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Indices of vectorfields and residues of singular foliations after Nash transformation

(about joint works with M.H. Schwartz and T. Suwa)

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Introduction.

The aim of this lecture is to give motivations for Nash residues of singular holomorphic foliations. We will use the Nash transformation, introduced by R. MacPherson in the definition of characteristic classes of algebraic complex varieties and also used by various authors in the study of characteristic classes (M. H. Schwartz, W. Fulton and R. MacPherson...).

We recall the relation between the index of radial vectorfields defined by M.H. Schwartz [Sc] and the "index" of their lifting in the Nash transformation. The formula uses the notion of local Euler obstruction (see [McP] [BSc]). Following the definition of T.Suwa [Su] and the construction of S. Sertöz [Se], we define the Nash transformation and the Nash residue for singular holomorphic foliations. Finally we give the Nash residue theorem obtained by T. Suwa and the author [BSu]. Basic references are [Sc] and [Su].

1.Nash transformation of an analytic complex variety.

Let X be an analytic complex variety of complex dimension n, embedded in an analytic complex manifold M. Consider the Grassmann bundle $\mu : G_n(TM) \to M$ associated to the tangent bundle $\pi : TM \to M$. The fiber of $G_n(TM)$ over $x \in M$ is the set of n-complex planes in T_xM. Over the regular part X_{reg} of X, there is a section σ of μ, given by $\sigma(x) = T_x(X_{\text{reg}})$. The Nash transformation $\widetilde{X}$ is the closure of Imσ in $G_n(TM)$. It is an analytic variety and the restriction of μ to $\widetilde{X}$ is an analytic map $\nu : \widetilde{X} \to X$. We have diagrams :

$$\begin{array}{ccc} & & G_n(TM) \\ & \sigma \nearrow & \downarrow \mu \\ X_{reg} & \hookrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{X} = \overline{Im(\sigma)} & \hookrightarrow & G_n(TM) \\ \downarrow \nu & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

and we write $\widetilde{X}_{reg} = \nu^{-1}(X_{reg}) \cong X_{reg}$.

We denote by ξ the restriction to $\widetilde{X}$ of the tautological bundle over $G_n(TM)$:

$$\xi = \{\widetilde{v}(\widetilde{x}) = (v(x), \widetilde{x}) : \widetilde{x} \in \widetilde{X},\ x = \nu(\widetilde{x}),\ v(x) \in \widetilde{x} \subset T_x M\}$$

2. Index of vectorfields and Nash transformation.

Let (X_α) denote a stratification of M, i.e. a partition of M in manifolds X_α (called the strata) such that M is a locally finite union of strata and every $\overline{X}_\alpha$ and $\overline{X}_\alpha - X_\alpha$ are union of strata. We suppose that X itself is union of strata. The stratification is required to satisfy the following Whitney conditions (a) and (b) for each pair of strata $X_\alpha \subset \overline{X_\beta}$:

(a) let (x_n) be a sequence of points in X_β converging to $y \in X_\alpha$ and suppose that the sequence of tangent spaces $T_{x_n}(X_\beta)$ has a limit L. Then $T_y(X_\alpha) \subset L$.

(b) let $(x_n) \in X_\beta$ be a sequence of points converging to $y \in X_\alpha$ and $(y_n) \in X_\alpha$ a sequence of points converging to y, if the sequence of tangent spaces $T_{x_n}(X_\beta)$ has a limit L and if the sequence of lines $\overline{x_n y_n}$ has a limit λ, Then $\lambda \subset L$.

Definition. A stratified vectorfield v on a subset $A \subset M$ is a section of $TM|_A$ such that $v(x) \in T_x(X_\alpha)$ for $x \in X_\alpha$.

In [Sc], MH Schwartz proved the existence of so-called radial vectorfields, they are suitable stratified vectorfields with isolated singular points. The construction is made by induction on the dimension of strata and is the sum of two extensions :

The (a)-condition shows that one can extend a vectorfield v given on a stratum X_α in a neighbourhood of X_α as a parallel extension v' which is a stratified vectorfield.

The vectorfield $g(x) = \mathrm{grad}(d(x, X_\alpha))$ is tangent to the geodesic curves pointing outward X_α, relatively to the given Riemannian metric. The (b)-condition implies the existence of a stratified vectorfield w, called transversal, satisfying the following condition : For every $\varepsilon > 0$, there exists a (geodesic) tubular neighbourhood of X_α such that angle $\langle w, g\rangle \leq \varepsilon$. Furthermore $w|_{X_\alpha} = 0$ and w is growing according to the distance to X_α.

The local radial extension.

