数理解析研究所講究録
955 巻 1996 年 93-108

On the irregular singularities of confluent hypergeometric $\mathcal{D}$-modules

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1 Intoduction

In this expository paper, I will explain the irregularity at a singular point of differential equation. At first, I will give you a review of study on ordinary linear differential equations. Secondly, I will talk about holonomic $\mathcal{D}$-modules, especially, confluent hypergeometric differential modules in two variables.

2 Index theorems of ordinary differential operator and its irregularity.

Consider a linear ordinary differential operator with coefficients in holomorphic functions at the origin in the Riemann Sphere:

$$Pu = (\sum_{i=0}^{m} a_i(x)(d/dx)^i)u.$$

where a_m is supposed not to be identically zero. Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ be the ring of convergent power-series and the ring of formal power-series in x, respectively. Then, we see the following isomorphism of linear spaces due to Deligne (cf. [24], etc.) :

$$H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \simeq \mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}),$$

where $\mathcal{A}_0$ is the sheaf of germs of functions asymptotically developable to the formal power-series 0 on the circle S^1, for, from the existence theorem of asymptotic solutions due to Hukuhara (cf. [27]) (and other many contributers), we have the short exact sequence

$$0 \to \mathcal{K}er(P : \mathcal{A}_0) \to \mathcal{A}_0 \xrightarrow{P} \mathcal{A}_0 \to 0,$$

from which, we get the exact sequence,

$$0 \to H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \to H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \xrightarrow{P} H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \to 0.$$

The dimension is finite and is equal to

$$\begin{aligned} i_0(P) &= \sup\{i - v(a_i) : i = 0, ..., m\} - (m - v(a_m)) \\ &= (v(a_m) - m) - \inf\{v(a_i) - i : i = 0, ..., m\}, \end{aligned}$$

which is called the irregularity by Malgrange [17], [18], the invariant of Fuchs by Gérard-Levelt [3], [4] or the irregular index by Komatsu (in a private communication), where,

$$v(a) = \sup\{p : x^{-p}a(x) \quad \textit{is} \quad \textit{holomorphic} \quad \textit{at} \quad \textit{the} \quad \textit{origin.}\}.$$

Remark 0: Let $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$ be the ring of the ring of convergent Laurent series with finite negative order terms, the ring of formal, the ring of formal Laurent series with finite negative order terms and the ring of convergent Laurent series, respectively. Denote by F one of $\mathcal{O}$, $\hat{\mathcal{O}}$, $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$. We consider P as an operator from F to itself. Then, $\mathrm{Ker}(P; F)$ and $\mathrm{Coker}(P; F)$ are finite dimensional, and has a index $\chi(P; F) = \dim_C \mathrm{Ker}(P; F) - \dim_C \mathrm{Coker}(P; F)$, which can be calculated as follows:

$$\begin{aligned} \chi(P; \mathcal{O}) &= m - v(a_m), \\ \chi(P; \hat{\mathcal{O}}) &= \sup\{i - v(a_i) : i = 1, ..., m\}, \\ \chi(P; \mathcal{K}) &= m - v(a_m) - \sup\{i - v(a_i) : i = 1, ..., m\}, \\ \chi(P; \hat{\mathcal{K}}) &= 0, \\ \chi(P; \mathcal{E}) &= 0. \end{aligned}$$

The quantity $i_0(P)$ is also equal to the followings [17], [18] :

$$\begin{aligned} &\chi(P; \hat{\mathcal{O}}) - \chi(P; \mathcal{O}), \\ &\chi(P; \hat{\mathcal{K}}) - \chi(\mathcal{K}), \\ &-\chi(P; \mathcal{K}), \\ &\chi(P; \hat{\mathcal{K}}/\mathcal{K}), \\ &\chi(P; \mathcal{E}) - \chi(P; \mathcal{K}), \\ &\chi(P; \mathcal{E}/\mathcal{K}), \\ &\chi(P; \mathcal{E}/\mathcal{O}) - \chi(P; \mathcal{K}/\mathcal{O}), \\ &\dim_C \mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}), \\ &\dim_C \mathrm{Ker}(P; \hat{\mathcal{K}}/\mathcal{K}), \\ &\dim_C \mathrm{Ker}(P; \mathcal{E}/\mathcal{K}), \\ &\dim_C \mathrm{Ker}(P; (\mathcal{E}/\mathcal{O})/(\mathcal{K}/\mathcal{O})). \end{aligned}$$

