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Bifurcations of Nusse-Yorke's family in the quadratic rational functions

Masayo FUJIMURA (藤村雅代) Dept. of Math., College of Sci. and Tech., Nihon Univ.
masayo@math.cst.nihon-u.ac.jp
Kiyoko NISHIZAWA (西沢清子) Department of Math., Josai Univ.
kiyoko@euclides.josai.ac.jp

1 Introduction

let $\{f_\lambda\}_\Lambda$ be an one-parameter family of discrete dynamical systems on $\mathbf{R}$, where Λ is an interval of $\mathbf{R}$.

As the parameter increased, a bifurcation parameter value λ_0 is called orbit creating if new periodic orbits are created, and no periodic orbits are annihilated, at λ_0. A bifurcation parameter value λ_0 is called orbit annihilating if periodic orbits are annihilating, and no new periodic orbits are created , at λ_0. A bifurcation parameter value λ_0 is called neutral if no periodic orbits are annihilating, and no periodic orbits are created , at λ_0.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing if every bifurcation parameter value in Λ is neutral or orbit creating, and it is said to be monotone decreasing if every bifurcation parameter value is neutral or orbit annihilating.

A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values.

A parameter value λ_0 is called anti-monotone if every neighborhood of λ_0 contains both orbit creating and orbit annihilating parameter values.

In [Jon93], they studied heuristically that the existence of anti-monotone parameter values in one-dimensional one-parameter multi-modal systems is a generic property and they supported this by numerical experiments.

It is not known if anti-monotone parameter values exist in all generic bi-modal families with a chaotic attractor.

For one-parameter family of dissipative plane C^3-diffeomorphisms, the situation is different:

ANTI-MONOTONICITY THEOREM([I. 92]) In any neighborhood of a nondegenerate, homoclinic-tangency parameter value of an one-parameter family of dissipative C^3-diffeomorphisms of the plane, there must be both infinitely many orbit-creation and infinitely many orbit-annihilation parameter values.

Subject to certain mild nondegeneracy restrictions, anti-monotonic creation and annihilation of periodic orbits must occur in all one-parameter families of dissipative, chaotic planar diffeomorphisms. The bifurcation diagram of any such family will have pitchforks with opposite orientations ([I. 92]).

The creation and annihilation of periodic orbits is the most fundamental bifurcation process in one-parameter family of maps.

Here we discuss an one-parameter real quadratic family with pitchforks with opposite orientations in its bifurcation diagram, and explain in the moduli space of the real quadratic rational maps why we should expect such behavior. This result is a complete answer to the paper of Nusse and Yorke ([NY88]). Namely they denoted that a 2-parameters family given as

$$\left\{ f_{m,r}(x) = m\,\frac{rx^2 + x + r}{1 + x^2} \right\},$$

by fixing the parameter r, does not exhibit periodic-halving bifurcation as the multiplier m is increasing. In other words, this family does not have orbit annihilating parameter values. But this statement is false. We show precisely our result later as Theorem 2 in Section 3, and further in this section that for a suitably chosen parameter $a = 1/m$, period-doubling and period-halving bifurcation both occur (pitchfork bifurcation with opposite orientations in its diagram), as a parameter r varies monotonely. For example, this phenomenon can occur at $a = -0.2$, see Figure 6.

2 Moduli space $\mathcal{M}_2(\mathbf{R})$ of real quadratic rational maps

$\mathrm{Rat}_2(\mathbf{R})$ is the space of all real quadratic rational maps $f : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$,

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0x^2 + a_1x + a_2}{b_0x^2 + b_1x + b_2}.$$

Two maps $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{R})$ are **holomorphically conjugate** if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{R})$ with $g \circ f_1 \circ g^{-1} = f_2$, denoted by $f_1 \sim f_2$.

Definition 1 $\mathcal{M}_2(\mathbf{R}) = \mathrm{Rat}_2(\mathbf{R})/\mathrm{PSL}_2(\mathbf{R})$ is called the **moduli space** of holomorphic conjugacy class $\langle f \rangle$ of real quadratic rational maps f.

Remark 1 The definitions of moduli space $\mathcal{M}_2(\mathbf{C})$ for the complex quadratic maps, $\mathrm{Rat}_2(\mathbf{C})$, is identified with $\mathrm{Rat}_2(\mathbf{C})/\mathrm{PSL}_2(\mathbf{C})$.