Let $a \in X_\alpha$, we will denote by b_α a small open ball in X_α with center a and by $\Theta_\varepsilon(b_\alpha)$ the "geodesic" neighbourhood of b_α in M, with "radius" $\leq \varepsilon$. Let v denote a section of $T(X_\alpha)$ over b_α, we can define in $\Theta_\varepsilon(b_\alpha)$ a parallel extension v' of v and a transversal vectorfield w. The extension $v^{rad} = v' + w$ is called radial vectorfield, it satisfies the following fondamental property :

Proposition [Sc] Let $a \in X_\alpha$ be an isolated singularity of the vectorfield v, then a is also an isolated singularity for v^{rad} and we have the "conservation of indices" property :

$$I(v^{rad}, a) = I(v^{rad}|_{X_\alpha}, a) \quad \text{where } v^{rad}|_{X_\alpha} = v$$

If ε is small enough, v^{rad} is pointing outward of $\Theta_\varepsilon(b_\alpha)$ on $\partial\Theta_\varepsilon(b_\alpha) - \Theta_\varepsilon(\partial b_\alpha)$ [Sc].

Considering ξ as a subspace of $\Lambda = TM \times_M G_n TM$, let us denote by $\nu_* : \xi \to T(M)|_X$ the restriction to ξ of the first projection $\Lambda \to T(M)$.

Let $a \in X_\alpha$ be an isolated singularity of the radial vectorfield v and let b be a ball (in M) centered in a, small enough so that ∂b is transverse to all strata X_β such that $X_\alpha \subset \overline{X_\beta}$.

Proposition [BSc] The restriction $v|_{\partial b \cap X}$ has a canonical lift $\widetilde{v}$ as a section of ξ over $\nu^{-1}(\partial b \cap X)$, i.e. $\nu_*(\widetilde{v}(\widetilde{x})) = v(\nu(\widetilde{x}))$.

The section $\widetilde{v}$ defined on $\nu^{-1}(\partial b \cap X)$ can be extended to $\nu^{-1}(b \cap X)$ as a section of ξ with isolated singularities. Let us denote by $Obs(\widetilde{v}; \nu^{-1}(b \cap X), \nu^{-1}(\partial b \cap X))$ the obstruction to the extension of $\widetilde{v}$ inside $\nu^{-1}(b \cap X)$, i.e. the evaluation of the obstruction cocycle on the fundamental cycle.

Definition [McP],[BSc] If v_0 is a vectorfield pointing outward on ∂b, then $I(v_0, a) = +1$. We define $Eu_a(X) = Obs(\widetilde{v}_0; \nu^{-1}(b \cap X), \nu^{-1}(\partial b \cap X))$ and call it local Euler obstruction in $a \in X$.

Proposition. [McP],[BSc] The local Euler obstruction $Eu_a(X)$ is well defined, i.e. independant of the different choices. It is a constructible function, i.e. constant along the strata of a Whitney stratification of X.

Theorem. [BSc] Under the above assumptions, we have :

$$Obs(\widetilde{v}; \nu^{-1}(b \cap X), \nu^{-1}(\partial b \cap X)) = Eu_a(X) \times I(v, a)$$

Nash transformation of a foliation.

Let M be a complex analytic manifold of (complex) dimension m. Write $\mathcal{O} = \mathcal{O}_M$ the structure sheaf (of germs of holomorphic functions) and $\Theta_M = \mathcal{O}_M(TM)$ the holomorphic tangent sheaf of M. We recall the classical definitions :

A singular foliation on M is a (full) integrable coherent subsheaf $\mathcal{F}$ of Θ_M :

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_M \longrightarrow \underline{Q} \longrightarrow 0$$

The integrability condition means that $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$.

The singular set of a coherent $\mathcal{O}_M$-module $\mathcal{G}$ is defined by :

$$\mathrm{Sing}(\mathcal{G}) = \{x \in M : \mathcal{G}_x \text{ is not a free } \mathcal{O}_x\text{-module }\} .$$

The singular set of the foliation $\mathcal{F}$, denoted $S = S(\mathcal{F})$ is defined by $\mathrm{Sing}(\underline{Q})$. It is a rare subanalytic set. We remark that $\mathrm{Sing}(\mathcal{F}) \subset S(\mathcal{F})$.

We recall that $\mathcal{F}$ is *full* if, for any open subset U in M,

$$\Gamma(U; \Theta_M) \cap \Gamma(U - S(\mathcal{F}); \mathcal{F}) = \Gamma(U; \mathcal{F})$$

In the present situation, we obtain an ordinary foliation of dimension $k = \mathrm{rk}\mathcal{F}$ on $M - S(\mathcal{F})$.

We will write, as above, $G_k(TM)$ the Grassmann bundle of k-complex planes in TM, and define $\varphi: M - S(\mathcal{F}) \longrightarrow G_k(TM)$ by $\varphi(x) = [\mathcal{F}_x]$, the k-plane generated by $\mathcal{F}_x$.

Definition. The Nash transformation $\widetilde{M}_{\mathcal{F}}$ of the foliation $\mathcal{F}$ is the closure of $\mathrm{Im}(\varphi)$ in $G_k(TM)$. It is a complex analytic variety ([Sc],[K]).

Let us denote by $\xi_{\mathcal{F}}$ the restriction of the tautological bundle over $\widetilde{M}_{\mathcal{F}}$ and by $\nu_{\mathcal{F}}$ the restriction to $\widetilde{M}_{\mathcal{F}}$ of the natural projection $\mu : G_k(TM) \to M$. There is an exact sequence of vector bundles on $\widetilde{M}_{\mathcal{F}}$:

$$0 \longrightarrow \xi_{\mathcal{F}} \longrightarrow \nu^*TM \longrightarrow W \longrightarrow 0$$

where W is the quotient bundle. Away from $S(\mathcal{F})$, the sequence is equivalent to the previous one.

