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The structure of the singular set of a complex analytic foliation

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Abstract

For stratified subsets or stratified maps, the local topological triviality has been studied by a number of people and it is generally known that if the stratification satisfies the "Whitney condition" or the "Thom condition", then we have the local topological triviality along each stratum (the Isotopy Lemmas of Thom). We consider here this type of local (analytical and topological) triviality for the case of a complex analytic singular foliation.

First we introduce the fundamental "Tangency Lemma" for a complex analytic singular foliation (Theorem (2.7)), which says that every vector field defining the foliation is "tangential" to the singular set. Next we discuss and summarize some implications of this lemma, which include the existence of the integral submanifold (leaf) through each point (even on the singular set) and the local analytical triviality of the foliation along each leaf.

As another application of the Tangency Lemma, we study the local topological triviality along each stratum of a stratification of the singular set as given in (3.3). This kind of triviality argument can be applied to the case where a stratum consists of (infinitely) many leaves. A.Kabila studied this problem for the case where the codimension of the foliation is one and the singular set is non-singular submanifold ([K]). We generalize his result in this talk.

In the process of this work, I received many helpful suggestions and advices, especially from T.Suwa. I would like to thank him for answering my questions and for supporting me in various ways.

1 Complex analytic singular foliations

First of all, we recall some general facts about singular foliations on complex manifolds and fix the notation used in this talk. For further details, see [BB] and [Suw].

Let M be a (connected) complex manifold of (complex) dimension n, and let $\mathcal{O}_M$ and Θ_M be, respectively, the sheaf of germs of holomorphic functions on M and the holomorphic tangent sheaf.

Now, let E be a coherent subsheaf of Θ_M. Note that, in this case, E is *coherent* if and only if E is locally finitely generated, since Θ_M is locally free. Then the *singular set* of E is defined by

$$S(E) = \{p \in M \mid (\Theta_M/E)_p \text{ is not } (\mathcal{O}_M)_p\text{-free}\},$$

and each point in $S(E)$ is called a *singular point* of E. Restricting E to a sufficiently small coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$, we can express E on U explicitly as follows:

$$(1.1) \qquad E = (v_1, v_2, \ldots, v_r) \qquad \left(\quad v_i = \sum_{j=1}^{n} f_{ij}(z) \frac{\partial}{\partial z_j} \qquad (1 \leq i \leq r) \quad \right),$$

where $f_{ij}(z)$ are holomorphic functions defined on U, and r is a non-negative integer. Then the singular set $S(E)$ is paraphrased on U as

$$S(E) \cap U = \{p \in U \mid \operatorname{rank}(f_{ij}(p)) \text{ is not maximal}\}.$$

Next, let us introduce the *integrability condition.* A coherent subsheaf E of Θ_M is said to be *integrable* (or *involutive*) if for every point p on $M - S(E)$,

$$(1.2) \qquad [E_p, E_p] \subset E_p$$

holds ([,] means the *Lie bracket* of smooth vector fields). Moreover, we define the *rank* (we sometimes call it *dimension*) of E to be the rank of locally free sheaf $E\,|_{M-S(E)}$, and denote it rank$E$.

Definition 1.3 *A* singular foliation *on M is an integrable coherent subsheaf E of Θ_M.*

It is clear that a singular foliation E induces a non-singular foliation on $M - S(E)$.

Definition 1.4 *Let E be a singular foliation on M. We say that E is* reduced *if*

$$\Gamma(U, \Theta_M) \cap \Gamma(U - S(E), E) = \Gamma(U, E)$$

holds for every open set U in M.

Remark 1.5 We can check the following facts about reduced foliations:

(i) If a singular foliation E is locally free,

$$E \text{ is reduced} \iff \operatorname{codim} S(E) \geq 2\ .$$

(ii) If E is reduced, then the "integrability condition" holds not only on $M - S(E)$ but on $S(E)$, i.e., (1.2) holds for every point $p \in M$.

It is generally known that we can also define singular foliations from the viewpoint of holomorphic 1-forms, but in this talk we do not mention this way of definition in detail. The two definitions are related by taking their "annihilator" each other, and it is also a general fact that if we consider only reduced foliations then the two definitions of singular foliation stated above are equivalent.

2 Singular set of a singular foliation

Next, let us recall some basic properties of the singular set of a singular foliation, and summarize the "tangency lemma" and its consequences which have been studied by P.Baum, D.Cerveau, Y.Mitera, T.Suwa and, for the real case, by T.Nagano, P.Stefan, H.Sussmann, et al. Hereafter, we assume $E(\subset \Theta_M)$ to be a singular foliation on a complex manifold M and set $r = \operatorname{rank} E$.