For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be fixed points of f, μ_i the multiplier of z_i $(1 \le i \le 3)$; $\mu_i = f'(z_i)$. Now consider elementary symmetric functions of three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

Milnor introduces coordinates of $\mathcal{M}_2(\mathbf{C})$ as follows [Mil92].

Lemma 1 (lemma 3.1 of [Mil92]) *These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that*

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \tag{1}$$

or in other words

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$, with coordinates σ_1 and σ_2.

Here after we treat only the real case. σ_i $(1 \le i \le 3)$ are all real, because three fixed points and multipliers are either all real or one real and a pair of complex conjugate numbers.

Proposition 1 *$\mathcal{M}_2(\mathbf{R}) \setminus \{\langle a(x \pm \frac{1}{x})\rangle\}_{a \in \mathbf{R}^\times}$ is isomorphic to $\mathbf{R}^2$ except on the cubic algebraic curve,*

$$F(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \tag{2}$$

There is two to one map between $\{\langle a(x \pm \frac{1}{x})\rangle\}$ and the curve of (2).

In Figure 15 [Mil92] the curve of (2) is drawn. Here we can give a defining equation (2) of this cubic curve.

Proof. There is a following relation between a point (σ_1, σ_2) on the moduli space and a multiplier μ of a fixed point of f, (compare lemma 3.4 of [Mil92])

$$\sigma_2 = (\mu + \frac{1}{\mu})\sigma_1 - (\mu^2 + \frac{2}{\mu}).$$

Hence,

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0. \tag{3}$$

Let μ_1, μ_2, μ_3 be the three roots of (3). Notice that if two multipliers μ_i, μ_j are different, corresponding fixed points z_i and z_j are distinct.

For $(\sigma_1, \sigma_2) \in \mathrm{R}^2$, the following four combinatorial cases of $\{\mu_1, \mu_2, \mu_3\}$ are possible.

If f has three distinct fixed points, we can assume that these points are 0, 1, and infinity.

Case 1. : μ_1, μ_2, μ_3 are real numbers and $\mu_i \neq \mu_j$, $(i \neq j)$

Corresponding quadratic map can be chosen as the following form,

$$f(x) = x\frac{x + \mu_1}{\mu_2 x + 1}.$$

Case 2. : μ_1, μ_2, μ_3 are one real number and a pair of complex conjugate numbers.

Let μ_1 be unique real root and μ_2, μ_3 a pair of complex conjugate roots, namely $\mu = \mu_3 = \bar{\mu}_2$. In this case corresponding fixed point z_1 is real and z_2, z_3 are a pair of complex conjugate numbers. Same as Case 1. corresponding quadratic map can chosen as the following form,

$$f(x) = x\frac{x + \mu}{\bar{\mu}x + 1}.$$

By using Möbius transformation again, real representative of the class $\langle f \rangle$ is obtained as follows,

$$i\frac{(\mu + \bar{\mu} + 2)x^2 + (\mu + \bar{\mu} - 2)}{(-\mu + \bar{\mu})x^2 + 4ix + (-\mu + \bar{\mu})}$$

Case 3. : $\mu_1 = \mu_2 = \mu_3$.

This case occurs only on the line: $\sigma_2 - 2\sigma_1 + 3 = 0$.

Now let $\hat{\mu} = \mu_1 = \mu_2 = \mu_3$.

If $\hat{\mu} \neq 1$, corresponding fixed points z_i $(i = 1, 2, 3)$ are distinct each other. Calculating the index of fixed points, we have $\hat{\mu} = 2$. Hence f has the following form,

$$f(x) = x\frac{x + 2}{2x + 1}.$$

If $\hat{\mu} = 1$, corresponding fixed points are same: $z_1 = z_2 = z_3$. The other case: $z_1 = z_2 \neq z_3$ can't occur because of $\deg f = 2$.

Assuming that only one fixed point is infinity, we have

$$f(x) = x + \frac{1}{x}.$$

Case 4. : $\mu_1 = \mu_2 \neq \mu_3$.

Case 4. occurs if and only if discriminant of the equation of (3) is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0. \qquad (4)$$

If corresponding fixed points z_1, z_2 satisfies $z_1 = z_2$, then μ_1 and μ_2 equal to 1 and μ_3 can be arbitrary.

