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THE DEGREE OF APPROXIMATION BY UNITAL POSITIVE LINEAR OPERATORS

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1. Introduction

Let X be a compact Hausdorff space and let $B(X)$ denote the Banach lattice of all real-valued bounded functions on X with the supremum norm $\|\cdot\|$. $C(X)$ denotes the closed sublattice of $B(X)$ consisting of all real-valued continuous functions on X. Let $p > 0$ and let G be a subset of $C(X)$ separating the points of X. For a bounded linear operator L of $C(X)$ into $B(X)$ and a function $g \in G$, we define

$$\mu^{(p)}(L;g)(y) = L(|g - g(y)1_X|^p)(y) \quad (y \in X),$$

where 1_X is the unit function defined by $1_X(t) = 1$ for all $t \in X$. Also, L is said to be unital if $L(1_X) = 1_X$. Let $\{L_\alpha\colon \alpha \in D\}$ be a net of positive linear operators of $C(X)$ into $B(X)$ and put

$$\mu_\alpha^{(p)}(g) = \mu^{(p)}(L_\alpha;g) \quad (\alpha \in D,\ g \in G),$$

whose norm is called the p-th moment for L_α with respect to g.

In [13] we proved the following convergence theorems, which may play an important role in the study of saturation property for $\{L_\alpha\}$:

Theorem A. *If* $\lim_\alpha \|\mu_\alpha^{(p)}(g)\| = 0$ *for some* $p > 0$ *and for all* $g \in G$, *and if there exists a strictly positive function* $u \in C(X)$ *such that* $\lim_\alpha \|L_\alpha(u) - u\| = 0$, *then* $\lim_\alpha \|L_\alpha(f) - f\| = 0$ *for every* $f \in C(X)$.

Theorem B. *Let* T *be a unital positive projection operator on* $C(X)$ *with* $T \neq I$ *(identity operator), such that* $L_\alpha T = T$ *for all* $\alpha \in D$. *If* $\lim_\alpha \|L_\alpha(\mu^{(p)}(T;g))\| = 0$ *for some* $p > 0$ *and for all* $g \in G$, *then* $\lim_\alpha \|L_\alpha(f) - T(f)\| = 0$ *for every* $f \in C(X)$.

These results establish a generalized Korovkin-type convergece theorem, and the Korovkin-type approximation theory is extensively treated in the books of Altomare and Campiti [1], Donner [3] and Keimel and Roth [5].

Now, in [14] we gave a quantitative version of Theorems A and B by using suitable moduli of continuity of f under certain requirements motivated by the work of the author [12], whose results can be improved by means of the higher order moments in [15].

The purpose of this paper is to refine these results for approximation of functions having certain smoothness properties by nets of unital positive linear operators of $C(X)$ into $B(X)$. Actually, the results of the author [10, 11] can be improved by means of the higher order moments. Concrete examples of approximating operators can be provided by the multidimensional Bernstein operators. Further related results and applications can be also found in [16].

2. Results

Let X be a compact convex subset of a real locally convex Hausdorff vector space E and let $G = A(X)$ denote the space of all real-valued continuous affine functions on X. If $f \in B(X)$, $\delta \geq 0$ and if $\{g_1, g_2, \cdots, g_m\}$ is a finite subset of G, then we define

$$\omega(f;g_1,\cdots,g_m,\delta) = \sup\{|f(x) - f(y)|: x,y \in X, d(x,y) \leq \delta\},$$

where

$$d(x,y) = \max\{|g_i(x) - g_i(y)|: i = 1, 2, \cdots, m\}.$$

This quantity is called the modulus of continuity of f with respect to $g_1, g_2, \cdots, g_m$.

Obviously, $\omega(f;g_1,\cdots,g_m,\cdot)$ is a monotone increasing function on $[0, \infty)$, and there holds

$$\text{(1)} \qquad \omega(f;g_1,\cdots,g_m,\xi\delta) \leq (1 + \xi)\omega(f;g_1,\cdots,g_m,\delta)$$

for all ξ, $\delta \geq 0$ (cf. [10; Lemma 1]).