Definition 2.1 *For each point p in M, we set*

$$E(p) = \{v(p) \mid v \in E_p\},$$

where $v(p)$ denotes the evaluation of the vector field germ v at p. Note that $E(p)$ is a sub-vector space of the tangent space T_pM.

Definition 2.2 *For an integer k with $0 \leq k \leq r$, we set*

$$L^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) = k\},$$

$$S^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) \leq k\},$$

and set $L^{(-1)} = S^{(-1)} = \emptyset$ for convenience. Clearly we have

$$L^{(k)} = S^{(k)} - S^{(k-1)}, \qquad S^{(k)} = \bigcup_{i=0}^{k} L^{(i)}$$

for $k = 0, 1, 2, \ldots, r$.

Remark 2.3 *$L^{(k)}$ and $S^{(k)}$ are analytic sets for every integer k with $0 \leq k \leq r$.*

By the remark stated above, we have the following *natural filtration* which consists of analytic sets:

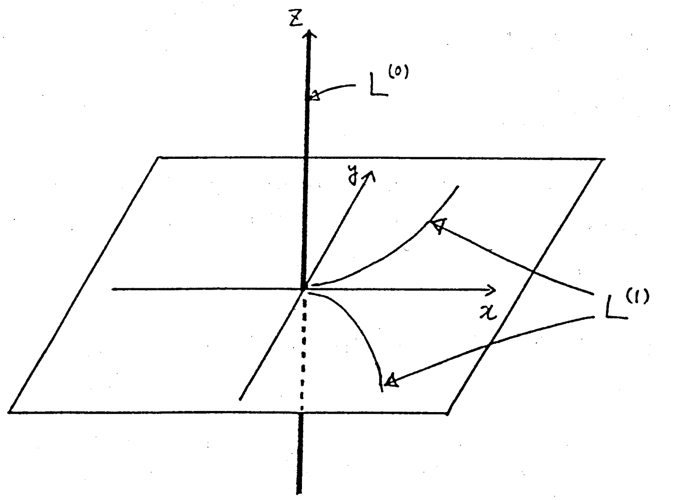
$$\begin{array}{ccccccccccccc} & S^{(r)} & \supset & S^{(r-1)} & \supset & S^{(r-2)} & \supset & \cdots\cdots & \supset & S^{(1)} & \supset & S^{(0)} & \supset & S^{(-1)}. \\ (2.4) & \| & & \| & & & & & & & & & & \| \\ & M & & S(E) & & & & & & & & & & \emptyset \end{array}$$

This filtration seems to give us information only about the "dimension" of the space $E(p)$ at p. In fact, however, if E is integrable at every point $p \in M$ then all $S^{(k)}$ appearing in (2.4) controll the "direction" of $E(p)$ at each point $p \in S^{(k)}$. Let us give an example here to grasp the meaning of this claim.

Example 2.5 Let v_1, v_2, v_3 be holomorphic vector fields on $M = \mathbf{C}^3 = \{(x, y, z)\}$ defined by

$$(2.6) \qquad \begin{cases} v_1 = 3y^2\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = (x^2 - y^3)\dfrac{\partial}{\partial y} + 3y^2 z\dfrac{\partial}{\partial z} \\ v_3 = (x^2 - y^3)\dfrac{\partial}{\partial x} - 2xz\dfrac{\partial}{\partial z}. \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2, v_3. We can easily check that E is integrable (at every point of $\mathbf{C}^3$), so E defines a singular foliation on $\mathbf{C}^3$. Since the rank of E is two, all $S^{(k)}$ appearing in (2.4) are given by $S(E) = S^{(1)} = \{xz = yz = x^2 - y^3 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^3 = 0\}$ and $S^{(0)} = \{x = y = 0\}$.



$$S^{(1)} = L^{(0)} \cup L^{(1)}$$

$$S^{(0)} = L^{(0)}$$

Now let us observe the analytic set $S^{(1)}$ in this example. If a point p belongs to $S^{(1)}$, the dimension of the space $E(p)$ should be less than one by the definition of $S^{(1)}$.

However we can obtain more information about $E(p)$ from just looking at the local structure of $S^{(1)}$. In fact, the direction of $E(p)$ is always "tangential" to $S^{(1)}$, in other words, $E(p)$ is always contained in the tangent cone of $S^{(1)}$ at p. This property can be stated precisely as follows.

Theorem 2.7 (Tangency Lemma) *Suppose $E(\subset \Theta_M)$ is integrable on the whole M. Let k be an integer with $0 \leq k \leq r$ and p a point in $S^{(k)}$. Then we have*

$$E(p) \subset C_p S^{(k)},$$

where $C_p S^{(k)}$ denotes the tangent cone of $S^{(k)}$ at p.

