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Quantum Jacobi Trudi formula and analytic Bethe ansatz

東大教養物理 国場敦夫 (Atsuo Kuniba)

1. Introduction

The main message of this note, which is based on the works [KS1,KS2,KOS] with Y.Ohta and J.Suzuki, is, 'Analytic Bethe ansatz is a character theory of finite dimensional representations of quantum affine algebras'. Analytic Bethe ansatz originates in solvable lattice models in statistical mechanics [B1]. It is a hypothetical prescription to produce an eigenvalue formula for row-to-row transfer matrices of the models. As for its validity, no general proof is known neither any counter example. It was invented by Reshetikhin in [R] by extracting the idea from Baxter's solution of the 8-vertex model [B2]. Let us explain it with a simplest example from $sl(2)$.

Consider the 6-vertex model on a square lattice [B1] with the Boltzmann weights $R_u(\pm,\pm,\pm,\pm) = [2+u]$, $R_u(\pm,\mp,\pm,\mp) = [u]$ and $R_u(\pm,\mp,\mp,\pm) = [2]$, where the local states $+$ or $-$ are ordered anti-clockwise from the left edge of the vertex. The function $[u]$ is defined by

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}. \tag{1.1}$$

Here u is a spectral parameter and q is a generic constant (not a root of unity). The Boltzmann weights can be arranged in an R-matrix $R_{W_1,W_1}(u)$ satisfying the Yang-Baxter equation and the model is solvable. Here the indices indicate that it is an intertwiner of the tensor product of the 2-dimensional $U_q(A_1^{(1)})$ module W_1. (We let W_m denote the $m+1$ dimensional irreducible one.) The row-to-row transfer matrix of the 6-vertex model is the $m=1$ case of the following more general matrix

$$T_m(u) = \mathrm{Tr}_{W_m}\big(R_{W_m,W_1}(u-w_1)\cdots R_{W_m,W_1}(u-w_N)\big). \tag{1.2}$$

Here N is the system size, $w_1,\ldots,w_N$ are complex parameters representing the inhomogeneity of local interactions, $m \in \mathbf{Z}_{\geq 0}$. Following the QISM terminology [QISM], we say that (1.2) is the row-to-row transfer matrix with the *auxiliary space* W_m that acts on the *quantum space* $W_1^{\otimes N}$. (More precisely, $W_m(u)$ and $\otimes_{j=1}^N W_1(w_j)$, respectively.) Due to the

Yang-Baxter equation, $[T_m(u), T_{m'}(u')] = 0$ holds. So they can be simultaneously diagonalized and we shall often write their eigenvalues also by the same symbol $T_m(u)$. One of the main subject in statistical mechanics is to study the spectrum of $T_m(u)$ (especially in the limit $N \to \infty$). Let us quote an eigenvalue formula for $T_1(u)$ [B1];

$$T_1(u) = \frac{Q(u-1)}{Q(u+1)}\phi(u+2) + \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.3a}$$

$$Q(u) = \prod_{j=1}^{n}[u - v_j], \quad \phi(u) = \prod_{j=1}^{N}[u - w_j]. \tag{1.3b}$$

Here, $0 \leq n \leq N/2$ is the number of the $-$ states in the eigenvector, which is preserved under the action of $T_1(u)$. $u_j \in \mathbf{C}$ are any solution of the Bethe ansatz equation (BAE)

$$-\frac{\phi(v_k+1)}{\phi(v_k-1)} = \frac{Q(v_k+2)}{Q(v_k-2)}. \tag{1.4}$$

On the result (1.3-4), one makes a few observations.

(i) The eigenvalue has the "dressed vacuum form (DVF)", which means the following. The "vacuum vector" $+, +, \ldots, +$ is the obvious eigenvector with the vacuum eigenvalue

$$\prod_{j=1}^{N} R_{u-w_j}(+,+,+,+) + \prod_{j=1}^{N} R_{u-w_j}(-,+,-,+) = \phi(u+2) + \phi(u). \tag{1.5}$$

Eq.(1.3) tells that general eigenvalues can still be expressed with the modifying "dress" factors Q/Q which is certainly 1 when $n = 0$. In particular, the number of the terms in $T_1(u)$ is the dimension of the auxiliary space $\dim W_1 = 2$.

