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BRANCH LOCI AND MONODROMY OF NORMAL SINGULARITIES

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1. Introduction

We denote by $\Delta^n(O,\epsilon)$ the n-dimensional polydisc in $\mathbf{C}^n$ with the center the origin O and the (multi)radius $\epsilon=(\epsilon',\epsilon_n)$, where $\epsilon'=(\epsilon_1,\ldots,\epsilon_{n-1})$.

In this talk, we prove the following theorem:

Theorem 1. *Let (X,x) be an n-dimensional normal singular point. Then there exists a surjective proper finite holomorphic mapping*

$$\mu:(X,x)\longrightarrow(\Delta^n(O,\epsilon),O)$$

for a sufficiently small ϵ, whose branch locus is contained in the hypersurface

$$B=\{(x',x_n)\in\Delta^n(O,\epsilon)\quad|\quad(x_n-g_1(x'))\ldots(x_n-g_N(x'))=0\},$$

where $g_j(x')$ are holomorphic functions of $x'=(x_1,\ldots,x_{n-1})$ such that $g_j(O)=0$.

Let B be a hypersurface of $\Delta^n(O,\epsilon)$ defined as in the theorem. From the theorem, we can construct a lot of normal singular points if we compute the fundamental group $\pi_1(\Delta^n(O,\epsilon)-B)$ and construct homomorphisms

$$\varphi:\pi_1(\Delta(O,\epsilon)-B)\longrightarrow S_d$$

(S_d is the d-th symmetric group), whose images are transitive. In fact, by the theorem of Grauert anf Remmert ([1]), there exists a (unique up to isomorphisms) normal singular point (X,x) and a surjective proper holomorphic mapping

$$\mu:(X,x)\longrightarrow(\Delta^n(O,\epsilon),O)$$

of degree d whose branch locus is contained in the hypersurface B and the monodromy representation is φ.

We carry this program out for two dimensional normal singular points.

2. Proof of Theorem 1

Let (X, x) be an n-dimensional normal singular point. It is known (see Gunning-Rossi ([2]) that there exists a surjective proper finite holomorphic mapping

$$\pi : (X, x) \longrightarrow (\Delta^n(O, \epsilon), O)$$

for a sufficiently small ϵ, whose branch locus B_π is given by

$$B_\pi = \{(x', x_n) \in \Delta^n(O, \epsilon) \quad | \quad f(x', x_n) = 0\},$$

where

$$x' = (x_1, \ldots, x_{n-1}),$$

$$f(x', x_n) = {x_n}^N + c_{N-1}(x'){x_n}^{N-1} + \cdots + c_0(x'),$$

$$c_j(x') \quad \text{are holomorphic functions on} \quad \Delta^n(O, \epsilon) \quad \text{with} \quad c_j(O) = 0.$$

(That is, $f(x', x_n)$ is a Weierstrass polynomial.)

The holomorphi mapping μ in the theorem is defined to be the composition

$$\mu = G_{N-1} \circ \cdots \circ G_1 \circ \pi$$

where G_j are polynomial type mappings and N is the degree of the above Weierstrass polynomial $f(x', x_n)$.

We assume for simplicity

$$N = 4.$$

(The proof for general N is similar.)

(i) Put

$$G_1 : (x', x_n) \longmapsto (z', z_n) = (x', f(x', x_n)).$$

This is a surjective proper finite holomorphic mapping from an open neighborhood of O onto an open neighborhood of O. The properness follows from the fact that the roots of an algebraic equation are (multi-valued) continuous functions of the coefficients.

The branch locus of $\pi \quad (= B_\pi)$ is mapped by G_1 to $\{z_n = 0\}$. Consider the mapping $G_1 \circ \pi$. The branch locus of this mapping is contained in the union of $\{z_n = 0\}$ and the branch locus of G_1, which is contained in the hypersurface $\{R = 0\}$, where R is the resultant of

$$f(z', x_n) - z_n \quad \text{and} \quad \frac{\partial f}{\partial x_n}(z', x_n)$$

as polynomials of x_n. Note that R can be written as

$$R = -4^4 {z_n}^3 + (\text{the lower terms}).$$

Put

$$f_1 = \frac{R}{-4^4} = {z_n}^3 + d_2(z'){z_n}^2 + d_1(z')z_n + d_0(z').$$

Note that f_1 is again a Weierstrass polynomial, that is $d_j(O) = 0$.