Assuming that z_1, z_3 are zero and infinity, we obtain the form,

$$f(x) = x\frac{x+1}{\mu_3 x + 1}.$$

In this case, one-parameter family $\left\{x\frac{x+1}{\mu x+1}\right\}_\mu$ corresponds with the first factor of (4). This line, $\sigma_2 - 2\sigma_1 + 3 = 0$, called $\mathrm{per}_1(1)$.

If $z_1 \neq z_2$ then $\mu_3 = \frac{2}{1+\mu_1}$, and f has following two real forms

$$\frac{1}{\mu_3}(x + \frac{1}{x}), \quad \frac{1}{\mu_3}(x - \frac{1}{x}),$$

These representatives are conjugate by $x \mapsto ix$, however, these are not "real" conjugate. In this case one-parameter families $\{\frac{1}{\mu_3}(x \pm \frac{1}{x})\}$ correspond with the second factor of (4). ∎

From now on, we regard moduli space $\mathcal{M}_2(\mathbf{R})$ as $\mathbf{R}^2$.

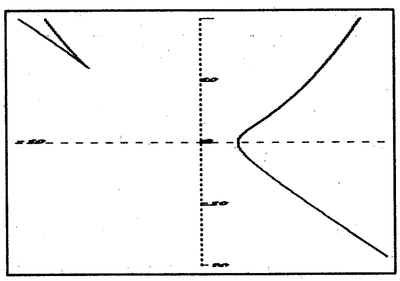

Figure 1: Moduli space with the cubic curve : $2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0$.

3 A quadratic family with non-monotone bifurcation

Now, we treat a real 2-parameter family given by Nusse and Yorke [NY88],

$$\left\{f_{m,r}(x) = m\frac{rx^2 + x + r}{1 + x^2}\right\}_{(m,r)\in\mathbf{R}^2}.$$

M. Bier and T. C. Bountis studied "period-bubbling" bifurcation [BB84]. Their purpose is to demonstrate that monotone bifurcation commonly arise in some of the simplest nonlinear dynamical systems involving the variation of more than one parameter. As a simple example of non-monotone bifurcation, they treat quadratic rational mapping, $x_{t+1} = Q + Ax_t/(x_t^2+1)$, $(A, Q > 0)$.

H. E. Nusse and J. A. Yorke gave an example of exponential function family that has non-monotone bifurcation, even though it has negative Schwarzian derivative [NY88]. Their question was arisen of whether having a negative Schwarzian derivative rules out non-monotone bifurcation. They describe in [NY88] that if the above quadratic rational family is written in the following form,

$$\left\{ f_{m,r}(x) = m\frac{rx^2+x+r}{1+x^2} \right\},$$

it does not exhibit non-monotone bifurcation as the parameter m is increased. But we can show that this family exhibit non-monotone bifurcation for suitable parameter r.

3.1 The case of the parameter r fixed

In this section, we show that this family exhibit non-monotone bifurcation for suitable parameter r.

Since $f_{m,r} \sim f_{m,-r}$, we can restrict parameter r positive: $r \geq 0$.

In general, we obtain next results for a fixed parameter r.

Theorem 1 *On $\mathcal{M}_2(\mathbf{R})$, one parameter family $\{f_{m,r}(x)\}_m$ for a fixed $r \neq \frac{1}{2}, 0$ is characterized as the following irreducible algebraic curve of degree 4,*

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -4096r^6 + (-128\sigma_1^2 + 512\sigma_1 + 512\sigma_2 + 1536)r^4 \\ &\quad +(-\sigma_1^4 + 8\sigma_1^3 + (8\sigma_2+8)\sigma_1^2 + (-32\sigma_2 - 96)\sigma_1 - 16\sigma_2^2 - 96\sigma_2 - 144)r^2 \\ &\quad -2\sigma_1^3 + (-\sigma_2+1)\sigma_1^2 + (8\sigma_2 - 12)\sigma_1 + 4\sigma_2^2 - 12\sigma_2 + 36 = 0. \end{aligned} \tag{5}$$

For $r = \frac{1}{2}$, following irreducible algebraic curve of degree 3.

$$H_{\frac{1}{2}}(\sigma_1,\sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0.$$

For $r = 0$,

$$H_0(\sigma_1,\sigma_2) = F(\sigma_1,\sigma_2).$$

Proof. Three fixed points z_1, z_2, z_3 of f are the roots of the equation

$$x^3 - mrx^2 + (1-m)x - mr = 0.$$

From the relation between coefficients and solutions, following equations hold.