(ii) The BAE (1.4) ensures that the eigenvalues are free of poles for finite u. The apparent pole at $u = v_k - 1$ in (1.3a) is spurious as the residues from the two terms cancel due to (1.4). The eigenvalues must actually be pole-free because the local Boltzmann weight, hence the matrix elements of $T_1(u)$ are so.

(iii) Properties inherited from the asymptotic behavior in $|u| \to \infty$ and the first/second inversion relations of the R-matrix (vertex Boltzmann weights). For example, one has

$$(q - q^{-1})^N q^{-N+\sum_j w_j} \lim_{q^u \to \infty} q^{-Nu} T_1(u) = q^{N-2n} + q^{2n-N}, \tag{1.6}$$

which is certainly an expected result from the definition of $T_1(u)$.

The analytic Bethe ansatz is the hypothesis that the postulates (i)-(iii) essentially determine a function of u uniquely and that the so obtained is the actual transfer matrix eigenvalue. As the input data, it only uses the BAE and the special components of the R-matrix (or the vacuum eigenvalue (1.5)) which should be normalized to be an entire function of u. Its validity can only be assured in general by a proper diagonalization, most notably, by the algebraic Bethe ansatz which yields the eigenvectors as well.

In (1.6), one notices already that the RHS is an $sl(2)$ character of the 2-dimensional representation space W_1. Thus $T_1(u)$ is a u-dependent version of it. This view point becomes even more natural if one considers the eigenvalues for general $T_m(u)$ and observes the following functional relations that generalize the usual character identities.

$$\begin{aligned} T_m(u+1)T_m(u-1) &= T_{m+1}(u)T_{m-1}(u) + g_m(u)\mathrm{Id}, \\ g_m(u) &= \prod_{k=0}^{m-1} \phi(u+2k-m)\phi(u+4+2k-m). \end{aligned} \tag{1.7}$$

where $m \geq 0$. Regarding (1.7) as an equation for the eigenvalues one can easily solve it under the initial condition (1.3a) and $T_0(u) = 1$ to find

$$T_m(u) = \Bigl(\prod_{k=1}^{m-1} \phi(u+m+1-2k)\Bigr) \sum_{j=0}^{m} \frac{Q(u-m)Q(u+m+2)\phi(u+m+1-2j)}{Q(u+m-2j)Q(u+m+2-2j)}. \tag{1.8}$$

To observe a representation theoretical content, we now set

$$\boxed{1} = \frac{Q(u-1)}{Q(u+1)}\phi(u+2), \quad \boxed{2} = \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.9}$$

where we assume on the LHS that the spectral parameter u is implicitly attached to the single box as well. In this notation (1.3a) reads as $\Lambda_1(u) = \boxed{1} + \boxed{2}$. Moreover, the result (1.8) for general m can be expressed as follows.

$$T_m(u) = \sum_{j=0}^{m} \overbrace{\boxed{1}\,\cdots\,\boxed{1}}^{m-j}\overbrace{\boxed{2}\,\cdots\,\boxed{2}}^{j}. \tag{1.10}$$

Here we interpret the tableau as the product of the m functions (1.9) with the spectral parameter u shifted to $u-m+1, u-m+3, \ldots, u+m-1$ from the left to the right. Notice that the tableaux appearing in (1.10) are exactly the semi-standard ones that label the weight vectors in the $(m+1)$-dimensional irreducible representation W_m of $U_q(\hat{sl}(2))$

