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Parabolic fixed points of two dimensional complex dynamical systems
(2次元複素力学系の放物型不動点について)

Shigehiro Ushiki (宇敷重広)
Graduate School of Human and Environmental Studies
Kyoto University (京都大学大学院 人間・環境学研究科)
Kyoto, 606-01, Japan

0. Introduction

Let T be a holomorphic mapping of a neighborhood, V, of the origin, $O=(0,0)\in\mathbb{C}^2$, into $\mathbb{C}^2$ with $T(O)=O$. The germ of such a mapping is called a *local analytic transformation.*

Let T denote the set of all local analytic transformations. Local analytic transformations T and T' are said to be *r-equivalent* if their power series expansion at the origin coincide up to order r. The equivalence class is called the *r-jet* of the local analytic transformation.

Local analytic transformations T and T' are said to be *r-conjugate* if there is an invertible local analytic transformation S such that $S^{-1}\circ T\circ S$ and T' are r-equivalent. Let $\mathsf{T}_I=\{T\in\mathsf{T}\mid dT(O)=id\}$, where dT denotes the differential of T and id denotes the identity map. The elements of T_I are called *parabolic local analytic transformations.* Ueda[2] gave a classification of 2-jets of T_I.

Let $\mathsf{E}=\{P\in\mathbb{C}^2\mid T^n(P)\to O \text{ as } n\to\infty\}$, and $\mathsf{D}=\{P\in\mathbb{C}^2\mid T^n$ converge uniformly to O in some neighborhood of P as $n\to\infty\}$. If $\mathsf{D}\neq\emptyset$, then we say O has a basin of attraction.

In Ueda's list of normal forms, the case of $N_{2,1}(\lambda)$ (case I-B in our classification) :

$$(0.1)\qquad \begin{cases} x_1 = x + \lambda x^2 + xy + \cdots \\ y_1 = y + (\lambda+1)xy + y^2 + \cdots \end{cases}$$

has a parabolic basin if $\mathrm{Re}(\lambda)>0$. In this note, we shall prove that the fixed point of the above type has another attractive basin of a different type. The author does not know if they are analytically conjugate or not in the basins. Since this new type of attractive basin appears as a degenerate case of parabolic basin, we call such a basin a weakly-parabolic basin.

1. 2-jets of parabolic local analytic transformations

Let $f : \mathbb{C}^2 \to \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ be homogeneous polynomials of degree 2, and let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a parabolic analytic transformation defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + f(x,y) \\ y + g(x,y) \end{pmatrix}.$$

Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

denote the homogeneous part of degree 2. We have $F = id + H$.

If an invertible local analytic transformation S has a linear part $L \in GL(2,\mathbb{C})$, then the 2-jet of $S^{-1} \circ F \circ S$ is given by

$$L^{-1} \circ F \circ L = id + L^{-1} \circ H \circ L.$$

Hence, if parabolic local transformations $F = id + H$ and $F' = id + H'$ are 2-equivalent, then there exists a linear isomorphism $L \in GL(2,\mathbb{C})$ such that

$$L^{-1} \circ H \circ L = H'$$

and vice versa. Thus, the classification of 2-jets is reduced to the classification of homogeneous polynomial maps $H : \mathbb{C}^2 \to \mathbb{C}^2$ under the conjugacy $L^{-1} \circ H \circ L$ with $L \in GL(2,\mathbb{C})$. We have several cases.

CASE I : $f(x,y)$ and $g(x,y)$ are mutually prime.

CASE II : $f(x,y)$ and $g(x,y)$ have a common factor of degree one.

CASE III : $f(x,y)$ or $g(x,y)$ is a scalar multiple of the other (and not both zero).

CASE IV : both $f(x,y)$ and $g(x,y)$ are 0.

First, let us consider the case I. Let $\pi : \mathbb{C}^2 \setminus \{O\} \to \overline{\mathbb{C}}$ denote the natural projection of $\mathbb{C}^2 \setminus \{O\}$ to the Riemann sphere $\overline{\mathbb{C}}$. Homogeneous maps H and H' induce rational maps of degree 2 on the Riemann sphere. We denote the induced rational maps by $[H] : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $[H'] : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ respectively.