A function $f \in C(X)$ is said to have the property (*MVP*) if there exist a finite subset $\{f_1, f_2, \cdots, f_r\}$

of $C(X)$ and a finite subset $\{h_1, h_2, \cdots, h_r\}$ of G such that

$$(2) \qquad f(x) - f(y) = \sum_{i=1}^{r} f_i(\xi_i)(h_i(x) - h_i(y))$$

for all $x, y \in X$, where each point ξ_i is an internal point of the segment joining x and y. In this event, we sometimes say that f has the property (MVP) associated with the system

$$(3) \qquad \{f_1, f_2, \cdots, f_r; h_1, h_2, \cdots, h_r\}.$$

Remark 1. Let $E = \mathbb{R}^r$, the r-dimensional Euclidean space equipped with the metric

$$d(x,y) = \max\{|x_i - y_i|: i = 1, 2, \cdots, r\}$$

for $x = (x_1, \cdots, x_r)$, $y = (y_1, \cdots, y_r) \in \mathbb{R}^r$ and let e_i, $i = 1, 2, \cdots, r$, be the i-th coordinate functions on X defined by

$$e_i(x) = x_i \qquad (x = (x_1, x_2, \cdots, x_r) \in X).$$

Then we have

$$\omega(f;e_1,\cdots,e_r,\delta) = \omega(f,\delta),$$

which is the usual modulus of continuity of f. Furthermore, every continuously differentiable function f on X has the property (MVP) associated with the system $\{f_1, f_2, \cdots, f_r; e_1, e_2, \cdots, e_r\}$, where

$$f_i(x) = \frac{\partial f}{\partial x_i}(x) \qquad (x = (x_1,\cdots,x_r) \in X, i = 1,\cdots,r)$$

is the i-th partial derivative of f.

From now on, we suppose that $f \in C(X)$ has the property (MVP) associated with the system (3).

Lemma 1. *Let φ be a positive linear functional on $C(X)$ with $\varphi(1_X) = 1$ and $y \in X$. Let $\{g_1, g_2, \cdots, g_m\}$ be a finite subset of G, $p > 1$ and $\delta > 0$. Then*

$$(4) \qquad |\varphi(f) - f(y)| \le \sum_{i=1}^{r} |f_i(y)||\varphi(h_i) - h_i(y)|$$

$$+ \left(1 + \delta^{-1}\left(\varphi(\Phi(\cdot, y))\right)^{1/p}\right)$$

$$\times \sum_{i=1}^{r} \left(\varphi(|h_i - h_i(y)1_X|^{p/(p-1)})\right)^{1-1/p} \omega(f_i; g_1, \cdots, g_m, \delta),$$

where

$$(5) \qquad \Phi(x, y) = \sum_{i=1}^{m} |g_i(x) - g_i(y)|^p \quad (x,\ y \in X).$$

Proof. For all $x \in X$, we define

$$F(x) = f(x) - f(y) - \sum_{i=1}^{r} f_i(y)(h_i(x) - h_i(y)).$$

Then we have

$$(6) \quad |\varphi(f) - f(y)| \le \sum_{i=1}^{r} |f_i(y)||\varphi(h_i) - h_i(y)| + |\varphi(F)|.$$