Thus the branch locus of $G_1 \circ \pi$ is contained in the union of the hypersurfaces $\{z_n = 0\}$ and $\{f_1 = 0\}$.

(ii) Next put

$$G_2 : (z', z_n) \longmapsto (w', w_n) = (z', f_1(z', z_n)).$$

A similar argument to (i) shows that the branch locus of the composition $G_2 \circ G_1 \circ \pi$ is contained in the union of the hypersurfaces

$$w_n = 0, \quad w_n = f_1(w', 0) = d_0(w') \quad \text{and} \quad f_2 = 0,$$

where f_2 is the resultant of

$$f_1(w', z_n) - w_n \quad \text{and} \quad \frac{\partial f_1}{\partial z_n}(w', z_n)$$

(as polynomials of z_n) divided by 3^3. This is again a Weierstrass polynomial:

$$f_2 = {w_n}^2 + e_1(w')w_n + e_0(w').$$

Note that $\{w_n = 0\}$ and $\{w_n = d_0(w')\}$ contain $G_2(\{f_1 = 0\})$ and $G_2(\{z_n = 0\})$, respectively.

(iii) Finally put

$$G_3 : (w', w_n) \longmapsto (v', v_n) = (w', f_2(w', w_n)).$$

A similar argument to (i) shows that the branch locus of the composition

$$\mu = G_3 \circ G_2 \circ G_1 \circ \pi$$

is contained in the union of the hypersurfaces

$$\{v_n = 0\}, \quad \{v_n = e_0(v')\}, \quad \{v_n = f_2(v', d_0(v')) := h_0(v')\} \quad \text{and} \quad \{v_n = h_1(v')\},$$

where $v_n - h_1(v')$ is the resultant of

$$f_2(v', w_n) - v_n \quad \text{and} \quad \frac{\partial f_2}{\partial w_n}(v', w_n)$$

(as polynomials of w_n) divided by -2^2:

$$h_1(v') = e_0(v') - \frac{{e_1}^2(v')}{4}.$$

Note that $\{v_n = 0\}$ and $\{v_n = e_0(v')\}$ contain $G_3(\{f_2 = 0\})$ and $G_3(\{w_n = 0\})$, respectively. Note also that the equation $v_n = h_0(v')$ is obtained by eliminating w_n from the equations

$$v_n = f_2(v', w_n) \quad \text{and} \quad w_n = d_0(v').$$

This proves Theorem 1.

A similar method to the proof of Theorem 1 shows the following theorem:

Theorem 2.

Let V be an n dimensional algebraic variety. Then there exists a projective normal algebraic variety W which is birational to V, and a surjective proper finite morphism F of W to the complex projective space $\mathbb{P}^n$ such that the branch locus of F is contained in the union of the hyperplane H_∞ at infinity and hypersurfaces whose defining equations in the affine coordinate system are

$$x_n = f_j(x_1, \ldots, x_{n-1}), \quad (j = 1, \ldots, N),$$

where f_j are polynomials of $n-1$ variables.

3. Fundamental Groups

In the rest of this talk, we assume

$$n = 2.$$

Let B be the curve in $\Delta^2(O, \epsilon)$ defined by

$$B = \{(y - g_1(x)) \ldots (y - g_N(x)) = 0\},$$

where (x, y) is the coordinate system and $g_j(x)$ are holomorphic functions with $g_j(0) = 0$.

We can compute the fundamental group $\pi_1(\Delta^2(O, \epsilon) - B)$ by the method of Zariski-van Kampen. That is, we take a sufficiently small positive number r, which is smaller than ϵ and we consider the line $x = r$. The line meets with the curve B at N points $q_j = (r, y_j)$, $\quad 1 \leq j \leq N$. Taking a reference point o on the line with $o \neq q_j$, we consider the lassos (meridians) γ_j, $\quad 1 \leq j \leq N$, which start from the point o and round the points q_j. Next consider the circle $\{re^{it} \quad | \quad 0 \leq t \leq 2\pi\}$. When a point moves on the circle counterclockwisely, the N intersection points of the curve B and the line $x = re^{it}$ induces a braid, which induces the braid monodromy on the lassos γ_j, which gives the generating relations between them. The fundamental group $\pi_1(\Delta^2(O, \epsilon) - B)$ is the group generated by γ_j, $\quad 1 \leq j \leq N$, with the generating relations.

We describe the fundamental group dividing into several cases depending on the forms of the power series expansions at $x = 0$ of the holomorphic functions g_j.