(plainly, the spin $\frac{m}{2}$ representation of $sl(2)$). In this sense $T_m(u)$ is an analogue ("Yang-Baxterizations") of the character of the auxiliary space W_m, which may be natural from (1.2). The functional relation (1.7) thereby plays the role of a character identity. Under the BAE (1.4), is $T_m(u)$ pole-free for general $m \geq 1$? This is a crucial check for (1.10) be a correct DVF. To answer it, solve (1.7) keeping $T_1(u)$. The result reads

$$T_m(u) = det \begin{pmatrix} T_1(u-m+1) & g_1(u-m+2) & 0 & \cdots & & 0 \\ 1 & T_1(u-m+3) & & & & \\ 0 & 1 & \ddots & & & 0 \\ \vdots & \vdots & & & T_1(u+m-3) & g_1(u+m-2) \\ 0 & 0 & \cdots & & 1 & T_1(u+m-1) \end{pmatrix}. \tag{1.11}$$

which expresses T_m in terms of the fundamental T_1. Obviously, this reduces to a Jacobi Trudi formula [M] for Schur functions if the u-dependence is absent (or in the limit $u \to \infty$). In this sense (1.11) may be called a quantum Jacobi Trudi formula. It manifestly tells that $T_m(u)$ is pole-free, which is by no means so obvious from the expression (1.10). One can also check the character limit $(q-q^{-1})^{mN} q^{-mN+m\sum_j w_j} \lim_{q^u \to \infty} q^{-mNu} T_1(u) = \sum_{j=0}^{m} q^{(N-2n)(m-2j)}$.

To summarize so far, the functional relation (1.7), the tableau representation (1.10) and the quantum Jacobi Trudi formula (1.11) are typical features in transfer matrices and analytic Bethe ansatz in solvable lattice models.

2. Bethe ansatz equation

Having seen the $sl(2)$ example, a natural question is a generalization to other algebras. For simplicity, we shall consider vertex models associated with the Yangian $Y(X_r)$ for $X_r = A_r, B_r, C_r, D_r, E_{6,7,8}, F_4$ and G_2. Let $W^{(i)}, 1 \leq i \leq N$ be a finite dimensional irreducible $Y(X_r)$ module and $P_a^{(i)}(\zeta)$, $1 \leq a \leq r$ be the characterizing Drinfel'd polynomials [D]. The BAE relevant to the transfer matrices acting on the quantum space $\otimes_{i=1}^N W^{(i)}$ has been conjectured as follows.

$$-\prod_{i=1}^{N} \frac{P_a^{(i)}(v_k^{(a)} + \frac{(\alpha_a|\alpha_a)}{2})}{P_a^{(i)}(v_k^{(a)} - \frac{(\alpha_a|\alpha_a)}{2})} = \prod_{b=1}^{r} \frac{Q_b(v_k^{(a)} + (\alpha_a|\alpha_b))}{Q_b(v_k^{(a)} - (\alpha_a|\alpha_b))} \quad 1 \leq a \leq r, 1 \leq k \leq N_a. \tag{2.1}$$

Here α_a's are the simple roots (normalization $|$ long root $|^2 = 2$), $Q_a(u) = \prod_{j=1}^{N_a}[u - v_j^{(a)}]$ and we understand that $q \to 1$ in (1.1). (On the other hand, for generic q, we suppose that (2.1) is valid if $P_a^{(i)}(\zeta)$ is replaced by a natural q-analogue.) The RHS of the conjecture (2.1) is due to [RW] and the LHS is due to [KOS] and [ST]. It has been formulated purely from the representation theoretical data, the root system and the Drinfel'd polynomial. As for the functional relations, an analogue of (1.7), called T-system, has been proposed for arbitrary X_r in [KNS].