LEMMA 1.1 H and H' are conjugate by an element of $GL(2, \mathbb{C})$ if and only if $[H]$ and $[H']$ are conjugate by a Möbius transformation.

The classification of rational functions of degree 2 under the conjugacy of Möbius transformations is well known(*e.g.* see Milnor[1]). A conjugacy class of rational functions of degree two is characterized by the set of three multipliers of the fixed points. The three multipliers, say μ_1, μ_2, μ_3, are subject to the restriction

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0.$$

These values are invariant under the conjugacies.

If $\mu_i \neq 1$ $(i = 1, 2, 3)$, then the residues at each of the fixed points

$$\lambda_i = \frac{1}{2\pi\sqrt{-1}} \int \frac{dz}{[H](z) - z} = \frac{1}{\mu_i - 1}$$

give another set of holomorphic invariants. The values λ_i are called "translation numbers" in the normal forms studied by Ueda[2]. $\lambda_1, \lambda_2, \lambda_3$ are subject to the restriction

$$\lambda_1 + \lambda_2 + \lambda_3 = -1.$$

Ueda[2] proved the following.

THEOREM (Ueda) If Re $\lambda_i > 0$, then F has a (parabolic) basin of attraction of the fixed point O which corresponds to λ_i. If F is an automorphism of a complex manifold, then the basin of attraction is isomorphic to $\mathbb{C}^2$ and the dynamics in the basin is analytically conjugate to a translation.

This theorem holds also in the cases I-B and II-A-2 below. See Ueda[2] for the proof. Our case I is divided into three sub-cases.

CASE I-A : $[H]$ has three distinct fixed points.
CASE I-B : $[H]$ has a double fixed point and a simple fixed point.
CASE I-C : $[H]$ has a triple fixed point.

Normal forms as 2-jets for these cases are as follows.

$$\text{(I-A)} \qquad \begin{cases} x_1 = x + \lambda_1 x^2 + (\lambda_2 + 1)xy \\ y_1 = y + (\lambda_1 + 1)xy + \lambda_2 y^2. \end{cases}$$

Note that in our case I-A, we exclude the case where $\lambda_i = 0$ holds for some i . This case is treated as case II-A-1 and III-A-1, since in this case the components of H have a common factor.

The parameter λ in the following normal form is given by $\lambda = \frac{1}{\mu_1 - 1}$, if $\mu_1 \neq 1$ and $\mu_2 = \mu_3 = 1$, for example.

$$\text{(I-B)} \qquad \begin{cases} x_1 = x + \lambda x^2 + xy \\ y_1 = y + (\lambda+1)xy + y^2. \end{cases}$$

Note that in our case I-B, we exclude the case of $\lambda = 0$, in which case the induced map $[H]$ degenerates to a Möbius transformation with an indeterminate point. This case will be treated as case II-B-1.

In case I-C, we have $\mu_1 = \mu_2 = \mu_3 = 1$.

$$\text{(I-C)} \qquad \begin{cases} x_1 = x + xy \\ y_1 = y + x^2 + y^2. \end{cases}$$

Next, considr the case II, where $f(x,y)$ and $g(x,y)$ have a common factor and the induced map $[H]$ defines a Möbius transformation except at the indeterminate point corresponding to the common factor. We have three possibilities for the Möbius transformation $[H]$.

CASE II-A : $[H]$ has two distinct fixed points.
CASE II-B : $[H]$ has a double fixed point.
CASE II-C : $[H]$ is the identity.

And taking the indeterminate point, originating from the common factor, into considerations, we have sub-cases as follows.

CASE II-A-1 : the indeterminate point is different from the fixed points.

CASE II-A-2 : the indeterminate point coincides with one of the fixed points of the Möbius transformation.

CASE II-B-1 : the indeterminate point is different from the double fixed point.

CASE II-B-2 : the indeterminate point coincides with the double fixed point.