Case 1. $g_j(x) = a_j x +$ higher terms, $\quad (a_j \neq a_k \quad \text{for} j \neq k)$.

In this case,

$$\pi_1(\Delta^2(O, \epsilon) - B) = < \gamma_1, \ldots, \gamma_N \quad | \quad \gamma_j \gamma_0 = \gamma_0 \gamma_j, \quad \text{for} 1 \leq j \leq N >,$$

where

$$\gamma_0 = \gamma_N \cdots \gamma_1.$$

Case 2. $g_j(x) = a_0x + a_jx^2 + \text{higher terms}, \quad (a_j \neq a_k \quad \text{for} j \neq k)$.
In this case,

$$\pi_1(\Delta^2(O,\epsilon) - B) = < \gamma_1, \ldots, \gamma_N \quad | \quad \gamma_j{\gamma_0}^2 = {\gamma_0}^2\gamma_j \quad \text{for} 1 \leq j \leq N >,$$

where

$$\gamma_0 = \gamma_N \cdots \gamma_1.$$

Case 3.

$$\begin{aligned} g_1(x) &= a_1x + b_1x^2 + \text{higher terms}, \\ g_2(x) &= a_1x + b_2x^2 + \text{higher terms}, \\ g_3(x) &= a_1x + b_3x^2 + \text{higher terms}, \\ g_4(x) &= a_2x + c_1x^2 + \text{higher terms}, \\ g_5(x) &= a_2x + c_2x^2 + \text{higher terms}, \\ (a_1 &\neq a_2, \quad c_1 \neq c_2, \quad b_j \quad \text{are distinct}). \end{aligned}$$

In this case,

$$\begin{aligned} &\pi_1(\Delta^2(O,\epsilon) - B) = < \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \quad | \\ &\gamma_j\delta_1\gamma_0 = \delta_1\gamma_0\gamma_j \quad (j = 1,2,3), \quad \gamma_j\delta_2\gamma_0 = \delta_2\gamma_0\gamma_j \quad (j = 4,5) >, \end{aligned}$$

where

$$\gamma_0 = \gamma_5\gamma_4\gamma_3\gamma_2\gamma_1, \quad \delta_1 = \gamma_3\gamma_2\gamma_1, \quad \delta_2 = \gamma_5\gamma_4.$$

Case 4.

$$\begin{aligned} g_1(x) &= a_1x + b_1x^2 + c_1x^3 + \text{higher terms}, \\ g_2(x) &= a_1x + b_1x^2 + c_2x^3 + \text{higher terms}, \\ g_3(x) &= a_1x + b_2x^2 + c'_1x^3 + \text{higher terms}, \\ g_4(x) &= a_1x + b_2x^2 + c'_2x^3 + \text{higher terms}, \\ g_5(x) &= a_2x + b'_1x^2 + d_1x^3 + \text{higher terms}, \\ g_6(x) &= a_2x + b'_1x^2 + d_2x^3 + \text{higher terms}, \\ g_7(x) &= a_2x + b'_2x^2 + d'_1x^3 + \text{higher terms}, \\ g_8(x) &= a_2x + b'_2x^2 + d'_2x^3 + \text{higher terms}, \\ (a_1 &\neq a_2, \quad b_1 \neq b_2, \quad b'_1 \neq b'_2, \quad c_1 \neq c_2, \quad c'_1 \neq c'_2, \quad d_1 \neq d_2, \quad d'_1 \neq d'_2). \end{aligned}$$

In this case,

$$\pi_1(\Delta^2(O,\epsilon)-B) =< \gamma_1,\ldots,\gamma_8 \quad |$$
$$\gamma_j\epsilon_1\delta_1\gamma_0 = \epsilon_1\delta_1\gamma_0\gamma_j \quad (j=1,2), \quad \gamma_j\epsilon_2\delta_1\gamma_0 = \epsilon_2\delta_1\gamma_0\gamma_j \quad (j=3,4),$$
$$\gamma_j\epsilon_3\delta_2\gamma_0 = \epsilon_3\delta_2\gamma_0\gamma_j \quad (j=5,6), \quad \gamma_j\epsilon_4\delta_2\gamma_0 = \epsilon_4\delta_2\gamma_0\gamma_j \quad (j=7,8) >$$

where

$$\epsilon_1 = \gamma_2\gamma_1, \quad \epsilon_2 = \gamma_4\gamma_3, \quad \epsilon_3 = \gamma_6\gamma_5, \quad \epsilon_4 = \gamma_8\gamma_7,$$
$$\delta_1 = \gamma_4\gamma_3\gamma_2\gamma_1, \quad \delta_2 = \gamma_8\gamma_7\gamma_6\gamma_5, \quad \gamma_0 = \gamma_8\cdots\gamma_1.$$

The fundamental group in the general case can be written in a similar way.