In the rest of the paper we shall also consider the case $X_r = A_r$ exclusively. See [KOS] for B_r case and [KS2] for the twisted quantum affine algebra case. For simplicity, we shall further concentrate on the case where the quantum space is formally trivial ($N = 0$ or $\forall P_a^{(i)} = 1$) and set the LHS of the BAE (2.1) to -1. This corresponds to considering the dress part only, which does not lose the essential features. To recover the vacuum part for a given LHS is easy. In the next section, we shall introduce a wide class of the DVFs $T_{\lambda \subset \mu}(u)$ associated with any skew Young diagrams $\lambda \subset \mu$. According to the analytic Bethe ansatz, it is natural to expect that the $T_{\lambda \subset \mu}(u)$ is the eigenvalue formula for a certain transfer matrix whose auxiliary space is labelled by $\lambda \subset \mu$ and u. Denoting it by $W_{\lambda \subset \mu}(u)$, one should be able to characterize it completely as an irreducible finite dimensional module over $Y(A_r)$. As is well known, this can be done by specifying the associated Drinfel'd polynomial. In section 4, we shall explain our empirical prescription to extract the Drinfel'd polynomial from a given DVF. This is yet hypothetical but works for all the known examples. We will actually apply it to our $T_{\lambda \subset \mu}(u)$ and give the conjectural Drinfel'd polynomial.

3. Construction of the DVF $T_{\lambda\subset\mu}(u)$

Put

$$J = \{1, 2, \ldots, r+1\}. \tag{3.1}$$

For $a \in J$, define the function

$$\boxed{a}_u = \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)}, \tag{3.2}$$

where we have set $Q_0(u) = Q_{r+1}(u) = 1$. We shall often suppress the argument u.

Let $\mu = (\mu_1, \mu_2, \ldots)$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ be a Young diagram and $\mu' = (\mu'_1, \mu'_2, \ldots)$ be its transpose. By a skew-Young diagram we mean a pair of Young diagrams $\lambda \subset \mu$. It is depicted by the region corresponding to the subtraction $\mu - \lambda$. For definiteness, we assume that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$. A Young diagram μ is naturally identified with a skew-Young diagram $\phi \subset \mu$. By a semi standard tableau b on a skew-Young diagram $\lambda \subset \mu$ we mean an assignment of an element $b(i,j) \in J$ to the (i,j)-th box in $\lambda \subset \mu$ under the following rule: (We locate $(1,1)$ at the top left corner of μ, $(i+1,j)$ and $(i,j+1)$ to the below and the right of (i,j), respectively.)

$$b(i,j) \leq b(i,j+1), \qquad b(i,j) < b(i+1,j). \tag{3.3}$$

Denote by $SST(\lambda \subset \mu)$ the set of semi standard tableaux on $\lambda \subset \mu$.

Given a skew-Young diagram $\lambda \subset \mu$, we define a function $T_{\lambda\subset\mu}(u)$ as the following sum over the semi standard tableaux.

$$T_{\lambda\subset\mu}(u) = \sum_{b\in SST(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} \boxed{b(i,j)}_{u+\mu'_1-\mu_1-2i+2j}. \tag{3.4}$$

This actually gives 0 unless $\mu'_i - \lambda'_i \leq r+1$ for all i since $SST(\lambda \subset \mu) = \phi$ otherwise. In the limit $u \to \infty$, $T_{\lambda\subset\mu}(u)$ is just the skew Schur function $S_{\mu/\lambda}(x_1 = q^{-2N_1}, x_2 = q^{2N_1-2N_2}, \ldots, x_r = q^{2N_{r-1}-2N_r}, x_{r+1} = q^{2N_r})$ [M]. $T_m(u)$ in (1.10) corresponds to the case A_1 and $\lambda = \phi, \mu = (m)$. For later convenience we introduce the notation

$$\begin{aligned} e_k(u) &= T_{(1^k)}(u), \\ h_k(u) &= T_{(k)}(u). \end{aligned} \tag{3.5}$$

By the definition, $e_k(u)$ and $h_k(u)$ are non-zero only for $0 \leq k \leq r+1$ and $k \geq 0$, respectively.

Now we proceed to the pole-freeness of the DVFs introduced above.

Proposition. *For any $k \in \mathbf{Z}$, $e_k(u)$ is pole-free under the BAE (2.1) (LHS set to -1).*

This can be proved as in [KS1]. Namely, for each $1 \le a \le r$, one just has to keep track of the "color a poles" $(\cdots)/Q_a(u+\cdots)$, hence the appearance of the boxes $\boxed{a}$ and $\boxed{a+1}$.

