c_i|$. Furthermore we have the finite velocity propagation. Finally, if we suppose the condition ($\mu r Q$), then, for the Cauchy data such that u^0 are supported in $[-R,R]$ and verifying $\sum_i\int_{\mathbf{R}}u_i^0\log u_i^0(x)dx<\infty$, we have, for $t\in[0,\theta]$, $H(t)+K\leqq e^{C_*t}(H(0)+K)$ with C_* which depends only on the equation, where $H(t)=\sum_i\int_{\mathbf{R}}u_i\log u_i(x,t)dx$ and K depends only on the equation and R .*

Proof. By virtue of the finite velocity propagation shown in the preceding theorem, we can restrict the data in the interval $[a,a+h]$. Then the solution exists in small triangles of base $[a,a+h]$ and of height $\min\{\delta,h/C\}$. Pasting the solutions defined in these triangles, we have a solution in $\mathcal{B}$ up to the time θ with $\theta=\min\{\delta,h/C\}$.

For the calculus on the increase of $H(t)$, we make approach the data by $u_n^0=\inf(u^0,n)$. Then we have $\sum_i\int_{\mathbf{R}}u_{n,i}^0\log u_{n,i}^0(x,t)dx\leqq\sum_i\int_{\mathbf{R}}u_i^0\log u_i^0(x,t)dx<\infty$. By virtue of the theorem 2, the solutions u_n corresponding to the Cauchy data u_n^0 exist up to the time θ. Furthermore they are bounded and positive up to the time θ. Especially the solutions have their support in x included in $[-R',R']$ with $R'=R+C_*\theta$. Therefore the quantity

$H_n(t) = \sum_i \int_{\mathbf{R}} u_{n,i} \log u_{n,i}(x,t)dx$ are well-defined. We write

$$\begin{aligned}
&\sum_i \left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) (u_{n,i} \log u_{n,i} e^{-Ct}) \\
&= -C u_{n,i} \log u_{n,i} e^{-Ct} \\
&- \frac{1}{2} e^{-Ct} \sum_{ijk\ell} A_{ij}^{k\ell} (u_{n,k} u_{n,\ell} \log \frac{u_{n,k} u_{n,\ell}}{u_{n,i} u_{n,j}} - u_{n,k} u_{n,\ell} + u_{n,i} u_{n,j}) \\
&- e^{-Ct} \sum_{ik} \alpha_i^k u_{n,k} (\log u_{n,k} - \log u_{n,i}) \\
&\leqq -C e^{-Ct} \sum_i u_{n,i} \log u_{n,i} - e^{-Ct} \sum_{ik} \alpha_i^k u_{n,k} (\log u_{n,k} - \log u_{n,i}) \ ,
\end{aligned}$$

where C is a constant which we choose later. Integrating this inequality over $[0,t]$ then over $\mathbf{R}$, we obtain

$$\begin{aligned}
&e^{-Ct} H_n(t) - H_n(0) \\
&\leqq -C e^{-Ct} \int_0^t H_n(\tau) d\tau - e^{-Ct} \sum_{ik} \alpha_i^k \int_0^t \int_{\mathbf{R}} u_{n,k} \log u_{n,k} d\tau dx \\
&+ e^{-Ct} \sum_{ik} \alpha_i^k \int_0^t \int_{\mathbf{R}} u_{n,k} \log u_{n,i} d\tau dx \ .
\end{aligned}$$

We have

$$\begin{aligned}
-e^{-Ct} \sum_{ik} \alpha_i^k \int_0^t \int_{\mathbf{R}} u_{n,k} \log u_{n,k} d\tau dx &\leqq C_* t e^{-Ct} R' \\
&\leqq C_* R' t
\end{aligned}$$

with a constant C_* which depends only on the equation, because we have $-x \log x \leqq 1/e$ for any $x \in \mathbf{R}$. Because $u_k \log u_i \leqq \max\{0, u_i \log u_i, u_k \log u_k\}$, we obtain

$$\begin{aligned}
\sum_{ik} \alpha_i^k \int_0^t \int_{\mathbf{R}} u_k \log u_i d\tau dx &\leqq C_* e^{-Ct} \int_0^t H_n(\tau) d\tau + C_* t e^{-Ct} R' \\
&\leqq C_* e^{-Ct} \int_0^t H_n(\tau) d\tau + C_* R' t
\end{aligned}$$

with a constant C_* which depends only on the equation. We obtain finally

$$e^{-Ct} H_n(t) - H_n(0) \leqq -(C - C_*) e^{-Ct} \int_0^t H_n(\tau) d\tau + C_* R' t \ .$$