Let $1/p + 1/q = 1$. Since by (1), (2) and (5),

$$|F(x)| \le \sum_{i=1}^{r} |f_i(\xi_i) - f_i(y)||h_i(x) - h_i(y)|$$

$$\le \sum_{i=1}^{r} (1 + \delta^{-1} d(\xi_i, y))\omega(f_i; g_1, \cdots, g_m, \delta)|h_i(x) - h_i(y)|$$

$$\le \sum_{i=1}^{r} (1 + \delta^{-1} d(x, y))|h_i(x) - h_i(y)|\omega(f_i; g_1, \cdots, g_m, \delta)$$

$$\le \sum_{i=1}^{r}\left(1 + \delta^{-1}(\Phi(x,y))^{1/p}\right)|h_i(x) - h_i(y)|\omega(f_i;g_1,\cdots,g_m,\delta),$$

applying φ to both sides of this inequality with respect to the variable x and using Hölder's inequality, we get

$$|\varphi(F)| \le \sum_{i=1}^{r}\Big((\varphi(|h_i - h_i(y)1_X|^q))^{1/q}$$
$$+ \delta^{-1}(\varphi(\Phi(\cdot,y)))^{1/p}\ (\varphi(|h_i - h_i(y)1_X|^q))^{1/q}\Big)$$
$$\times\ \omega(f_i;g_1,\cdots,g_m,\delta),$$

which together with (6) implies (4).

Let $\{L_\alpha\colon \alpha \in D\}$ be a net of unital positive linear operators of $C(X)$ into $B(X)$. If $f \in B(X)$ and $g \in G$, then we define

$$\omega_\alpha(f,g) = \inf\{(1 + \varepsilon^{-1})\|\mu_\alpha^{(p/(p-1))}(g)\|^{1-1/p}$$
$$\times\ \omega\left(f;g_1,\cdots,g_m,\ \varepsilon\left\|\sum_{i=1}^{m}\mu_\alpha^{(p)}(g_i)\right\|^{1/p}\right)\colon\ p > 1,\ \varepsilon > 0,$$
$$g_1,\cdots,g_m \in G,\ \left\|\sum_{i=1}^{m}\mu_\alpha^{(p)}(g_i)\right\| > 0,\ m = 1,2,\cdots\}.$$

Using this quantity, we are now in a position to recast Theorem A in a quantitative form with the rate of convergence of $L_\alpha(f)$ for a function f having the property (*MVP*) associated with the system (3).

Theorem 1. *For all* $\alpha \in D$,

(7) $$\|L_\alpha(f) - f\| \le \sum_{i=1}^{r}\|f_i\|\|L_\alpha(h_i) - h_i\| + \sum_{i=1}^{r}\omega_\alpha(f_i,h_i).$$

Proof. Making use of Lemma 1 with $\varphi(\cdot) = L_\alpha(\cdot)(y)$ and taking the supremum over all $y \in X$, we have

$$\|L_\alpha(f) - f\| \le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(h_i) - h_i\| + \left(1 + \delta^{-1}\left\| \sum_{i=1}^{m} \mu_\alpha^{(p)}(g_i)\right\|^{1/p}\right) \times \sum_{i=1}^{r} \|\mu_\alpha^{(p/(p-1))}(h_i)\|^{1-1/p} \omega(f_i; g_1, \cdots, g_m, \delta).$$

Therefore, putting $\delta = \varepsilon\|\sum_{i=1}^{m}\mu_\alpha^{(p)}(g_i)\|^{1/p} > 0$ in the right hand side of the above inequality, we establish the desired estimate (7).

Let T be as in Theorem B. If $f \in B(X)$ and $g \in G$, then we define

$$\omega_\alpha(T; f, g) = \inf\{(1 + \varepsilon^{-1})\|L_\alpha(\mu^{(p/(p-1))}(T; g))\|^{1-1/p} \times \omega\left(f; g_1, \cdots, g_m, \varepsilon\left\| \sum_{i=1}^{m} L_\alpha(\mu^{(p)}(T; g_i))\right\|^{1/p}\right) : p > 1,\ \varepsilon > 0,\ g_1, \cdots, g_m \in G,\ \left\| \sum_{i=1}^{m} L_\alpha(\mu^{(p)}(T; g_i))\right\| > 0,\ m = 1, 2, \cdots\}.$$

Concerning the degree of convergence in Theorem B, we have the following:

Theorem 2. *For all* $\alpha \in D$,

$$\|L_\alpha(f) - T(f)\| \le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(|T(h_i) - h_i|)\| + \sum_{i=1}^{r} \omega_\alpha(T; f, h_i). \tag{8}$$

Proof. Applying Lemma 1 to $\varphi(\cdot) = T(\cdot)(y)$ with any fixed $y \in X$, we get

$$(9)\qquad |T(f) - f| \le \sum_{i=1}^{r} \|f_i\| \, |T(h_i) - h_i| + \left(1 + \delta^{-1}\left(\sum_{i=1}^{m} \mu^{(p)}(T;g_i)\right)^{1/p}\right)$$

$$\times \sum_{i=1}^{r} \left(\mu^{(p/(p-1))}(T;h_i)\right)^{1-1/p} \omega(f_i;g_1,\cdots,g_m,\delta).$$

Now let ψ be a positive linear functional on $C(X)$ with $\psi(1_X) = 1$. Applying ψ to both sides of (9) and using Hölder's inequality, we obtain

$$|\psi(T(f)) - \psi(f)| \le \sum_{i=1}^{r} \|f_i\| \psi(|T(h_i) - h_i|) + \left(1 + \delta^{-1}\left(\sum_{i=1}^{m} \psi(\mu^{(p)}(T;g_i))\right)^{1/p}\right)$$

$$\times \sum_{i=1}^{r} \left(\psi(\mu^{(p/(p-1))}(T;h_i))\right)^{1-1/p} \omega(f_i;g_1,\cdots,g_m,\delta).$$

Take $\psi(\cdot) = L_\alpha(\cdot)(x)$, where x is an arbitrary fixed point of X. Then, since $L_\alpha T = T$, we have

$$|T(f)(x) - L_\alpha(f)(x)| \le \sum_{i=1}^{r} \|f_i\| L_\alpha(|T(h_i) - h_i|)(x) + \left(1 + \delta^{-1}\left(\sum_{i=1}^{m} L_\alpha(\mu^{(p)}(T;g_i))(x)\right)^{1/p}\right)$$

$$\times \sum_{i=1}^{r} \left(L_\alpha(\mu^{(p/(p-1))}(T;h_1))(x)\right)^{1-1/p} \omega(f_i;g_1,\cdots,g_m,\delta),$$

which implies

$$\|L_\alpha(f) - T(f)\| \le \sum_{i=1}^{r} \|f_i\| \|L_\alpha(|T(h_i) - h_i|)\| + \left(1 + \delta^{-1}\left\|\sum_{i=1}^{m} L_\alpha(\mu^{(p)}(T;g_i))\right\|^{1/p}\right)$$

$$\times \sum_{i=1}^{r} \|L_\alpha(\mu^{(p/(p-1))}(T;h_i))\|^{1-1/p} \omega(f_i;g_1,\cdots,g_m,\delta).$$

Thus putting $\delta = \varepsilon\|\sum_{i=1}^{m} L_\alpha(\mu^{(p)}(T;g_i))\|^{1/p} > 0$ in the right hand side of the above inequality, we establish the desired estimate (8).

In the rest of this section it is moreover assumed that

$$T(g^i) = g^i \quad (g \in G,\ i = 1, 2, \cdots, k - 1),$$

where k is an even positive integer. In addition, we suppose that each L_α maps $C(X)$ into itself and

$$L_\alpha(g^k) = g^k + \xi_\alpha(T(g^k) - g^k)$$

for all $\alpha \in D$ and all $g \in G$, where $\{\xi_\alpha : \alpha \in D\}$ is a net of real numbers with $0 < \xi_\alpha < 1$. For $f \in B(X)$ and $\delta > 0$, we define

$$\Omega(f,\delta) = \inf\{(1 + \varepsilon^{-1}) \times \omega\left(f;g_1,\cdots,g_m,\ \delta\varepsilon\left\|\sum_{i=1}^{m}(T(g_i^k) - g_i^k)\right\|^{1/k}\right) : \varepsilon > 0,$$
$$g_1,\cdots,g_m \in G,\ \left\|\sum_{i=1}^{m}(T(g_i^k) - g_i^k)\right\| > 0,\ m = 1, 2, \cdots\}.$$

Using this quantity, we have the following result which is more convenient for later applications.