Theorem (Quantum Jacobi Trudi formula).

$$T_{\lambda\subset\mu}(u) = det_{1\le i,j\le\mu_1}(e_{\mu'_i-\lambda'_j-i+j}(u+\mu'_1-\mu_1-\mu'_i-\lambda'_j+i+j-1)). \qquad (3.6a)$$

$$= det_{1\le i,j\le\mu'_1}(h_{\mu_j-\lambda_i+i-j}(u+\mu'_1-\mu_1+\mu_j+\lambda_i-i-j+1)). \qquad (3.6b)$$

Eq.(3.6a) can be verified, for example, by induction on μ_1, i.e., by showing the same recursive relation for the tableau sum (3.4) as an expansion of the determinant. Then (3.6b) can be derived by a similar argument to [M]. Obviously, (3.6) is a quantum ($Y(A_r)$ or $U_q(A_r^{(1)})$) analogue of the classical Jacobi Trudi formula [M]. For the usual Young diagram case $\lambda = \phi \subset \mu$, it first appeared in [BR]. A representation theoretical account in terms of resolutions is available in [C]. From Proposition and Theorem, one has

Corollary. *$T_{\lambda\subset\mu}(u)$ is pole-free provided the BAE (2.1) (LHS set to -1) holds.*

Combining this with $\lim_{u\to\infty} T_{\lambda\subset\mu}(u) = \sharp SST(\lambda\subset\mu)$ ($\lim_{q^u\to\infty} T_{\lambda\subset\mu}(u) = S_{\mu/\lambda}$ in $U_q(A_r^{(1)})$ case), we see that $T_{\lambda\subset\mu}(u)$ is in fact a constant independent of u. This is a rather special feature owing to the fact that vacuum part is taken trivially. In general, $T_{\lambda\subset\mu}(u)$ is a polynomial in u (Laurent polynomial in q^u in $U_q(A_r^{(1)})$ case). By using Sylvester's theorem, one can further rewrite (3.6) into a determiant involving T_{hook} Young diagram as well. The result can be viewed as a quantum analogue of the Giambelli formula. See theorem 3.1 in [KOS] for the B_r case.

4. Drinfel'd polynomials

The analytic Bethe ansatz indicates that $T_{\lambda\subset\mu}(u)$ (3.4) describes the spectrum of the transfer matrix whose auxiliary space is labeled by the skew-Young diagram $\lambda \subset \mu$ and u. Denote the space by $W_{\lambda\subset\mu}(u)$. We suppose it is an irreducible finite dimensional module over $Y(B_r)$ (or $U_q(B_r^{(1)})$ in the trigonometric case) in view that all the terms in (3.4) seem coupling to make the apparent poles spurious under BAE. Now we shall specify the Drinfel'd polynomial $P_a(\zeta)$ [D] that characterizes $W_{\lambda\subset\mu}(u)$ based on some empirical procedure. Our convention slightly differs from the original one in Theorem 2 of [D] in such a way that

$$1 + \sum_{k=0}^{\infty} d_{ik}\zeta^{-k-1} = \frac{P_i(\zeta+1)}{P_i(\zeta-1)}. \tag{4.1}$$

For any $b \in SST(\lambda \subset \mu)$, the corresponding summand (3.4) has the form

$$\prod_{a=1}^{r} \frac{Q_a(u+x_1^a)\cdots Q_a(u+x_{i_a}^a)}{Q_a(u+y_1^a)\cdots Q_a(u+y_{i_a}^a)}. \tag{4.2}$$

where x_j^a, y_j^a and i_a are specified from b. This summand carries the A_r-weight

$$wt(b) = \sum_{a=1}^{r}\Bigl(\frac{1}{2}\sum_{j=1}^{i_a}(y_j^a - x_j^a)\Bigr)\Lambda_a \tag{4.3}$$