$$\begin{cases} z_1 + z_2 + z_3 = mr \\ z_1z_2 + z_2z_3 + z_3z_1 = 1 - m \\ z_1z_2z_3 = mr \end{cases}$$

Let μ_i $(i = 1,2,3)$ be multiplier of each fixed point z_i $(i = 1,2,3)$ given by,

$$\mu_i = m\frac{z_i^2 - 1}{(z_i^2+1)^2}.$$

By using "Gröbner basis" of Risa/Asir, symbolic and algebraic computation system, we can obtain $\sigma_1(= \mu_1 + \mu_2 + \mu_3)$ and $\sigma_2(= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$ as functions of m and r:

$$\begin{cases} 4m^2r^2 - m^2 + (\sigma_1 + 2)m - 4 = 0 \\ -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases} \tag{6}$$

Using "Gröbner basis" again, we can remove m from (6), and we have (5).

In the case of $r = \frac{1}{2}$, $-\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0$.

In the case of r equal to 0, algebraic curve of (5) coincides with the curve of (2). ∎

Example 1 For example, bifurcation diagram of one parameter family $\{f_{m,0.54}\}_m$ is given in Figure 2, and its characteristic curve is Figure 3.

Example 2 We can analyze the non-monotone bifurcation by overwriting the algebraic curve of $H_r = 0$ on the $\mathcal{M}_2(\mathbf{R})$.

One parameter family $\{f_{m,0.58}\}_m$ has non-monotone (period-bubbling) bifurcation. See Figure 4.

In Figure 5, the thick line indicates this family, and the gray belt is the region on which each map has attracting period 2 cycle. When algebraic curve of degree 4 through this gray belt, period-doubling bifurcation occurs. In this case, the curve intersects the gray belt (period-doubling occurs) and intersects again the period 1 region (period-halving occurs). Hence period-bubbling bifurcation occurs, as in Figure 4.

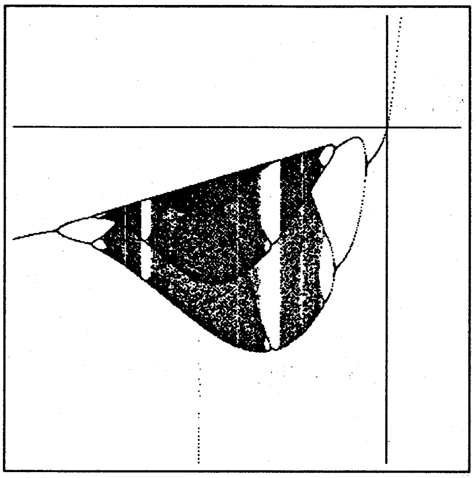

Figure 2: Non-monotone bifurcation, $-25.0 \leq m \leq 5.0$, $-3.0 \leq x \leq 1.0$, $r = 0.54$

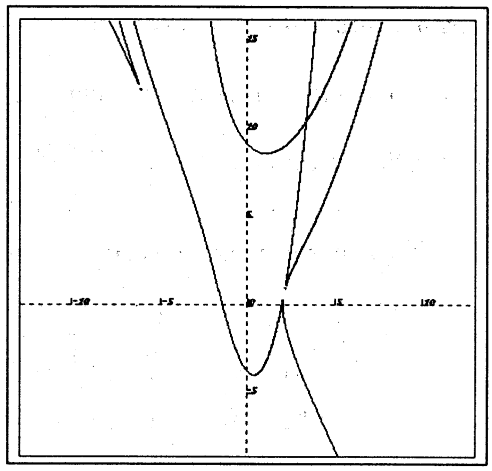

Figure 3: algebraic curve of degree 4 and cubic curve in the moduli space. In the case of $r = 0.54$.

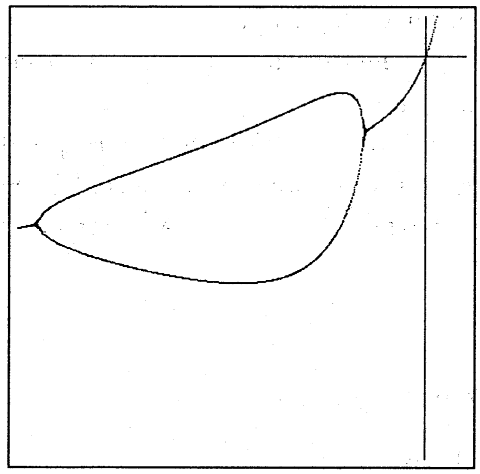

Figure 4: Period-bubbling bifurcation: $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$, Parameter $r = 0.58$.