Remark 1: If we consider a linear ordinary differential operator with coefficients in holomorphic functions at the infinity in the Riemann Sphere and we do not use the variable $t = \frac{1}{x}$, the quantity is equal to

$$\begin{aligned} i_\infty(P) &= \sup\{v'(a_i) - i : i = 0, ..., m\} - (v'(a_m) - m) \\ &= (m - v'(a_m)) - \inf\{i - v'(a_i) : i = 0, ..., m\}, \end{aligned}$$

where

$$v'(a) = \sup\{p : x^{-p}a(x) \quad is \quad holomorphic \quad at \quad the \quad infinity.\}.$$

Remark 2: We have also another important quantity associated with the linear ordinary differential operator $P = (\sum_{i=0}^{m} a_i(x)(d/dx)^i)$. At the origin, we set

$$k = \sup\{0, \frac{(v(a_m) - m) - (v(a_i) - i)}{m - i} : i = 0, ..., m - 1\},$$

and at the infinity, we set

$$k = \sup\{0, \frac{(m - v'(a_m)) - (i - v'(a_i))}{m - i} : i = 0, ..., m - 1\},$$

which is called the invariant of Katz by Gérard-Levelt [3], [4] or the order by Sibuya [29], and $k + 1$ is called the irregularity by Komatsu [9], [10]. In order to understand the importance of this quantity, see the above references and also Ramis [25], [26], Komatsu [11], Malgrange [21]. In adding a word,

$$i_0(P) \geq k \geq \frac{i_0(P)}{m}, \quad mk \geq i_0(P) \geq k.$$

Consider for example the generalized confluent hypergeometric differential operator

$$\frac{d^2}{dz^2}w + (A_0 + \frac{A_1}{z})\frac{d}{dz}w + (B_0 + \frac{B_1}{z} + \frac{B_2}{z^2})w = 0.$$

where A_0, A_1, B_0, B_1 and B_2 are complex numbers. The value of irregularity in the sense of Malgrange may be equal to 0, 1 or 2 and the value of order may be equal to 0, $\frac{1}{2}$ or 1. Here, we give a list of irregularities, orders and bases of

$$H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0)) \simeq \mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}),$$

for Kummer, Bessel and Airy differential equations.

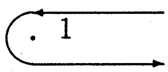
2.1 Confluent Hypergeometric(Kummer) Equation.

$$A_0 = -1, A_1 = c, B_0 = 0, B_1 = -a, B_2 = 0, k = 1, i_\infty(P) = 1.$$

Denote by $G_2(z)$ the confluent hypergeometric function, namely,

$$G_2(z) = \frac{2}{1-e^{2\pi i(\gamma-\alpha)}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_C e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

where, for $-\pi < \theta < \pi$, and $\frac{1}{2}\pi - \theta < \arg z < \frac{3}{2}\pi - \theta$, $C = C(1;\theta)$ is the path of integral on which $\arg(\zeta - 1)$ is taken to be initially θ and finally $\theta + 2\pi$, and so $G_2(z)$ is defined for $-\frac{1}{2}\pi < \arg z < -\frac{5}{2}\pi$, in particular, for $\theta = 0$ and $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, the path of integral is as follows,



and

$$G_2(z) = \frac{2}{1-e^{2\pi i(\gamma-\alpha)}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty}(e^{\pi i(\gamma-\alpha-1)} - e^{-\pi i(\gamma-\alpha-1)})e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty} e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{-\infty} -e^{z(1-\zeta)}(1-\zeta)^{\alpha-1}\zeta^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}e^z\int_0^{-\infty} e^{-z\zeta}(1-\zeta)^{\alpha-1}\zeta^{\gamma-\alpha-1}d\zeta,$$

$$G_2(z) = -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}e^z z^{\alpha-\gamma}\int_0^{+\infty} e^{-t}(1-\frac{t}{z})^{\alpha-1}t^{\gamma-\alpha-1}dt,$$

by using the Newton's binomial expansion

$$(1 \pm \frac{t}{z})^{\alpha-1} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-1-n)\Gamma(n+1)}(\pm\frac{t}{z})^n,$$

or

$$(1 \pm \frac{t}{z})^{\alpha-1} = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}(\pm\frac{t}{z})^n.$$