The normal form of case II-A-1 is same as the case I-A. There is a restriction on the parameters. Let $\gamma \in \mathbb{C} \setminus \{0,1\}$ denote the multiplier at one of the fixed point of the Möbius transformation. The parameters in the normal form are given by $\lambda_1 = \frac{\gamma}{1-\gamma}$, $\lambda_2 = \frac{1}{\gamma-1}$, and $\lambda_3 = 0$.

$$\text{(II-A-1)} \qquad \begin{cases} x_1 = x + \frac{\gamma}{1-\gamma}x^2 + \frac{\gamma}{\gamma-1}xy \\ y_1 = y + \frac{1}{1-\gamma}xy + \frac{1}{\gamma-1}y^2. \end{cases}$$

$$\text{(II-A-2)} \qquad \begin{cases} x_1 = x + \lambda x^2 \\ y_1 = y + (\lambda+1)xy. \end{cases}$$

Here, the parameter (translation number) λ is given by $\lambda = \frac{\gamma}{1-\gamma}$, for multiplier $\gamma \in \mathbb{C}\setminus\{0,1\}$ of the Möbius transformation at the indeterminate fixed point. Note that the cases $\lambda = 0$ and $\lambda = -1$ are omitted here. These cases will be treated as cases III-A-2 and III-B-1 below.

The case II-B-1 corresponds to the exceptional case of I-B with $\lambda = 0$.

$$\text{(II-B-1)} \qquad \begin{cases} x_1 = x + xy \\ y_1 = y + xy + y^2. \end{cases}$$

$$\text{(II-B-2)} \qquad \begin{cases} x_1 = x + x^2 \\ y_1 = y + x^2 + xy. \end{cases}$$

$$\text{(II-C)} \qquad \begin{cases} x_1 = x + x^2 \\ y_1 = y + xy. \end{cases}$$

In the case III, the induced map $[H]$ yields a constant function on the Riemann sphere. We have the following sub-cases according to the common factors of the components of H .

CASE III-A : the components $f(x,y)$ and $g(x,y)$ have two mutually prime common factors.
CASE III-B : the components $f(x,y)$ and $g(x,y)$ have a double common factor.

The common factor defines the indeterminate points of the induced map $[H]$. The value of the constant function $[H]$ is defined except at these indeterminate points. Let $v([H])$ denote the value. Taking these points into considerations, we have following sub-cases.

CASE III-A-1 : $v([H])$ is different from the indeterminate points.
CASE III-A-2 : $v([H])$ coincides with one of the indeterminate points.
CASE III-B-1 : $v([H])$ is different from the double indeterminate point.

CASE III-B-2 : $v([H])$ coincides with the double indeterminate point.

The case III-A-1 falls into the normal form I-A with excepted parameters $\lambda_1 = \lambda_2 = 0$, and a simpler normal form is given by

$$\text{(III-A-1)} \qquad \begin{cases} x_1 = x \\ y_1 = y + xy + y^2. \end{cases}$$

The normal form for case III-A-2 is obtained by setting $\lambda = 0$ in II-A-2.

$$\text{(III-A-2)} \qquad \begin{cases} x_1 = x \\ y_1 = y + xy. \end{cases}$$

The normal form for case III-B-1 is obtained by setting $\lambda = -1$ in II-A-2.

$$\text{(III-B-1)} \qquad \begin{cases} x_1 = x \\ y_1 = y + y^2. \end{cases}$$

$$\text{(III-B-2)} \qquad \begin{cases} x_1 = x \\ y_1 = y + x^2. \end{cases}$$

Finally, the case IV has the 2-jet normal form

$$\text{(IV)} \qquad \begin{cases} x_1 = x \\ y_1 = y. \end{cases}$$

Here, we note the correspondence between our classification of 2-jet normal forms of parabolic analytic transformations and that of Ueda's classification[2].