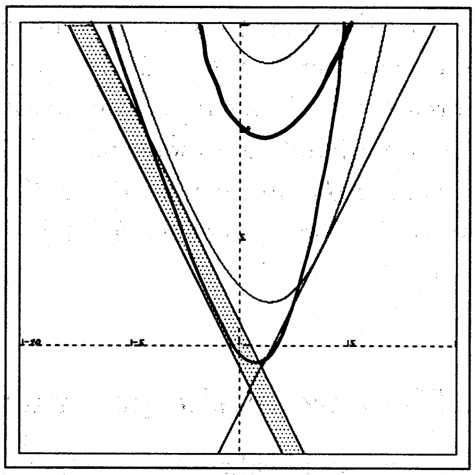

Figure 5: algebraic curve of degree 4 in the "classified" moduli space. Thick curve corresponds with $r = 0.58$, thin curve corresponds with $r = 0.7$.

Theorem 2 *For a fixed parameter r, there are following three possibilities;*

1. *various bifurcations occur, if* $0 < r \leq \frac{1}{2}$,
2. *non-monotone bifurcations occur, if* $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$, *or*
3. *any bifurcation can't occur, if* $\frac{3\sqrt{3}}{8} \leq r$.

Proof. Now introduce the escape locus E;

$$\begin{aligned} E &= E_1 \bigcup E_2 \bigcup E_3, \\ E_1 &= \{(\sigma_1, \sigma_2) \mid \sigma_2 > -2\sigma_1 + 1, \sigma_2 > 2\sigma_1 - 3\}, \\ E_2 &= \{(\sigma_1, \sigma_2) \mid \sigma_2 < 2\sigma_1 - 3, \sigma_1 < -1\}, \\ E_3 &= \{(\sigma_1, \sigma_2) \mid \sigma_2 < \frac{-2\sigma_1^2 - 7\sigma_1 - 10}{2 + \sigma_1}, \sigma_1 > -1\}. \end{aligned}$$

E is the region on which there doesn't real periodic orbits other than the attracting fixed point. Therefor if the algebraic curve H_r corresponding to $\{f_{m,r}\}_m$ contained in the escape locus, any bifurcation don't occur.

Now consider $H_r(2, \sigma_2) = 4(2r - 1)(2r + 1)(16r^2 + 8r - \sigma_2)(-16r^2 + 8r + \sigma_2) = 0$. If $r \neq \pm\frac{1}{2}$, at least one of the $\sigma_2 = 16r^2 + 8r$ or $\sigma_2 = 16r^2 - 8r$ is positive number. Hence, it is impossible that the curve H_r contained in the component $E_2 \cup E_3$.

From calculation, if $64r^2 - 27 > 0$, the curve H_r contained in the escape locus, that is, any bifurcation can't occur. Hence possibility 3. is determined.

On the other hand, non-monotone bifurcation can occur on condition that corresponding algebraic curve H_r intersects transversally the line $\mathrm{Per}_1(1)$, the boundary of period 2 region and the escape locus, and again intersects the line $\mathrm{Per}_1(1)$. $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$ satisfies this condition.

∎

3.2 The case of the parameter m fixed

For simplification of calculation, we rewrite the family as follows

$$\left\{ f_{a,r}(x) = \frac{1}{a} \frac{rx^2 + x + r}{1 + x^2} \right\}.$$

For a fixed parameter a, one parameter family $\{f_{a,r}\}_r$ becomes parabola in $\mathcal{M}_2(\mathbf{R})$. Hence it is easy to analyze the dynamical behavior along the algebraic curve in $\mathcal{M}_2(\mathbf{R})$, in comparison with algebraic curve of degree 4.

It is known that for a suitably chosen parameter a, period-doubling and period-halving bifurcation both occur, as the parameter r varies monotonely. For example, this phenomenon can occur at $a = -0.2$, see Figure 6.

The bifurcation diagram is symmetry as $f_{-0.2,-r}(x) = -f_{-0.2,r}(-x)$. The family $\{f_{-0.2,r}(x)\}_r$ coincides with a part of parabolic curve in $\mathcal{M}_2(\mathbf{R})$, see Figure 7.