The asymptotic behaviours at the infinity for $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, is as follows (cf. [2] etc.) :

$$G_2(z) \approx -2e^{-\pi i(\gamma-\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z^{-(\gamma-\alpha)}\exp(-(-z))\sum_{n=0}^{\infty}\frac{\Gamma(n+\gamma-\alpha)\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}z^{-n},$$

Therefore, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ in the following way: Put $U_1 = \{z \in \mathbf{C} : |z| > R, \frac{\pi}{2} < \arg z < \frac{5}{2}\pi\}$, and $U_2 = \{z \in \mathbf{C} : |z| > R, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\}$

for a positive real number R. Then, $\{U_1,\ U_2\}$ forms an open sectorial covering at $z=\infty$ and put

$$u_{12}(z)=u(z)\quad(\frac{1}{2}\pi<\arg\ z<\frac{3}{2}\pi),\ u_{12}(z)=0\quad(\frac{3}{2}\pi<\arg\ z<\frac{5}{2}\pi).$$

In this situation, the cohomology classes of $\{u_{12}\}$ forms a basis of $H^1(S^1,\mathcal{K}er(P:\mathcal{A}_0))$. By the original vanishing theorem due to [17] in asymptotic analysis, we have 0-cochains $\{u_1,\ u_2\}$ such that

$$u_{12}(z)=u_2(z)-u_1(z),$$

where $u_j(z)$ are defined in U_j for $j=1,\ 2$ and asymptotically developable to a formal power-series $\hat{u}=\sum_{r=0}^{\infty}u_r z^{-r}$ at the first. The coefficient u_r is given by the following:

$$u_r=\frac{-1}{2\pi i}\int_0^{-\infty}z^{r-1}G_2(z)dz$$

$$u_r=\frac{-1}{2\pi i}\int_0^{-\infty}z^{r-1}(-2)e^{-\pi i(\gamma-\alpha-1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{+\infty}e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta dz,$$

$$u_r=\frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{-\infty}z^{r-1}\int_1^{+\infty}e^{z\zeta}\zeta^{\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta dz,$$

$$u_r=\frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_1^{+\infty}\zeta^{\alpha-r-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$u_r=\frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_0^{1}\zeta^{r-\gamma}(\zeta-1)^{\gamma-\alpha-1}d\zeta,$$

$$u_r=\frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)\int_0^{1}\zeta^{r-\gamma}(-1)^{\gamma-\alpha-1}(1-\zeta)^{\gamma-\alpha-1}d\zeta,$$

$$u_r=\frac{e^{-\pi i(\gamma-\alpha-1)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^r\Gamma(r)(-1)^{\gamma-\alpha-1}\frac{\Gamma(r-\gamma+1)\Gamma(\gamma-\alpha)}{\Gamma(r-\alpha+1)},$$

$$u_r=\frac{1}{\pi i}\frac{\Gamma(\gamma)\Gamma(r-\gamma+1)}{\Gamma(\alpha)\Gamma(r-\alpha+1)}(-1)^r\Gamma(r).$$

By the vanishing theorem in asymptotic analysis with Gevrey estimates due to [24], we can assert secondly that $\hat{u}$ and $\hat{v}$ are formal power-series with Gevrey order $\sigma=1$. Our new theorem [16] claims thirdly that we can have asymptotic estimates for the coefficients of $\hat{u}$, more precise than Gevrey estimates: for any sufficiently large number r,

$$u_r=\frac{e^{-\pi i(\gamma-\alpha)}}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\sum_{s=0}^{M-1}\frac{\Gamma(s+\gamma-\alpha)\Gamma(s+1-\alpha)}{\Gamma(1-\alpha)\Gamma(s+1)}(e^{\pi i})^{s-r+(\gamma-\alpha)}\Gamma(r-s-(\gamma-\alpha))$$
$$+O\{\Gamma(r-M-\Re(\gamma-\alpha))\}$$

$$u_r=\frac{1}{\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}(-1)^r\sum_{s=0}^{M-1}\frac{\Gamma(s+\gamma-\alpha)\Gamma(s+1-\alpha)\Gamma(r-s-(\gamma-\alpha))}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)\Gamma(s+1)}(-1)^s$$

$$+O\{\Gamma(r-M-\Re(\gamma-\alpha))\}$$

provided $1 \leq M < r$.