Ueda's notation	our clasification			
$N_1(\lambda_1, \lambda_2, \lambda_3)$	I-A,	II-A-1,	III-A-1	
$N_{2,1}(\lambda)$	I-B,	II-B-1		
$N_{2,2}(\lambda)$		II-A-2,	III-A-2, III-B-1	
$N_{3,1}$	I-C			
$N_{3,2}$		II-B-2		
$N_{3,3}$			III-B-2	
N_4		II-C		
N_0				IV

2. Pseudo-parabolic fixed points

In this section, we consider the case I-B. In this case, the induced map $[H]$ has a simple fixed point and a double fixed point. The translation number λ in the normal form I-B is related to the simple fixed point. We call a fixed point of type I-B a *pseudo-parabolic* fixed point. We are interested in the double fixed point of $[H]$. In order to study the behavior of the local analytic transformation in the neighborhood of the pseudo-parabolic fixed point, we consider the blow-up $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ of $\mathbb{C}^2$ at O. we denote the exceptional curve by $\Theta = \pi^{-1}(O) \simeq \overline{\mathbb{C}}$. Let V be the domain of definition of the transformation T and let $\widehat{V} = \pi^{-1}(V)$. The transformation induces an analytic transformation $\widehat{T} : \widehat{V} \to \widehat{\mathbb{C}^2}$. As $dT(O) = id$, all points of the exceptional curve are fixed points of $\widehat{T}$.

Let us try a blow-up in our case I-B. The x-axis direction, $\{y = 0\}$, corresponds to the simple fixed point of $[H]$, and is related to the translation number λ. To see this, we may try a blow-up with $t = \frac{y}{x}$. We obtain the following local analytic transformation.

$$(2.3) \qquad \begin{cases} x_1 = x + (\lambda + t)x^2 + \cdots \\ t_1 = t + tx + \cdots . \end{cases}$$

The y-axis direction, $\{x = 0\}$, corresponds to the parabolic fixed point of $[H]$. We try a blow-up with $u = \frac{x}{y}$ and obtain the following.

$$(2.4) \qquad \begin{cases} y_1 = y + (1 + (\lambda + 1)u)y^2 + \cdots \\ u_1 = u - u^2 y + \cdots . \end{cases}$$

Local analytic transformations arising from such a blow-up leaves the exceptional curve invariant, and all the points in the exceptional curve are fixed points. By taking a system of local coordinates around the point in the exceptional curve, we can assume, in general, that the local analytic transformation is of the following form.

$$(2.5) \qquad \begin{cases} x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\ y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots . \end{cases}$$

Local analytic transformations of the form (2.5) is called a transformation of class S_ν, $\nu = 0, 1, 2, \cdots$ [resp. class S_∞] if $g_1(y)$ vanishes at $y = 0$ exactly with order ν [resp. vanish identically]. For $T \in S_1$, we define the *translation number* λ by

$$\lambda = \frac{f_2(0)}{g_1'(0)}.$$

The translation number λ and the multiplier μ of the corresponding simple fixed point of $[H]$ are related by $\lambda = \frac{1}{\mu-1}$. The translation number is also a holomorphic invariant in class S_1.

For $T \in \mathsf{S}$, the order of vanishing of $g_1(y)$ at $y = 0$ is invariant under those holomorphic change of coordinates which transforms the transformation of the form (2.5) into the same form.

Let T be a local analytic transformation, and $T \in \mathsf{S}_1$. The origin has a basin of attraction if the real part of the translation number is positive. We call this basin of attraction a *parabolic basin* of the parabolic fixed point.

Note that (2.3) is of class S_1 and its translation number is λ. The transformation for the double fixed point (2.4) is of class S_2, which shall be discussed in the following section.

3. Weakly-parabolic basin

In this section, we consider a local analytic transformation $T \in \mathsf{S}_2$ given by

$$(3.1) \qquad \begin{cases} x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\ y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots, \end{cases}$$

where $g_1(0) = 0$, $g_1'(0) = 0$, and $g_1''(0) \neq 0$.

THEOREM 3.1 If $f_2(0) \neq 0$, local analytic transformation (3.1) has a non-empty basin of attraction.

We call this attractive basin a *weakly-parabolic basin.* As a preliminary, we try to simplify the transformation by local change of coordinates.