4. Construction of Monodromy

We want to find homomorphisms

$$\varphi : \pi_1(\Delta - B) \longrightarrow S_d$$

such that the image is transitive, where

$$\Delta = \Delta^2(O,\epsilon)$$

and S_d is the d-th symmetric group. We discuss our method only for B in Case 1 in the last section. (As for B in the general case, our method can be discussed in a similar way.)

The fundamental group $\pi_1(\Delta - B)$ in Case 1 is generated by

$$\gamma_1,\ldots,\gamma_N$$

with the generating relations

$$\gamma_0\gamma_j = \gamma_j\gamma_0, \quad (j=1,\ldots,N),$$

where

$$\gamma_0 = \gamma_N\cdots\gamma_1.$$

The homomorphism φ is constructed if we find permutations $B_1,\ldots,B_N$ and A of d- letters such that

$$AB_j = B_jA, \quad (j=1,\ldots,N)$$

and

$$A = B_N \dots B_1.$$

In fact, we define φ by

$$\varphi(\gamma_j) = B_j, \quad (j = 1, \dots, N).$$

We can find such permutations as follows: Let A be any permutation of d-letters. Let $B_1, \dots, B_{N-1}$ be any permutations in $Z_A(S_d)$, the centralizer of A in S_d. Put

$$B_N = A(B_{N-1} \dots B_1)^{-1}.$$

However the subgroup G of S_d generated by $B_1, \dots, B_N$ and A is not transitive in general. We can easily show the following lemma, whose proof is omitted:

Lemma 1. *Let G be a subgroup of $Z_A(S_d)$ which contains A. If A is expressed as the product of cyclic permutations without common letters which are not of all equal length, then G is not transitive.*

Let

$$A = (a_1 \dots a_s)(b_1 \dots b_s) \dots (c_1 \dots c_s)$$

be the decomposition into the product of cyclic permutations of equal length s without common letters. Consider the t sets

$$a = \{a_1, \dots, a_s\}, b = \{b_1, \dots, b_s\}, \dots, c = \{c_1, \dots, c_s\}, \quad (d = st).$$

Then we can easily show the following two lemmas, whose proofs are omitted.

Lemma 2. *Every permutation B in $Z_A(S_d)$ induces naturally a permutation $\Psi(B)$ of t letters $a, b, \dots, c$. The mapping Ψ is a homomorphism of $Z_A(S_d)$ onto S_t whose kernel is isomorphic to the abelian group $(\mathbb{Z}/s\mathbb{Z})^t$.*

Lemma 3. *Let G be a subgroup of $Z_A(S_d)$ which contains A. Then G is a transitive subgroup of S_d if and only if $\Psi(G)$ is a transitive subgroup of S_t.*

Using these lemmas, we can construct a lot of homomorphisms

$$\varphi : \pi_1(\Delta(O, \epsilon), O) \longrightarrow S_d$$

and consequently a lot of two dimensional normal singularities (X, x) and covering mappings

$$\mu : (X, x) \longrightarrow (\Delta(O, \epsilon), O),$$

whose branch loci are contained in the curve B and the monodromies are φ.

Example. Put

$$A = (12)(34)(56).$$

Then $Z_A(S_6)$ consists of 48 permutations. Among them, we choose

$$B_1 = (146235), B_2 = (135246), B_3 = (145236), \quad (d = 6, N = 3).$$

Note that

$$A = B_3 B_2 B_1.$$

Let

$$\varphi : \pi_1(\Delta, O) \longrightarrow S_6$$

be the homomorphism defined by

$$\varphi(\gamma_j) = B_j \quad (j = 1, 2, 3).$$

Then the corresponding covering mapping

$$\mu : (X, x) \longrightarrow (\Delta(O, \epsilon), O)$$

is a non-Galois covering of mapping degree 6 which branches at 3 lines passing through O and the ramification indices are all 6.

References

1. H. Grauert and R. Remmert, *Komplexe Räume*, Math. Ann. **136** (1958), 245–318.
2. R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice Hall, New York, 1965.

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