In the intersection $U_1 \cap U_2$, $Pu_1(z) = Pu_2(z)$, which define holomorphic functions f at the infinity, and $P\hat{u} = f$, so the equivalence class of $\hat{u}$, forms a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$.

Of course, in this case, we can compute a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$ directly: for example, as a formal solution of the inhomogeneous linear ordinary differential equation $P\hat{w} = \frac{1-\gamma}{z^2}$, we have

$$\hat{w} = \sum_{r=0}^{\infty}(-1)^{r-1}\frac{\Gamma(r)\Gamma(r+1-\gamma)\Gamma(1-\alpha)}{\Gamma(1-\gamma)\Gamma(r+1-\alpha)}z^{-r}$$

and the equivalence class of $\hat{w}$ as a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$, of which coefficients admit asymptotic estimates by the result on Γ-function.

By a little more calculation, we find that $\hat{u}$ is equivalent to

$$\frac{-1}{\pi i}\frac{\Gamma(\gamma)\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)}\hat{w} = \frac{-1}{\pi i}\frac{\sin\pi\alpha}{\sin\pi\gamma}\hat{w},$$

modulo $\mathcal{O}$.

2.2 Bessel Equations.

$$A_0 = 0,\ A_1 = 1,\ B_0 = 1,\ B_1 = 0,\ B_2 = -\nu^2,\ k = 1,\ i_\infty(P) = 2.$$

Denote by $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ the Hankel functions, namely,

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\frac{e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}(1+\frac{it}{2z})^{\nu-\frac{1}{2}}dt,$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\frac{e^{-i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}(1-\frac{it}{2z})^{\nu-\frac{1}{2}}dt,$$

we know the asymptotic behaviours at the infinity (cf. [2] etc.)

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+\frac{1}{2})e^{i(z-\frac{1}{2}(\nu-n)\pi-\frac{1}{4}\pi)}}{\Gamma(\nu-n+\frac{1}{2})\Gamma(n+1)(2z)^n} \quad (-\pi < \arg\ z < 2\pi),$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+\frac{1}{2})e^{-i(z-\frac{1}{2}(\nu-n)\pi-\frac{1}{4}\pi)}}{\Gamma(\nu-n+\frac{1}{2})\Gamma(n+1)(2z)^n} \quad (-2\pi < \arg\ z < \pi).$$

by using the Newton's binomial expansion

$$(1 \pm \frac{it}{2z})^{\nu-\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-n+\frac{1}{2})\Gamma(n+1)}(\pm\frac{it}{2z})^n.$$

Therefore, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ in the following way: Put $U_1 = \{z \in \mathbf{C} :\ |z| > R,\ -\pi < \arg\ z < \pi\}$ and $U_2 = \{z \in \mathbf{C} :\ |z| > R,\ -2\pi < \arg\ z < 0\}$

for a positive real number R. Then, $\{U_1,\ U_2\}$ forms an open sectorial covering at $z=\infty$ and put

$$u_{12}(z)=H_\nu^{(1)}(z)\quad(0<\arg\ z<\pi),\quad u_{12}(z)=0\quad(-\pi<\arg\ z<0),$$

and

$$v_{12}(z)=0\quad(0<\arg\ z<\pi),\quad v_{12}(z)=H_\nu^{(2)}(z)\quad(-\pi<\arg\ z<0).$$

In this situation, the pair of cohomology classes of $\{u_{12}\}$ and $\{v_{12}\}$ forms a basis of $H^1(S^1,\mathcal{K}er(P:\mathcal{A}_0))$. By the original vanishing theorem due to [17] in asymptotic analysis, we have 0-cochains $\{u_1,\ u_2\}$ and $\{v_1,\ v_2\}$ such that

$$u_{12}(z)=u_2(z)-u_1(z),\quad v_{12}(z)=v_2(z)-v_1(z),$$

where $u_j(z)$ and $v_j(z)$ are defined in U_j for $j=1,\ 2$ and asymptotic developable to formal power-series $\hat{u}=\sum_{r=0}^{\infty}u_rz^{-r}$ and $\hat{v}=\sum_{r=0}^{\infty}v_rz^{-r}$, respectively, at the first. By the vanishing theorem in asymptotic analysis with Gevrey estimates due to [24], we can assert secondly that $\hat{u}$ and $\hat{v}$ are formal power-series with Gevrey order $\sigma=1$. Our new theorem [16] claims thirdly that we can have asymptotic estimates for the coefficients of $\hat{u}$ and $\hat{v}$ more precise than Gevrey estimates: for any sufficiently large number r,