PROPOSITION 3.2 For any $\delta \in \mathbb{C}$, by a change of coordinates $S_\alpha : (X, Y) \mapsto (x, y)$ of the form

$$(3.2) \qquad \begin{cases} x = \alpha(Y)X \\ y = Y, \end{cases}$$

where $\alpha(Y)$ is an analytic function of Y, transformation (3.1) can be transformed into the form

$$(3.3) \qquad \begin{cases} X_1 = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\ Y_1 = Y + G_1(Y)X + G_2(Y)X^2 + \cdots \end{cases}$$

with $F_2(Y) = 1 + \delta Y + \cdots$, $G_1(0) = G_1'(0) = 0$, and $G_1''(0) \neq 0$.

PROOF Let $\widetilde{T}$ denote the transformation (3.3). As S_α is a local automorphism, we have $\alpha(0) \neq 0$ and

$$T \circ S_\alpha = S_\alpha \circ \widetilde{T}$$

holds. Expand the both sides as power series in X with analytic functions in Y as coefficients. By comparing the coefficients of both sides, we have

$$f_2(Y)(\alpha(Y))^2 = \alpha(Y)F_2(Y) + \alpha'(Y)G_1(Y) \tag{3.4}$$

and

$$G_1(Y) = \alpha(Y)g_1(Y). \tag{3.5}$$

The function $\alpha(Y)$ must satisfy the differential equation

$$f_2(Y)\alpha(Y) - g_1(Y)\alpha'(Y) = F_2(Y), \tag{3.6}$$

with $\alpha(0) \neq 0$. Let

$$a_0 = \frac{1}{f_2(0)},$$

$$a_1 = \frac{1}{f_2(0)}(\delta - a_0 f_2'(0)) = \frac{1}{f_2(0)}(\delta - \frac{f_2'(0)}{f_2(0)})$$

and choose the analytic function $\alpha(Y)$ as, for example,

$$\alpha(Y) = a_0 + a_1 Y.$$

We obtain the desired change of coordinates of the proposition. As $\alpha(0) = a_0 \neq 0$, the conditions for $G_1(Y)$ are satisfied.

Especially, as we have $G_1''(0) = f_2(0)g_1''(0)$, we can take $\delta = G_1''(0)/2 = f_2(0)g_1''(0)/2$ to be used in the following proposition.

PROPOSITION 3.3 Assume $T \in \mathrm{S}_2$ and $f_2(y) = 1 + \delta y + O(y^2)$, with $\delta = \frac{g_1''(0)}{2}$. By a change of coordinates $S_\beta : (X, Y) \mapsto (x, y)$ of the form

$$\begin{cases} x = X \\ y = \beta(Y), \end{cases} \tag{3.7}$$

with $\beta(0) = 0$, $\beta'(0) \neq 0$, T can be transformed into $\widetilde{T} : (X, Y) \mapsto (X_1, Y_1)$,

$$\begin{cases} X_1 = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\ Y_1 = Y + G_1(Y)X + G_2(Y)X^2 + \cdots \end{cases} \tag{3.8}$$

with $F_2(Y) = 1 + Y + O(Y^2)$ and $G_1(Y) = Y^2 + O(Y^3)$.

PROOF Compare both sides of $T \circ S_\beta = S_\beta \circ \widetilde{T}$ as power series in X and obtain

$$f_2(\beta(Y)) = F_2(Y), \quad \text{and} \quad g_1(\beta(Y)) = \beta'(Y)G_1(Y).$$

Let $\beta(Y) = \frac{2}{g_1''(0)}Y$, for example, to get $G_1(Y) = Y^2 + O(Y^3)$. Note that, here, generally, a term of order 3 cannot be suppressed by an analytic change of coordinates. We have, also,

$$F_2(Y) = f_2(\beta(Y)) = 1 + Y + O(Y^2).$$

PROPOSITION 3.4 Let $T : (x, y) \mapsto (x_1, y_1)$ be a local analytic transformation of the form

$$(3.9) \qquad \begin{cases} x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\ y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots, \end{cases}$$

and let $S : (X, Y) \mapsto (x, y)$ be a change of local coordinates of the form

$$(3.10) \qquad \begin{cases} x = \alpha_1(Y)X + \alpha_2(Y)X^2 + \alpha_3(Y)X^3 + \cdots \\ y = \beta_0(Y) + \beta_1(Y)X + \beta_2(Y)X^2 + \cdots. \end{cases}$$