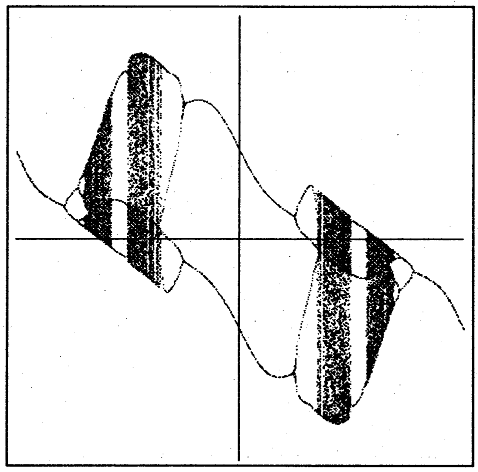

Figure 6: Period-doubling and period-halving bifurcation, $-1.0 \leq r \leq 1.0$, $-6.0 \leq x \leq 6.0$, $\quad a = -0.20$

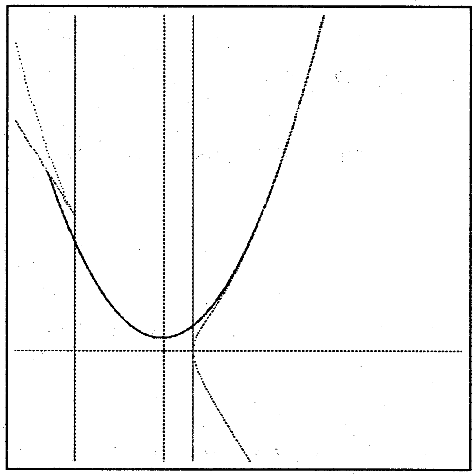

Figure 7: Parabolic curve in the moduli space. In the case of $a = -0.2$, $\quad -10 \leq \sigma_1 \leq 30$, $-10 \leq \sigma_2 \leq 40$

We obtain next results for a fixed parameter a.

Proposition 2 *On $\mathcal{M}_2(\mathbf{R})$, one parameter family $\{f_{a,r}(x)\}_r$ becomes the parabola*

$$\sigma_2 = \frac{1}{4}\left(\sigma_1 - (8a + 4 + \frac{1}{2a})\right)^2 - \frac{1}{a^2}\left(16a^3 + 3a^2 + \frac{a}{2} + \frac{1}{16}\right) \tag{7}$$

with the turn

$$\left(\frac{1}{a}(-4a^2 + 2a - 1), \frac{1}{a}(4a^3 - 4a^2 + 5a - 2)\right).$$

Any point excepts the turn on this parabola, corresponds with two maps $f_{a,r}, f_{a,-r}$.
The set of the turns parameterized by a coincides with the cubic algebraic curve of (2).

Proof.

Same as the proof of proposition 1, a, r varied function σ_1, σ_2 is obtained as follows,

$$\begin{cases} \sigma_1 = -\frac{1}{a}(4r^2 - 4a^2 + 2a - 1) \\ \sigma_2 = -\frac{1}{a^2}\{4a^4 + (8a^2 + 12a - 1)r^2 + 4a^4 - 4a^3 + 5a^2 - 2a\} \end{cases} \tag{8}$$

Removing r from previous two equations, we have

$$\sigma_2 = \frac{1}{4}\sigma_1^2 + (-4a - 2 - \frac{1}{4a})\sigma_1 + 16a^2 + 3 + \frac{1}{2a}. \tag{9}$$

For any a, a family $\{f_r(x)\}_r$ forms parabola on the moduli space. Since $r^2 \geq 0$, this parabola has the turn $(r = 0)$,

$$(\sigma_1, \sigma_2) = \left(\frac{1}{a}(-4a^2 + 2a - 1), \frac{1}{a^2}(4a^4 - 4a^3 + 5a^2 - 2a)\right)$$

It is clear that trace of turns parameterized a satisfies

$$\begin{cases} \sigma_1 = 4a - 2 + \frac{1}{a} \\ \sigma_2 = 4a^2 - 4a + 5 - \frac{2}{a} \end{cases}$$

Expressing a as a function of σ_1, and substituting second equation, we have the equation (2).

∎

Hence, we can take the principal part, $r \geq 0$ of the bifurcation diagram, again.
For $a = -0.2$, a bifurcation with two types occur in the principal part $r \geq 0$ in Figure 6.