$$u_r=\sqrt{\frac{2}{\pi}}\sum_{s=0}^{M-1}\frac{\Gamma(\nu+s+\frac{1}{2})e^{i(-\frac{1}{2}(\nu-s)\pi-\frac{1}{4}\pi)}}{\Gamma(\nu-s+\frac{1}{2})\Gamma(s+1)(2)^s}(-i)^{s-r+\frac{1}{2}}\Gamma(r-s-\frac{1}{2})+O\{\Gamma(r-M-\frac{1}{2})\}$$

$$v_r=\sqrt{\frac{2}{\pi}}\sum_{s=0}^{M-1}\frac{\Gamma(\nu+s+\frac{1}{2})e^{-i(-\frac{1}{2}(\nu-s)\pi-\frac{1}{4}\pi)}}{\Gamma(\nu-s+\frac{1}{2})\Gamma(s+1)(2)^s}(+i)^{s-r+\frac{1}{2}}\Gamma(r-s-\frac{1}{2})+O\{\Gamma(r-M-\frac{1}{2})\}$$

provided $1\leq M<r$.

In the intersection $U_1\cap U_2$, $Pu_1(z)=Pu_2(z)$ and $Pv_1(z)=Pv_2(z)$, which define holomorphic functions f and g at the infinity, and $P\hat{u}=f$, $P\hat{v}=g$, so the pair of equivalence classes of $\hat{u}$ and $\hat{v}$ forms a basis of $\mathrm{Ker}(P;\hat{\mathcal{O}}/\mathcal{O})$. Therefore, if $\hat{w}$ is a formal solution to an inhomogeneous equation $P\hat{w}=h\in\mathcal{O}$, we assert that $\hat{w}=\sum_{r=0}^{\infty}w_rz^{-r}$ should have the same kind of asymptotic estimates for coefficients.

Of course, in this case, we can compute a basis of $\mathrm{Ker}(P;\hat{\mathcal{O}}/\mathcal{O})$ directly: for example, as a formal solution of the inhomogeneous linear ordinary differential equation $P\hat{w}_j=z^{-j}$, we have

$$\hat{w}_1=\sum_0^{\infty}(-4)^n\frac{\Gamma(n+\frac{j+\nu}{2})}{\Gamma(\frac{j+\nu}{2})}\frac{\Gamma(n+\frac{j-\nu}{2})}{\Gamma(\frac{j-\nu}{2})}z^{-2n-j}$$

for $j=1,\ 2$ and the pair of equivalence classes of $\hat{w}_1$ and $\hat{w}_2$ as a basis of $\mathrm{Ker}(P;\hat{\mathcal{O}}/\mathcal{O})$, of which coefficients admit asymptotic estimates by the result on Γ-function.

2.3 Airy Equation.

The Airy equation is of the form

$$\frac{1}{z}\frac{d^2v}{dz^2} - v = 0,$$

which is transformed into the Bessel equation with the parameter $\nu = \frac{1}{3}$ by the transformation

$$v(z) = (z^{\frac{3}{2}})^{\frac{1}{3}} w(\frac{2}{3} i z^{\frac{3}{2}}).$$

$$k = \frac{3}{2},\ i_\infty(P) = 3.$$

Denote by $Ai(z)$ the Airy function, namely,

$$Ai(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \exp(zt - \frac{t^3}{3})dt,$$

The asymptotic behaviours at the infinity is as follows (cf. [2] etc.) :

$$Ai(z) \approx \frac{1}{2\pi} z^{-\frac{1}{4}} \exp(-\frac{2}{3} z^{\frac{3}{2}}) \sum_{n=0}^{\infty} \frac{\Gamma(3n+\frac{1}{2})}{(2n)!} (\frac{i}{3z^{\frac{3}{4}}})^{2n} \quad (|\arg z| < \frac{\pi}{3}),$$