Let $\widetilde{T} : (X, Y) \mapsto (X_1, Y_1)$ be the transformation given by $\widetilde{T} = S^{-1} \circ T \circ S$, with

$$(3.11) \qquad \begin{cases} X_1 = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\ Y_1 = Y + G_1(Y)X + G_2(Y)X^2 + \cdots. \end{cases}$$

Then we have the followings.

$$G_1(Y) = \frac{\alpha_1(Y)}{\beta_0'(Y)} g_1(\beta_0(Y))$$

and

$$F_2(Y) = \alpha_1(Y) f_2(\beta_0(Y)) - \frac{\alpha_1'(Y)}{\beta_0(Y)} g_1(\beta_0(Y)).$$

PROOF These are verified by an immediate computation.

PROPOSITION 3.5 Assume $T \in \mathrm{S}_2$ is of the form (3.9) with $f_2(y) = 1 + y + O(y^2)$ and $g_1(y) = y^2 + O(y^3)$. By a local change of coordinates S of the form (3.10), the transformation T can be transformed into $\widetilde{T}$ of the form (3.11) with $F_2(Y) = f_2(Y)$, $G_1(Y) = g_1(Y)$ and $G_2(Y) = 0$.

PROOF We set $\alpha_1(Y) = 1$ and $\beta_0(Y) = Y$. Then proposition 3.4 guarantees that $G_1(Y) = g_1(Y)$ and $F_2(Y) = f_2(Y)$. Compute $S \circ \widetilde{T}$ and $T \circ S$ to compare the coefficients of X^2 in y_1. We get

$$G_2(Y) = g_2(Y) + \beta_1(Y)(g_1'(Y) - f_2(Y)) + g_1(Y)(\alpha_2(Y) - \beta_1'(Y)).$$

Hence, if we set

$$\beta_1(Y) = \frac{g_2(Y)}{f_2(Y) - g_1'(Y)}$$

and

$$\alpha_2(Y) = \beta_1'(Y),$$

we get $G_2(Y) = 0$. As $f_2(Y) = 1+Y+O(Y^2)$ and $g_1'(Y) = O(Y), \beta_1(Y)$ is analytic near the origin.

4. Proof of theorem 3.1

By propositions in the previous section, we can asuume

$$f_2(y) = 1 + y + O(y^2),$$

$$g_1(y) = y^2 + O(y^3),$$

and

$$g_2(y) = 0$$

to prove theorem 3.1. Then, the transformation $T : (x, y) \mapsto (x_1, y_1)$, $T \in \mathrm{S}_2$, takes the following form

$$(4.1) \qquad \begin{cases} x_1 = x + (1+y)x^2 + O(y^2x^2) + O(x^3) \\ y_1 = y + y^2x + O(y^3x) + O(x^3), \end{cases}$$

where $O(\varphi(x, y))$ implies some analytic function, say $\psi(x, y)$, which can be written as $\psi(x, y) = \varphi(x, y)\rho(x, y)$ for some analytic function $\rho(x, y)$ in a neighborhood of the origin.

As we are interested in the behavior of the transformation in the y-axis direction near the origin, let us blow-up the origin along the y-axis. More precisely, we change the coordinates by

$$(4.2) \qquad u = \frac{x}{y}, \quad v = y$$

into new coordinates (u, v). The origin $(0, 0)$ of (x, y)-coordinates corresponds to the exceptional curve $\overline{\mathbb{C}} \times \{0\}$ in the (u, v)-coordinates.