Property. There are isomorphisms of sheaves :

$$\underline{\xi_{\mathcal{F}}}|_{\widetilde{M_{\mathcal{F}}}-\widetilde{S}} = \nu_{\mathcal{F}}^{*}\mathcal{F}|_{\widetilde{M_{\mathcal{F}}}-\widetilde{S}} \qquad \underline{W}|_{\widetilde{M_{\mathcal{F}}}-\widetilde{S}} = \nu_{\mathcal{F}}^{*}\left(\Theta_M/\mathcal{F}\right)|_{\widetilde{M_{\mathcal{F}}}-\widetilde{S}}$$

where $\widetilde{S} = \nu_{\mathcal{F}}^{-1}(S)$.

4. Residues.

Let us recall briefly Baum-Bott theory of residues :

Let Z denote a connected compact component of S and U an open neighbourhood of Z such that there is a deformation retraction $r: U \to Z$.

On $U - Z$, the sheaf $\underline{Q} = \Theta_M/\mathcal{F}$ is locally free and admits a connection ∇ which determines a closed $2i$-form $\sigma_i(K)$ on $U - Z$ for each i, $1 \leq i \leq n$. There exists a closed $2i$-form ω_i on U which coincides with $\sigma_i(K)$ outside of a subset of U containing Z in its interior.

Let $\varphi(X_1, \ldots X_n)$ be a symmetric, homogeneous polynomial of degree d and $\sigma_1, \ldots,$ σ_n the elementary symmetric functions of the variables $X_1, \ldots X_n$. There is a polynomial $\widetilde{\varphi}$ such that $\varphi = \widetilde{\varphi}(\sigma_1, \ldots, \sigma_n)$.

The differential form $\varphi(\underline{Q}) = \widetilde{\varphi}(\omega_1, \ldots, \omega_n)$ is a closed d-form in U. If $d > m - k$, then $\varphi(\underline{Q})$ has compact support and defines a class $[\varphi(\underline{Q})] \in H_c^{2d}(U; \mathbf{C})$.

Definition [BB] The residue of $\mathcal{F}$ with respect to φ along Z is

$$\mathrm{Res}_{\varphi}(\mathcal{F}; Z) = r_*([\varphi(\underline{Q})] \cap [U]) \in H_{2n-2d}(Z; \mathbf{C}).$$

Theorem [BB] If M is compact, then

$$\varphi(\underline{Q}) \cap [M] = \sum_{Z} i_* \mathrm{Res}_{\varphi}(\mathcal{F}, Z)$$

where i is the inclusion $Z \hookrightarrow M$.

The Baum-Bott residues are localized classes of Q and and $\varphi(\underline{Q}) \in H^{2d}(M; \mathbf{C})$ by the Bott vanishing theorem.

This is illustrated by the following diagram :

$$\begin{array}{ccccc}
H^{2d}(M - S(\mathcal{F}); \mathbf{C}) & \leftarrow & H^{2d}(M; \mathbf{C}) & \leftarrow & H^{2d}(M, M - S(\mathcal{F}); \mathbf{C}) \\
= 0 & & & & \downarrow\cong \text{ Alexander} - \text{Lefschetz} \\
 & & & & H_{2m-2d}(S(\mathcal{F}); \mathbf{C}) \\
 & & & & \downarrow= \\
 & & & & \oplus H_{2m-2d}(Z; \mathbf{C})
\end{array}$$

Theorem [BSu] (1) For a homogeneous symmetric polynomial φ of degree $d > m-k$ and a compact connected component $\widetilde{Z}$ of $\widetilde{S}$, there is a well-defined residue $\mathrm{Res}_\varphi(W, \widetilde{Z})$ in $H_{2m-2d}(\widetilde{Z}; \mathbf{C})$.

(2) If M is compact, we have the residue formula

$$\sum i_* \mathrm{Res}_\varphi(W, \widetilde{Z}) = \varphi(W) \cap [\widetilde{M_\mathcal{F}}] \quad \text{in} \quad H_{2m-2d}(\widetilde{M_\mathcal{F}}, \mathbf{C}),$$

where i denotes the embedding $\widetilde{Z} \hookrightarrow \widetilde{M_\mathcal{F}}$, the sum is taken over the components of $\widetilde{S}$ and $\varphi(W)$ is the characteristic class (in $H^{2d}(\widetilde{M_\mathcal{F}}; \mathbf{C})$) of W corresponding to the polynomial φ.

We call $\mathrm{Res}_\varphi(W, \widetilde{Z})$ the Nash residue of W with respect to φ at $\widetilde{Z}$. The Nash residues are localized characteristic classes of the vector bundle W (of rank $m-k$), while the Baum-Bott residues are those of the coherent sheaf $\underline{Q}$.

In [BSu], we give application of Nash residue to the rationality conjecture and examples of computations.

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