Therefore, we can choose a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$ in the following way: Put

$$\begin{aligned} U_1 &= \{z \in \mathbf{C} : \ |z| > R,\ -\pi < \arg\ z < \frac{1}{3}\pi\}, \\ U_2 &= \{z \in \mathbf{C} : \ |z| > R,\ -\frac{1}{3}\pi < \arg\ z < \pi\}, \\ U_3 &= \{z \in \mathbf{C} : \ |z| > R,\ \frac{1}{3}\pi < \arg\ z < -\frac{5}{3}\pi\} \end{aligned}$$

for a positive real number R. Then, $\{U_1,\ U_2,\ U_3\}$ forms an open sectorial covering at $z = \infty$ and put

$$\begin{aligned} u_{12}(z) &= Ai(z) \quad (-\frac{1}{3} < \arg\ z < \frac{1}{3}\pi), \\ u_{23}(z) &= 0 \quad (\frac{1}{3}\pi < \arg\ z < \pi), \\ u_{31}(z) &= 0 \quad (\pi < \arg\ z < \frac{5}{3}\pi), \\ v_{12}(z) &= 0 \quad (-\frac{1}{3} < \arg\ z < \frac{1}{3}\pi), \\ v_{23}(z) &= Ai(\exp(-\frac{2}{3}\pi i) z) \quad (\frac{1}{3}\pi < \arg\ z < \pi), \\ v_{31}(z) &= 0 \quad (-\pi < \arg\ z < \frac{5}{3}\pi), \end{aligned}$$

$$
\begin{aligned}
w_{12}(z) &= 0 \quad (-\frac{1}{3} < \arg\ z < \frac{1}{3}\pi), \\
w_{23}(z) &= 0 \quad (\frac{1}{3}\pi < \arg\ z < \pi), \\
w_{31}(z) &= Ai(\exp(\frac{2}{3}\pi i)z) \quad (-\pi < \arg\ z < \frac{5}{3}\pi),
\end{aligned}
$$

In this situation, the pair of cohomology classes of $\{u_{ij}\}$, $\{v_{ij}\}$ and $\{w_{ij}\}$ forms a basis of $H^1(S^1, \mathcal{K}er(P : \mathcal{A}_0))$. By the original vanishing theorem due to [17] in asymptotic analysis, we have 0-cochains $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ such that

$$
\begin{aligned}
u_{j\ell}(z) &= u_\ell(z) - u_j(z), \\
v_{j\ell}(z) &= v_\ell(z) - v_j(z), \quad ((j,\ell) = (1,2),(2,3),(3,1)) \\
w_{j\ell}(z) &= w_\ell(z) - w_j(z),
\end{aligned}
$$

where $u_j(z)$, $v_j(z)$ and $w_j(z)$ are defined in U_j for $j = 1,\ 2,\ 3$ and asymptotically developable to formal power-series $\hat{u} = \sum_{r=0}^{\infty} u_r z^{-r}$, $\hat{v} = \sum_{r=0}^{\infty} v_r z^{-r}$ and $\hat{w} = \sum_{r=0}^{\infty} w_r z^{-r}$, respectively, at the first. By the vanishing theorem in asymptotic analysis with Gevrey estimates due to [24], we can assert secondly that $\hat{u}$ and $\hat{v}$ are formal power-series with Gevrey order $\sigma = \frac{3}{2}$. Our new theorem [16] claims thirdly that we can have asymptotic estimates for the coefficients of $\hat{u}$, $\hat{v}$ and $\hat{w}$ more precise than Gevrey estimates: for any sufficiently large number r,

$$
u_r = \frac{1}{2\pi i}\sum_{s=0}^{M-1} \frac{\Gamma(3s+\frac{1}{2})}{(2s)!}(\frac{i}{3})^{2s}\Gamma(r - \frac{3}{2}s - \frac{1}{4}) + O\{\Gamma(r - M - \frac{1}{4})\}
$$

provided $1 \leq M < r$.

In the intersection $U_j \cap U_\ell$, $Pu_j(z) = Pu_\ell(z)$ and $Pv_j(z) = Pv_\ell(z)$, which define holomorphic functions f, g and h at the infinity, and $P\hat{u} = f$, $P\hat{v} = g$, $P\hat{w} = h$, so the triple of equivalence classes of $\hat{u}$, $\hat{v}$ and $\hat{w}$ forms a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$.