In the (u,v)-coordinates, (4.1) takes the form

$$\text{(4.3)} \qquad \begin{cases} u_1 = u + vu^2 + O(v^3u^2) + O(v^2u^3) \\ v_1 = v + v^3u + O(v^4u) + O(v^3u^3). \end{cases}$$

Let us take a new system of coordinates defined by

$$\text{(4.4)} \qquad z = \tfrac{1}{u}, \qquad w = \tfrac{1}{v}.$$

Then (4.3) is transformed into the form

$$\text{(4.5)} \qquad \begin{cases} z_1 = z - \frac{1}{w}h_1(z,w) \\ w_1 = w - \frac{1}{zw}h_2(z,w), \end{cases}$$

where $h_1(z,w) = 1+O(\frac{1}{zw})+O(\frac{1}{w^2})$ and $h_2(z,w) = 1+O(\frac{1}{z^2})+O(\frac{1}{w})$. We regard (4.5) as a transformation near $(\infty,\infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Take constants $\theta_0, \theta_1, \theta_2$ such that

$$0 < \theta_0 < \frac{1}{8}\pi, \ 0 < \theta_2 < \frac{1}{8}\theta_0, \ \text{and} \ \ \theta_0 + \theta_2 < \theta_1 < \frac{5}{4}\theta_0 - \theta_2.$$

Note that $0 < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3}$ holds.

Choose r_1 and r_2 such that $\frac{3}{4} < r_1 < 1 < r_2 < \frac{5}{4}$ and let

$$\Omega = \{z \in \mathbb{C} \mid |\arg z| < \theta_2, r_1 < |z| < r_2\}.$$

For $R_1, R_2 > 0$, let

$$U = \{z \in \mathbb{C} \mid |\arg(-z)| < \theta_1, \operatorname{Re} z < -R_1\}$$

and

$$V = \{w \in \mathbb{C} \mid |\arg w| < \theta_0, \operatorname{Re} w > R_2\}.$$

Choose sufficiently large R_1 and R_2 such that

$$h_1(z,w) \in \Omega \quad \text{and} \quad h_2(z,w) \in \Omega$$

holds for all $(z,w) \in U \times V$, and that

$$r_2 < R_1 R_2^2 \sin(\frac{\theta_0}{2}).$$

Let $\Phi : (z,w) \mapsto (z_1,w_1)$ denote the transformation (4.5) defined near $(\infty,\infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

PROPOSITION 4.1 If $(z,w) \in U \times V$, then $\Phi(z,w) = (z_1,w_1) \in U \times V$, Re $z_1 <$ Re z, and Re $w_1 >$ Re w.

PROOF Let $(z, w) \in U \times V$. Then

$$|\arg(\frac{1}{w}h_1(z,w))| < \theta_0 + \theta_2 < \theta_1$$

and

$$\mathrm{Re}(\frac{1}{w}h_1(z,w)) > 0.$$

Hence $z_1 \in U$ and $\mathrm{Re}\, z_1 < \mathrm{Re}\, z$ follow. Now, let $\theta = \arg w$. Then $-\theta_0 < \theta < \theta_0$. Note that

$$|\arg(-\frac{1}{zw}h_2(z,w))| < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3}$$

and

$$\mathrm{Re}(-\frac{1}{zw}h_2(z,w)) > 0.$$

First, consider the case where $\frac{\theta_0}{2} < \theta < \theta_0$. In this case, we have

$$\arg(-\frac{1}{zw}h_2(z,w)) < -\theta + \theta_1 + \theta_2 < \theta_0.$$

So, we have $\arg w_1 < \theta_0$ and $\mathrm{Re}\, w_1 > w > R_2$. On the other hand,

$$|w_1 - w| = |-\frac{1}{zw}h_2(z,w)| < \frac{r_2}{R_1 R_2} < R_2 \sin\frac{\theta_0}{2}.$$

Hence $w_1 \in V$ in this case.

Similarly, if $-\theta_0 < \theta < -\frac{\theta_0}{2}$, we have $w_1 \in V$.

Next, if $|\theta| \leq \frac{\theta_0}{2}$, we have

$$\mathrm{Re}(-\frac{1}{zw}h_2(z,w)) > 0 \quad \text{and} \quad |w_1 - w| < R_2 \sin\frac{\theta_0}{2},$$

which imply $w_1 \in V$ and $\mathrm{Re}\, w_1 > \mathrm{Re}\, w$. Thus proposition 4.1 is proved.

Theorem 3.1 is a corollary of this proposition.

References

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