in the sense that $\lim_{q^u\to\infty}(4.2) = q^{-2(wt(b)|\sum_{a=1}^{r} N_a\alpha_a)}$. ($\Lambda_a$: a-th fundamental weight.) From $SST(\lambda \subset \mu)$, take such b_0 that $wt(b_0)$ is highest with respect to the root system. In our case, such b_0 is unique and given by

$$b_0(i,j) = i - \lambda'_j, \quad 1 \le j \le \mu_1,\ \lambda'_j + 1 \le i \le \mu'_j. \tag{4.4}$$

It turns out that the corresponding 'highest' term b_0 in (4.2) can be expressed uniquely in the form

$$\prod_{a=1}^{r}\prod_{j=1}^{M_a} \frac{Q_a(u+z_j^a-1)}{Q_a(u+z_j^a+1)} \tag{4.5}$$

for some M_a and $\{z_j^a | 1 \le j \le M_a\}$ up to the permutations of z_j^a's for each a. We then propose that the Drinfel'd polynomial $P_a^{W_{\lambda\subset\mu}(u)}(\zeta)$ for $W_{\lambda\subset\mu}(u)$ is given by

$$P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = \prod_{j=1}^{M_a}(\zeta - u - z_j^a) \quad 1 \le a \le r. \tag{4.6}$$

In our case, it reads explicitly as follows.

$$P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = \prod_{\substack{1\le j\le \mu_1 \\ \mu'_j-\lambda'_j=a}} (\zeta - u - \mu'_1 + \mu_1 + 1 + a + 2\lambda'_j - 2j). \tag{4.7}$$

For example, in the case of the rectangular Young diagram $\lambda = \phi, \mu = (m^b)$, (4.7) reads

$$P_a^{W_{(m^b)}(u)}(\zeta) = \big((\zeta - u + m - 1)(\zeta - u + m - 3)\cdots(\zeta - u - m + 1)\big)^{\delta_{ab}}. \tag{4.8}$$

Thus the modules $W_{(1^b)}(u)$ are the fundamental representations in the sense of [CP].

References

[B1] R.J. Baxter, Exactly Solved Models in Statistical Mechanics. London: Academic 1982

[B2] R.J. Baxter, Ann. Phys. **70** (1972) 193

[BR] V.V. Bazhanov, N.Yu. Reshetikhin, J.Phys.A:Math.Gen. **23** (1990) 1477

[C] I. Cherednik, in Proc. of the XVII International Conference on Differential Geometric Methods in Theoretical Physics, Chester, ed. A.I. Solomon, World Scientific, Singapore, 1989

[CP] V. Chari and A. Pressley, J. reine angew.Math. **417** (1991) 87

[D] V.G. Drinfel'd, Sov.Math.Dokl.**36** (1988) 212

[KNS] A. Kuniba, T. Nakanishi and J. Suzuki, Int.J.Mod.Phys. **A9** (1994) 5215

[KOS] A. Kuniba, Y.Ohta and J. Suzuki, J. Phys. A:Math.Gen. **28** (1995) 6211

[KS1] A. Kuniba and J. Suzuki, Commun. Math. Phys. **173** (1995) 225

[KS2] A. Kuniba and J. Suzuki, J. Phys. A:Math.Gen. **28** (1995) 711

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995

[QISM] E.K. Sklyanin, L.A. Takhtajan and L.D. Faddeev, Theor.Math.Phys.**40** (1980) 688
P.P. Kulish and E.K. Sklyanin, Lect.Note.Phys. **151** 61, Springer 1982

[R] N.Yu. Reshetikhin, Sov.Phys.JETP **57** (1983) 691, Theor.Math.Phys.**63** (1985) 555, Lett.Math.Phys.**14** (1987) 235

[RW] N.Yu. Reshetikhin and P.B. Wiegmann, Phys.Lett.B**189** (1987) 125

[ST] E.K. Sklyanin and V.O. Tarasov, private communication