Of course, in this case, we can compute a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$ directly: for example, as a formal solution of the inhomogeneous linear ordinary differential equation $P\hat{w}_j = -z^{-j}$, we have

$$
\hat{w}_j = \sum_{n=0}^{\infty} \frac{\Gamma(3n+j)\Gamma(\frac{j-1}{3})}{3^{n+1}\Gamma(n+1+\frac{j-1}{3})}
$$

for $j = 2,\ 3,\ 4$ and the pair of equivalence classes of $\hat{w}_1$, $\hat{w}_2$ and $\hat{w}_3$ as a basis of $\mathrm{Ker}(P; \hat{\mathcal{O}}/\mathcal{O})$, of which coefficients admit asymptotic estimates by the result on Γ-function.

3 Indices of holonomic $\mathcal{D}$-modules and their irregularities

Let $\mathcal{D}_0$ be the stalk of germs of linear ordinary differential operators with holomorphic coefficients, and put $\mathcal{M}_0 = \mathcal{D}_0/\mathcal{D}_0 P$. Then, $\mathcal{M}_0$ has a projective resolution

$$0 \leftarrow \mathcal{M}_0 \leftarrow \mathcal{D}_0 \xleftarrow{P} \mathcal{D}_0 \leftarrow 0,$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}_0}(\cdot, \mathcal{F}_0)$, we have the solution complex with values in $\mathcal{F}$ at the origin,

$$\mathcal{S}ol(\mathcal{M}_0, \mathcal{F}_0) : \mathcal{F}_0 \xrightarrow{P} \mathcal{F}_0 \to 0.$$

We have the isomorphism:

$$\mathrm{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) \simeq \mathrm{Ker}(\mathcal{F}_0; P), \quad \mathrm{Ext}^1(\mathcal{M}_0, \mathcal{F}_0) \simeq \mathrm{Coker}(\mathcal{F}_0; P).$$

Therefore, the index as $\mathcal{D}$-module at the origin,

$$\chi(\mathcal{M}; \mathcal{F})_0 = \dim_C \mathrm{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) - \dim_C \mathrm{Ext}^1(\mathcal{M}_0, \mathcal{F}_0),$$

is equal to the index $\chi(P; F)$, and the irregularity as $\mathcal{D}$-module at the origin,

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{O}}) - \chi(\mathcal{M}_0; \mathcal{O}),$$

is equal to the irregularity $\mathrm{Irr}(P)_0$ and

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{K}}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\mathrm{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}/\mathcal{O}) - \chi(\mathcal{M}_0; \mathcal{K}/\mathcal{O}).$$

Let $\mathcal{D}$ be the sheaf of germs of linear partial differential operetors with coefficients of holomorphic functions on a manifold M and let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. The module $\mathcal{M}$ has a projective resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{m_0} \xleftarrow{P_0} \mathcal{D}^{m_1} \xleftarrow{P_1} \mathcal{D}^{m_2} \xleftarrow{P_2} \cdots \xleftarrow{P_{2n-1}} \mathcal{D}^{m_{2n}} \leftarrow 0$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}}(\cdot, \mathcal{F})$, we have the solution complex with values in $\mathcal{F}$,

$$\mathcal{S}ol(\mathcal{M}, \mathcal{F}) : \mathcal{F}^{m_0} \xrightarrow{P_0^t} \mathcal{F}^{m_1} \xrightarrow{P_1^t} \cdots \xrightarrow{P_{2n-1}^t} \mathcal{F}^{m_{2n}} \to 0.$$

For a point p, the index of holonomic $\mathcal{D}$-module $\mathcal{M}$with values in $\mathcal{F}$ is defined by

$$\chi(\mathcal{M}; \mathcal{F})_p = \sum_{i=0}^{2n} \dim_C (-1)^i \mathcal{E}xt^i(\mathcal{M}, \mathcal{F})_p.$$

For the point p, the irregularity of holonomic $\mathcal{D}$-module $\mathcal{M}$is defined by

$$\mathrm{Irr}(\mathcal{M})_p = \chi(\mathcal{M}; \mathcal{O}_{M\hat{|}H}))_p - \chi(\mathcal{M}; \mathcal{O}_{M|H})_p,$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M , H is the set of singular points of $\mathcal{M}$, $\mathcal{O}_{M|H}$ is the zero-extension of the restriction of $\mathcal{O}$ on H and $\mathcal{O}_{M\hat{|}H}$ is the Zariski completion of $\mathcal{O}$ along H.