Proposition 3 *The trace of vertices of parabola family in proposition 2 is*

$$G(\sigma_1, \sigma_2) = 2\sigma_1^3 + (\sigma_2 - 13)\sigma_1^2 + (4\sigma_2 + 24)\sigma_1 + 4\sigma_2^2 - 16\sigma_2 + 16 = 0. \tag{10}$$

Moreover, there is following relation between G and F,

$$G(2 - \sigma_1, 2 - \sigma_2) = F(\sigma_1, \sigma_2).$$

Proof. Here we take leave out of consideration to existence of the turn. From proposition 2, vertices of parabola family satisfies

$$\begin{cases} \sigma_1 = 8a + 4 + \frac{1}{2a} \\ \sigma_2 = -\frac{1}{a^2}(16a^3 + 3a^2 + \frac{1}{2}a + 16) \end{cases}$$

Removing the parameter a from previous equations, we have (10). It is clear from calculation that $G(2 - \sigma_1, 2 - \sigma_2) = F(\sigma_1, \sigma_2)$.

∎

Proposition 4 *$\mathcal{M}_2(\mathbf{R})$ is the disjoint union of the region of Nusse-York's family $\{f_{a,r}(x)\}$ and of the following set*

$$\{(\sigma_1,\sigma_2)\in\mathcal{M}_2(\mathbf{R})\,|F(\sigma_1,\sigma_2)<0\}\cup\{(\sigma_1,\sigma_2)\in\mathcal{M}_2(\mathbf{R})\,|\sigma_1=2,\sigma_2<-1\}.$$

Proof. Since for any a, r, the map $f_{a,r}$ always has critical points ± 1, $f_{a,r}$ cannot exists on $\{\pm\deg 2\}\subset\mathcal{M}_2(\mathbf{R})$.

From the parabolic curve (7)

$$16a^2-4a\sigma_1+\left(\frac{1}{4}\sigma_1^2-2\sigma_1+3-\sigma_2\right)+\left(\frac{1}{2}-\frac{\sigma_1}{4}\right)\frac{1}{a}=0, \qquad (11)$$

multiplying by a, ($a\neq 0$),

$$16a^3-4\sigma_1a^2+\left(\frac{1}{4}\sigma_1^2-2\sigma_1+3-\sigma_2\right)a+\frac{1}{2}-\frac{\sigma_1}{4}=0. \qquad (12)$$

For $(\sigma_1,\sigma_2)\in\mathbf{R}^2$, there certainly exists real value a because (12) is cubic equation with real coefficients. But, $a=0$ is not allowed. Therefore $(\sigma_1,\sigma_2)\in\mathbf{R}^2$ which gives $a=0$ are exceptional points on $\mathcal{M}_2(\mathbf{R})$.

Calculating the constant term of (12) equal to zero, we have $\sigma_1=2$. Substituting (11)

$$a(16a^2-8a-\sigma_2)=0.$$

Since real solution of this equation is only $a=0$,

$$\frac{D}{4}=16+16\sigma_2<0.$$

Hence we have $\sigma_2<-1$.

∎

Remark 2 The set $\{(\sigma_1,\sigma_2)\in\mathcal{M}_2(\mathbf{R})|\sigma_1=2,\sigma_2<-1\}$ corresponds with a part of the family of quadratic polynomials $\{x^2+c|c<-\frac{1}{4}\}$.

This proposition shows that 2-parameter family $\{f_{m,r}\}$ covers all intrinsic part of the quadratic rational maps.

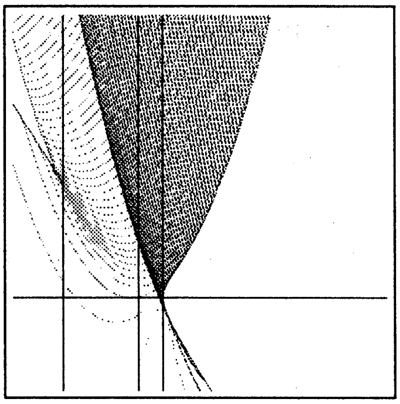

Figure 8: Moduli space with parabolic curves: $-10 \leq \sigma_1 \leq 20$, $-10 \leq \sigma_2 \leq 30$. The right white region is the degree +2 region, and the wedged white region in the left is the degree −2 region.

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