Corollary 1. *Let $\{n_\alpha : \alpha \in D\}$ be a net of positive integers and let $L_\alpha^{n_\alpha}$ denote the n_α-iteration of L_α for each $\alpha \in D$. Then for all $\alpha \in D$, we have:*

$$(10) \qquad \left\|L_\alpha^{n_\alpha}(f) - f\right\| \le \sum_{i=1}^{r} \nu_\alpha^{(k)}(h_i)\Omega(f_i, \theta_\alpha)$$

$$\leq \sum_{i=1}^{r} \nu_\alpha^{(k)}(h_i)\Omega(f_i, (n_\alpha\xi_\alpha)^{1/k}),$$

where

$$\nu_\alpha^{(k)} = \left\|\mu^{(k/(k-1))}\left(L_\alpha^{n_\alpha}; h_i\right)\right\|^{1-1/k}$$

and

$$\theta_\alpha = \left(1 - \left(1 - \xi_\alpha\right)^{n_\alpha}\right)^{1/k} ;$$

(11) $$\left\|L_\alpha^{n_\alpha}(f) - T(f)\right\| \leq \sum_{i=1}^{r} \nu_\alpha^{(k)}(T;h_i)\Omega\left(f_i, \left(1 - \xi_\alpha\right)^{n_\alpha/k}\right),$$

where

$$\nu_\alpha^{(k)}(T;h_i) = \left\|L_\alpha^{n_\alpha}\left(\mu^{(k/(k-1))}(T;h_i)\right)\right\|^{1-1/k}.$$

Indeed, by induction on the degree of iteration, it can be verified that $L_\alpha^{n_\alpha} T = T$ and

$$L_\alpha^{n_\alpha}(g^k) = T(g^k) + (1 - \xi_\alpha)^{n_\alpha}(g^k - T(g^k))$$

for all $\alpha \in D$ and all $g \in G$. Thus (10) and (11) follow from Theorems 1 and 2, respectively.

3. Applications

Let $\mathbb{N}$ denote the set of all non-nagative integers. Let $\{a_{\alpha,n}: \alpha \in D, n \in \mathbb{N}\}$ be a family of non-negative real numbers with $\sum_{n=0}^{\infty} a_{\alpha,n} = 1$ for each $\alpha \in D$, and let $\{L_n: n \in \mathbb{N}\}$ be a sequence of unital positive linear operators of $C(X)$ into $B(X)$. For any $f \in C(X)$, we define

$$T_\alpha(f) = \sum_{n=0}^{\infty} a_{\alpha,n} L_n(f) \qquad (\alpha \in D),$$

which converges in $B(X)$. Let $\{W(t)\colon t \geq 0\}$ be a family of unital positive linear operators of $C(X)$ into $B(X)$ such that for each $f \in C(X)$, the mapping $t \longmapsto W(t)(f)$ is strongly continuous on $[0, \infty)$. Let Ψ be a non-negative continuous function on $[0, \infty)$ and $\{v_\alpha\colon \alpha \in D\}$ a net of positive real numbers with $\lim_\alpha v_\alpha = 0$ or $\lim_\alpha v_\alpha = +\infty$. For any $f \in C(X)$, we define

$$C_\alpha(f) = \frac{1}{v_\alpha} \int_0^{v_\alpha} W(\Psi(t))(f)dt \qquad (\alpha \in D)$$

and

$$R_\alpha(f) = v_\alpha \int_0^{\infty} \exp(-v_\alpha t) W(\Psi(t))(f)dt \qquad (\alpha \in D),$$

which exist in $B(X)$.