4 Holonomic $\mathcal{D}$-module defined by confluent hypergeometric partial differential equations Φ_3

In the sequel, we consider the solution complexes of holonomic $\mathcal{D}$-module defined by confluent hypergeometric partial differential equations Φ_3 and give the calculation of the cohomology groups.

The system of confluent hypergeometric partial differential equations Φ_3 [2] is as follows:

$$\Phi_3 : \begin{cases} x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} + (c - x)\dfrac{\partial u}{\partial x} - bu = 0 & \text{(denoted by } L_1 u = 0) \\ y\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial^2 u}{\partial x \partial y} + c\dfrac{\partial u}{\partial y} - u = 0 & \text{(denoted by } L_2{}' u = 0) \\ x\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y} = 0 & \text{(denoted by } L_3{}' u = 0) \end{cases}$$

where b, c are not non-negative integers.
We consider the $\mathcal{D}$-module $\mathcal{M}_3$ defined by Φ_3, namely we put

$$\mathcal{M}_3 = \mathcal{D}/(\mathcal{D}L_1 + \mathcal{D}L_2{}').$$

We have a projective resolution

$$0 \longleftarrow \mathcal{M}_3 \longleftarrow \mathcal{D} \longleftarrow \mathcal{D}^3 \longleftarrow \mathcal{D}^2 \longleftarrow 0$$

and we have the solution complex $\mathcal{S}ol(\mathcal{M}_3, \mathcal{F})$ with values in $\mathcal{F}$

$$\mathcal{F} \xrightarrow{\nabla_0} \mathcal{F}^3 \xrightarrow{\nabla_1} \mathcal{F}^2 \longrightarrow 0,$$

where

$$\nabla_0 = \begin{pmatrix} L_1 \\ L_2{}' \\ L_3{}' \end{pmatrix},$$

$$\nabla_1 = \begin{pmatrix} -L_2{}' & L_1 & 0 \\ \dfrac{\partial}{\partial y} & -\dfrac{\partial}{\partial x} & 1 \end{pmatrix}.$$

by using Takayama's Kan [30] and we have the same result as Φ_2 in $H = \{(\infty, y); y \in P^1_C\}$ [5] [6].

Theorem 1 . *Let $M = P^1_C \times P^1_C$, $H = \{(\infty, y); y \in P^1_C\}$, $p \in H \backslash (\infty, \infty)$ be as above. The dimensions of chohomology groups of the solution complexes for the $\mathcal{D}$-module defined by Φ_3 are as folow:*

(1) *If* $1 \leq s < 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}$$

(2) *If* $s > 2$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s}$,

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2).$$

(3) *In the case of* $s = 2$,

(i) *if* $A > 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2).$$

(ii) *if* $0 < A < 1$,

for $\mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)}$,

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}$$

(iii) *if* $A = 1$,

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},2,1-})_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}.$$

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0, \quad (j = 0, 1, 2).$$

(iv) $\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = \begin{cases} 0, & (j = 0, 2) \\ 1, & (j = 1) \end{cases}.$

$\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0, \quad (j = 0, 1, 2).$

(4) $\dim_C \operatorname{Ext}^j((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0, \quad (j = 0, 1, 2).$

Corollary 1 . *The indexes of $\mathcal{D}$-module defined by Φ_3 are as follow:*
(1) *If* $1 \leq s < 2$,

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s},$$

$$\mathcal{X}((\mathcal{M}_3)_p, \mathcal{F}_p) = -1.$$

(2) *If* $s > 2$,

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s},$$

$$\mathcal{X}((\mathcal{M}_3)_p, \mathcal{F}_p) = 0.$$

(3) *In the case of* $s = 2$
(i) *if* $A > 1$,

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)},$$

$$\mathcal{X}((\mathcal{M}_3)_p, \mathcal{F}_p) = 0.$$

(ii) *if* $0 < A < 1$,

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)},$$

$$\mathcal{X}((\mathcal{M}_3)_p, \mathcal{F}_p) = -1.$$

(iii) *if* $A = 1$,

$$\mathcal{X}((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},2,1-})_p) = -1.$$

$$\mathcal{X}((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0.$$

(iv) $\mathcal{X}((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = -1.$
$\mathcal{X}((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0.$

(4) $\mathcal{X}((\mathcal{M}_3)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0.$

Corollary 2 . *The irregularity* $\mathrm{Irr}((\mathcal{M}_3)_p) = 1$.

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