Remark 2.8 The assumption of theorem (2.7) cannot be dropped. As a counterexample, you may consider the singular foliation on $\mathbf{C}^2 = \{(x, y)\}$ generated by $v_1 = \frac{\partial}{\partial x}$ and $v_2 = x\frac{\partial}{\partial y}$.

Theorem (2.7) can be showed as a corollary of an important theorem by D.Cerveau ([C]), but under a little stronger assumption we can draw a stronger result directly. The proof of the following proposition is originally due to T.Suwa.

Proposition 2.9 ((Strong) Tangency Lemma) *Suppose $E(\subset \Theta_M)$ is reduced (see* remark (1.5) (ii)*) and p is a point in M. Let v be a germ in E_p and let $\{\varphi_t = \exp tv\}$ be the local 1-parameter group of transformations induced by v. For all t sufficiently close to 0, we have*

$$(\varphi_t)_* E_p = E_{\varphi_t(p)},$$

where $(\varphi_t)_*$ denotes the differential map of φ_t.

Remark 2.10 Theorem (2.7) was proved by P.Baum under the hypotheses that E is reduced, $k = 1$ and p is a non-singular point of $S^{(1)}$ (see [B]). For the case of real singular foliations, see [N], [St] and [Sus].

Using theorem (2.7) we can prove the following results for a singular foliation E of dimension r on M. For details, we refer to [M].

Theorem 2.11 *Let k be an integer with $0 \leq k \leq r$ and $S^{(k)} = \{X_\alpha\}_{\alpha \in A}$ the natural Whitney stratification of the analytic set $S^{(k)}$. Then for any $\alpha \in A$ and $p \in X_\alpha$, we have $E(p) \subset T_p X_\alpha$. Moreover, E induces a non-singular foliation of dimension k on $X_\alpha - S^{(k-1)}$.*

Theorem 2.12 (Existence of Integral Submanifolds) *There exist integral submanifolds (whose dimensions are lower than r) also on $S(E)$. To be more precise, there is a family $\mathcal{L}$ of submanifolds of M such that $M = \bigcup_{L\in\mathcal{L}} L$ is a disjoint union and that any $L \in \mathcal{L}$ and $p \in L$, we have $E(p) = T_pL$.*

Each element L in $\mathcal{L}$ is often called a *leaf* of E.

Theorem 2.13 (Local Analytical Triviality) *Let k be an integer with $0 \leq k \leq r$ and p a point in $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Then there exist a small polydisk D of dimension $n-k$ transversal to $E(p)$ in T_pM, a singular foliation E' on D with $E'(p)=\{0\}$, a neighborhood U of p in M and a submersion $\pi : U \longrightarrow D$ such that*

$$E|_U \simeq \left(\pi^*(E'^a)\right)^a .$$

Theorem (2.13) says that the structure of a singular foliation E is locally analytically trivial along the leaf through each point p in M. Therefore, in the situation of example (2.5), if a point p belongs to $L^{(1)}$ then the singular foliation E is locally analytically trivial at p along $L^{(1)}$, since the leaf through p is $L^{(1)}$. If p belongs to $L^{(0)}$, however, theorem (2.13) does not say anything since the leaf through p consists of one point p. So the triviality along this type of singular set (along z-axis in example (2.5)) is another interesting topic. In fact, in order to obtain some triviality along z-axis in example (2.5), we must separate the origin from z-axis. In the following section we consider a stratification of the singular set $S(E)$ which gives a local triviality of E along each stratum.

3 Stratification and local topological triviality

Let E be a singular foliation on M. Since the singular set $S(E)$ is analytic, we can construct the "natural Whitney stratification" of $S(E)$ (see [W]), but this is not enough to achieve our purpose because the dimension of the leaf of E is not always constant on each stratum.

Example 3.1 Let v_1, v_2, v_3 be holomorphic vector fields on $M = \mathbf{C}^3 = \{(x, y, z)\}$ defined by

$$(3.2) \qquad \begin{cases} v_1 = y(3y+2z^2)\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = 2yz\dfrac{\partial}{\partial y} - (3y+2z^2)\dfrac{\partial}{\partial z} \\ v_3 = y^2z\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial z} . \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2, v_3. We can easily check that E is integrable, so E defines a singular foliation on $\mathbf{C}^3$. E is reduced, and $\operatorname{rank} E = 2$. By (3.2), $S(E) = S^{(1)} = \{x = yz = y(3y + 2z^2) = 0\} = \{x = y = 0\} = \{z\text{-axis}\}$ and $S^{(0)} = \{(0,0,0)\}$. Since $S(E)$ is non-singular, $S(E) = \{z\text{-axis}\}$ is the only stratum of the natural Whitney stratification of $S(E)$, but $\dim E(p)$ is not constant on the stratum.