All the operators given above are unital positive linear operators of $C(X)$ into $B(X)$ and our general results obtained in the preceding section are applicable to them. In particular, for applications of Corollary 1 it is convenient to make the following definition: Let S be a positive projection operator of $C(X)$ onto a closed linear subspace of $C(X)$ containing $A(X)$, $\{S_\alpha\colon \alpha \in D\}$ a net of unital positive linear operators of $C(X)$ into itself and $\{x_\alpha\colon \alpha \in D\}$ a net of non-negative real numbers. We say that $\{S_\alpha\}$ is of type $[S; x_\alpha]$ if

$$S_\alpha S = S \quad \text{and} \quad S_\alpha(g^2) = g^2 + x_\alpha(S(g^2) - g^2)$$

for all $\alpha \in D$ and all $g \in A(X)$.

Now we consider the case where X is a compact convex subset of $E = \mathbb{R}^r$, and let $C^{(1)}(X)$ denote the space of all continuously differentiable functions on X. Let $\{L_\alpha: \alpha \in D\}$ be a net of unital positive linear operators of $C(X)$ into $B(X)$. If $\|\sum_{i=1}^r \mu_\alpha^{(2)}(e_i)\| = 0$, then $L_\alpha = I$ (cf. [10; Lamma 2], [12; Lemma 1]). Thus we always consider the case where $\|\sum_{i=1}^r \mu_\alpha^{(2)}(e_i)\| > 0$ for each $\alpha \in D$. Then for all $f \in B(X)$, $\alpha \in D$ and for $j = 1, 2, \cdots, r$, we have

$$\omega_\alpha(f,e_j) \le \inf\{(1 + \varepsilon^{-1})\|\mu_\alpha^{(2)}(e_j)\|^{1/2} \times \omega\left(f, \varepsilon\left\| \sum_{i=1}^r \mu_\alpha^{(2)}(e_i)\right\|^{1/2}\right): \varepsilon > 0\}.$$

Therefore, in view of Remark 1, we extend the results of Censor [2] (cf. [7]) and give a quantitative version of Korovkin type convergence theorem due to Karlin and Ziegler [4] for all functions in $C^{(1)}(X)$.

In particular, we take $X = \mathbb{I}_r$, the unit r-cube, i.e.,

$$\mathbb{I}_r = \{x = (x_1,\cdots,x_r) \in \mathbb{R}^r: 0 \le x_i \le 1, \ i = 1, \cdots, r\},$$

and let F be the closed linear subspace of $C(\mathbb{I}_r)$ spanned by the set

$$\{e_1^{m_1} e_2^{m_2} \cdots e_r^{m_r}: m_i \in \{0,1\}, \ i = 1, 2, \cdots, r\}.$$

Let $\{B_n: n \geq 1\}$ be the sequence of the Bernstein operators on $C(\mathbb{I}_r)$, given by

$$B_n(f)(x) = \sum_{m_1=0}^{n} \cdots \sum_{m_r=0}^{n} f(m_1/n, \cdots, m_r/n) \times \prod_{i=1}^{r} \binom{n}{m_i} x_i^{m_i} (1 - x_i)^{n - m_i}$$

for $f \in C(\mathbb{I}_r)$ and $x = (x_1, \cdots, x_r) \in \mathbb{I}_r$ (see, e.g., [6]). Then it can be verified that B_1 is a positive projection operator of $C(\mathbb{I}_r)$ onto F and that $\{B_n\}$ is of type $[B_1; 1/n]$. Consequently, if $L_0 = I$, $L_n = B_n$, $n \geq 1$, then $\{T_\alpha\}$ is of type $[B_1; \sum_{n=1}^{\infty} a_{\alpha,n}/n]$, and so Corollary 1 can be applied to these operators. In particular, concerning the degree of approximation by iterations of the Bernstein operators we have the following estimates: Let $\{k_n: n \geq 1\}$ be a sequence of positive integers. Then for all $f \in C^{(1)}(\mathbb{I}_r)$ and all $n \geq 1$,