Taking the constant $C > C_*$, and using $-H_n(t) \leqq C_* R'$, we have

$$H_n(t) \leqq e^{Ct} (H(0) + C_* R' t) \ .$$

Putting $K = C_* R'$, we have

$$\begin{aligned} H_n(t) + K &\leqq e^{Ct}(H(0) + K + Kt) \\ &\leqq e^{Ct}\left(H(0) + Ke^t\right) \\ &\leqq e^{(C+1)t}(H(0) + K) \ . \end{aligned}$$

By virtue of the conservation of the mass, the $u_n(\cdot, t)$ converge to the solution $u(\cdot, t)$ in L^1 for each $t \in [0, \theta]$. Taking a suitable sub-sequence, the $u_n(\cdot, t)$ converge to $u(\cdot, t)$ almost everywhere. Because $u_{n,i} \log u_{n,i}(\cdot, t)$ are upper-bounded by $1/e$ and that they are supported in a fixed compact set, the Fatou's lemma give that

$$\begin{aligned} H(t) + K &= \sum_i \int_{\mathbf{R}} u_i \log u_i(x,t)\ dx + K \\ &\leqq \liminf \sum_i \int_{\mathbf{R}} u_{n,i} \log u_{n,i}(x,t)\ dx + K \\ &= \liminf H_n(t) + K \leqq e^{(C+1)t}(H(0) + K) \ . \blacksquare \end{aligned}$$

Corollary 4.— *Suppose the conditions (vd).*
1) Suppose that there exist two solutions u and v in $\mathcal{B}(\mathbf{R} \times [0,T])$ corresponding to the summable and positive data which coincident in an interval $[a,b]$. Then the solutions coincident in the triangle or trapezoid $\{(x,t)\ : t \in [0,T], x \in [a + Ct, b - Ct]\}$.
2) Let the Cauchy data be supported in $[-R, R]$, summable and positive. Suppose that there exists a solution u in $\mathcal{B}(\mathbf{R} \times [0,T])$. Then the support of $u(\cdot, t)$ is included in $[-R - Ct, R + Ct]$.

Proof. Let be $t_0 = \inf\{t\ : u(\cdot, t) \neq v(\cdot, t)\}$. We have then $u(\cdot, t_0) = v(\cdot, t_0) \in L^1$. Because u and v are in L^1, there exists p such that $\int_{\{x\ : u(x,t_0) \geqq p\}} u(x, t_0) dx < \delta/2$. Taking $h = \delta/(2p)$, we have, for any $a \in \mathbf{R}$,

$$\int_a^{a+h} u_i(x, t_0) dx \leqq \frac{\delta}{2} + hp \leqq \delta \ .$$

Using the preceding corollary, u and v coincident in small triangles of base $[a, a+h] \cap \{t = t_0\}$ and of height θ, which is a contradiction.
To prove the finite velocity propagation, we introduce

$$\begin{aligned} t_0 = \inf\{t \quad & \text{such that the support of } u(\cdot,t) \text{ is not} \\ & \text{included in } [-R - Ct, R + Ct] \quad \} \ . \end{aligned}$$

Then, similarly, there exists h such that $\int_a^{a+h} u_i(x, t_0) dx \leqq \delta$. By virtue of the preceding corollary, the support of $u(\cdot, t)$ is included in $[-R - Ct, R + Ct]$ up to the time $t_0 + \theta$, which is a contradiction. $\blacksquare$

Lemma 5.— *Suppose the conditions (vd) and (μrQ). Let $u(x,t)$ be a positive solution defined in* $\mathbf{R} \times [0,T]$ *and supported in* $[-R,R]$. *We suppose that, for any* $t \in [0,T]$, $\sum_i \int_{\mathbf{R}} u_i \log u_i(x,t)\, dx$ *is less than a constant* C *which don't depend on* t . *Then, for any* $\delta > 0$, *there exists a* h *which depends only on* R, C, *and* δ *such that* $\int_a^{a+h} u(x,t)\, dx \leqq \delta$ *for any* $a \in \mathbf{R}$ *and any* $t \in [0,T]$.