In the above example, in order to get a Whitney stratification such that the leaf dimension is constant on each stratum, we may separate the bad point $(0,0,0)$ from the z-axis. Generalizing this argument, it turns out that we must take a "good" stratification of $S(E)$ such that $\dim E(p)$ is constant on each stratum.

Definition 3.3 *Let $E(\subset \Theta_M)$ be a singular foliation of dimension r on M, and let $\mathcal{S}$ be a stratification of M. We say that $\mathcal{S}$ is* adapted to E *when, for any stratum $X \in \mathcal{S}$, there is an integer i with $0 \leq i \leq r$ such that $X \subset L^{(i)}$, i.e., the leaf dimension of E is constant on each stratum $X \in \mathcal{S}$.*

Proposition 3.4 *There exists at least one Whitney stratification $\mathcal{S}$ of M which is adapted to E.*

In the case of example (2.5), a stratification satisfying the condition in proposition (3.4) is given by

$$\left\{M - S(E),\ L^{(1)},\ L^{(0)} - \{0\},\ \{0\}\right\}.$$

Now let us introduce a regularity condition for stratifications which is adapted to E.

Definition 3.5 *Let E be a singular foliation on M and let X be a submanifold in M such that $X \subset L^{(k)}$, i.e., the leaf dimension of E is constant on X. Let p be a point in X. We say that X* satisfies the foliated Verdier condition for E at p *when there exist a tubular neighborhood $(T,\ \pi,\ \rho)$ of X, a neighborhood U_p around p contained in T, and a real number $\lambda > 0$ such that the following inequality holds for all $y \in U_p - X$:*

$$\delta\,(E(y),\ T_pX) \leq \lambda \cdot \rho(y)\,,$$

where $\delta(\ ,\ )$ denotes the angle between two vector subspaces. If X satisfies the foliated Verdier condition for E at every point $p \in X$, then we say simply that X satisfies the foliated Verdier condition for E. Moreover, a stratification $\mathcal{S}$ adapted to E is called foliated Verdier stratification for E *if every stratum $X \in \mathcal{S}$ satisfies the foliated Verdier condition for E.*

Then we have the following "isotopy lemma" for singular foliations, which corresponds to the isotopy lemmas of Thom.

Theorem 3.6 *Let E be a singular foliation on M and suppose $\mathcal{S}$ is a foliated Verdier stratification for E (Note that this assumption includes that $\mathcal{S}$ is adapted to E). Then the structure of E is topologically locally trivial along each stratum $X \in \mathcal{S}$.*

For a proof of this theorem, the precise definition of the local topological triviality for singular foliations and further details about this isotopy lemma, we refer to [Y].

REFERENCES

[B] P.Baum, *Structure of foliation singularities*, Adv. in Math. 18, pp 361-374, 1975.

[BB] P.Baum and R.Bott, *Singularities of holomorphic foliations*, J. of Diff. Geom. 7, pp 279-342, 1972.

[C] D.Cerveau, *Distributions involutives singulières*, Ann. Inst. Fourier 29, pp 261-294, 1979.

[GWPL] C.G.Gibson, K.Wirthmüller, A.A.du Plessis and E.J.N.Looijenga, *Topological stability of smooth mappings*, Springer-Verlag, Berlin, Heidelberg, 1976.

[K] A.Kabila, *Formes integrables a singularites lisses*, Thèse, Université de Dijon, 1983.

[M] Y.Mitera, *The structure of singular foliations on complex manifolds (in Japanese)*, Master's thesis, Hokkaido University, 1989.

[N] T.Nagano, *Linear differential systems with singularities and application to transitive Lie algebras*, J. Math. Soc. Japan 18, 1966.

[St] P.Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. London Math. Soc. 29, pp 699-713, 1974.

[Sus] H.J.Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. 180, pp 171-188, 1973.

[Suw] T.Suwa, *Unfoldings of complex analytic foliations with singularities*, Japanese J. of Math. 9, pp 181-206, 1983.

[TW] D.J.A.Trotman and L.C.Wilson, *Stratifications and finite determinacy*, Pré-publicaitions 94-9, Université de Provence.

[W] H.Whitney, *Tangents to an analytic variety*, Ann. of Math. 81, pp 496-549, 1965.

[Y] J.Yoshizaki, *On the structure of the singular set of a complex analytic foliation* , Preprint, Hokkaido University, 1995.

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