$$\text{(12)} \qquad \left\| B_n^{k_n}(f) - f \right\| \leq \frac{r}{2}\left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2}$$

$$\times \sum_{i=1}^{r} \inf\left\{(1 + \varepsilon^{-1})\omega\left(f_i, \left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2} \varepsilon\sqrt{r}/2\right) : \varepsilon > 0\right\}$$

$$\leq \frac{r}{2}\sqrt{k_n/n} \sum_{i=1}^{r} \inf\{(1 + \varepsilon^{-1})\omega(f_i, \varepsilon\sqrt{k_n/n}\, \sqrt{r}/2) : \varepsilon > 0\},$$

and

$$\text{(13)} \qquad \left\| B_n^{k_n}(f) - B_1(f) \right\| \le \frac{r}{2}\left(1 - \frac{1}{n}\right)^{k_n/2}$$

$$\times \sum_{i=1}^{r} \inf\{(1 + \varepsilon^{-1})\omega\left(f_i, \left(1 - \frac{1}{n}\right)^{k_n/2} \varepsilon\sqrt{r}/2\right) : \varepsilon > 0\},$$

where f_i stands for the i-th partial derivative of f.

Taking $\varepsilon = 2/\sqrt{r}$, (12) and (13) yield

$$\text{(14)} \qquad \left\| B_n^{k_n}(f) - f \right\| \le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right)\left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2}$$

$$\times \sum_{i=1}^{r} \omega\left(f_i, \left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2}\right)$$

$$\le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right)\sqrt{k_n/n} \sum_{i=1}^{r} \omega(f_i, \sqrt{k_n/n})$$

and

$$\text{(15)} \qquad \left\| B_n^{k_n}(f) - B_1(f) \right\| \le \frac{r}{2}\left(1 + \frac{\sqrt{r}}{2}\right)\left(1 - \frac{1}{n}\right)^{k_n/2}$$

$$\times \sum_{i=1}^{r} \omega\left(f_i, \left(1 - \frac{1}{n}\right)^{k_n/2}\right),$$

respectively. In particular, if $r = 1$, then (14) and (15) reduce to

$$\left\| B_n^{k_n}(f) - f \right\| \le \frac{3}{4}\left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2}$$

$$\times \ \omega\left(f', \left(1 - \left(1 - \frac{1}{n}\right)^{k_n}\right)^{1/2}\right) \le \frac{3}{4}\sqrt{k_n/n}\ \omega(f', \sqrt{k_n/n}),$$

which is given in [6; Theorem 1.6.2] for $\{k_n\} = \{1\}$ and

$$\left\| B_n^{k_n}(f) - B_1(f) \right\| \le \frac{3}{4}\left(1 - \frac{1}{n}\right)^{k_n/2} \omega\left(f', \left(1 - \frac{1}{n}\right)^{k_n/2}\right),$$

respectively (cf. [7], [8], [9]).

Statements analogous to the above-mentioned results may be derived for the case where B_n, $n \geq 1$, are the Bernstein operators on $C(\Delta_r)$ with the standard r-simplex

$$\Delta_r = \{x = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r:$$
$$x_i \geq 0, \ i = 1, 2, \cdots, r, \ x_1 + x_2 + \cdots + x_r \leq 1\},$$

given by

$$B_n(f)(x) = \sum_{m_1 \geq 0, \ m_1 + \cdots + m_r \leq n} f(m_1/n, \cdots, m_r/n)$$
$$\times \frac{n!}{m_1! m_2! \cdots m_r! (n - m_1 - m_2 - \cdots - m_r)!}$$
$$\times \ x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} (1 - x_1 - x_2 - \cdots - x_r)^{n - m_1 - m_2 - \cdots - m_r}$$

for $f \in C(\Delta_r)$ and $x = (x_1, \cdots, x_r) \in \Delta_r$ (see, e.g.,[6]). These can be obtained in the very general setting, and we refer to [16] for the details.

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