Proof. If not, for any $h > 0$, there should exist $a_* \in \mathbf{R}$ and $t_* \in [0,T^*]$ such that $\int_{a_*}^{a_*+h} u_i(x,t_*)dx > \delta$. Now we use the argument due to Toscani [8] and Tartar-Crandall [7]. We put, for $m \geqq 1$,

$$B_1 = \{x \in [a_*, a_* + h] \ : u_i(x,t_*) \geqq e^m\}$$

and

$$B_2 = [a_*, a_* + h] \setminus B_1 \ .$$

Then we should have

$$\int_{a_*}^{a_*+h} u_i(x,t_*)dx \leqq \frac{1}{m}\int_{B_1} u_i \log^+ u_i(y,t_*)dy + he^m$$

where $\log^+ y = \max\{\log y, 0\}$. On the other hand, we should have

$$\begin{aligned} C &\geqq \sum_i \int_{\mathbf{R}} u_i \log u_i(y,t_*)dy \\ &\geqq \sum_i \int_{-R}^{R} \left(u_i \log^+ u_i(y,t_*) - 1\right) dy \\ &= \sum_i \int_{-R}^{R} u_i \log^+ u_i(y,t_*)dy - 2pR \ , \end{aligned}$$

where $p = \sharp I$. Then we should obtain

$$\begin{aligned} \delta &< \sum_i \int_{a_*}^{a_*+h} u_i(y,t_*)dy \\ &\leqq \frac{1}{m}\sum_i \int_{a_*}^{a_*+h} u_i \log^+ u_i(y,t_*)dy + phe^m \\ &\leqq \frac{1}{m}(C + 2pR) + phe^m \ . \end{aligned}$$

Choosing m such that $\frac{1}{m}(C + 2pR) < \frac{\delta}{4}$, then h such that $phe^m < \frac{\delta}{4}$, we should have $\delta < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$, which is a contradiction. ∎

Corollary 6.— *Suppose the conditions (vd) and (μrQ). Let the Cauchy data be supported in $[-R, R]$, positive with finite entropy. Suppose that there exists a solution in $\mathcal{B}(\mathbf{R} \times [0, T])$. Then we have*

$$H(t) + K \leqq e^{C_* t}(H(0) + K)$$

where the constant C_* depends only on the equation and K depends only on the equation, R and T .

Proof. The solution have its support in x included in $[-R'R']$ with $R' = R + CT$. Let K be the constant associated to R' by the corollary 3. Let be $t_0 = \inf\{t$ such that the estimate is not valid at the time $t\}$. Taking a small $\varepsilon > 0$, we have $H(t_0 - \varepsilon) \leqq e^{C_*(t_0-\varepsilon)}(H(0) + K) \leqq e^{C_* T}(H(0) + K)$. Because $u(\cdot, t_0 - \varepsilon)$ is positive and supported in $[-R', R']$, the preceding lemma shows that there exists a h independent of ε such that we have, for any $a \in \mathbf{R}$, $\int_a^{a+h} u_i(x, t_0 - \varepsilon)dx \leqq \delta$. By virtue of the corollary 3, we have

$$\begin{aligned} H(t_0 - \varepsilon + \theta) + K &\leqq e^{C_* \theta}(H(t_0 - \varepsilon) + K) \\ &\leqq e^{C_* \theta} e^{C_*(t_0 - \varepsilon)}(H(0) + K) \,. \end{aligned}$$

The estimate is then verified up to the time $t_0 - \varepsilon + \theta$ with $\theta > 0$ independent of ε, which is a contradiction. ∎

Now let's show the proof of the main theorem.

Proof. Let T_0 be arbitrary number and we want to prove that the existence time of the solution is at lest T_0 . By virtue of the finite velocity propagation, we can suppose that the Cauchy data are supported in $[-R', R']$ and they have a finite entropy. Let T^* be the existence time of the solution. Suppose that $T^* < T_0$. By the corollary 6, the entropy is bounded for $t < T^*$: $H(t) \leqq H_T < \infty$. By virtue of the lemma 5, there exists $h > 0$ such that $\int_a^{a+h} u_i(x, t)dx \leqq \delta$ for any $a \in \mathbf{R}$ and $t < T^*$. For $t < T^*$, by the corollary 3, applied to the data $u_i(\cdot, t)$, there exists $\theta > 0$ independent of t such that the solution is prolonged in $\mathbf{R} \times [0, t + \theta]$. It is sufficient to choose $t > T^* - \theta$ for arriving at a contradiction. ∎

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