数理解析研究所講究録
963 巻 1996 年 124-141

On Riemann's Period matrix of $Y^2 = X^{2n+1} - 1$

YOSHIAKI TASHIRO* SEISHI YAMAZAKI† MINORU ITO‡
田代俶章（東京農工大） 山崎晴司（山梨大・教育） 伊藤実（横浜国大・教育）
TEIICHI HIGUCHI§
樋口禎一（横浜国大・教育）

Abstract

We are deeply interested in the theory of Abelian integrals which have an ample data concerning the moduli space of the Riemann surfaces. The theta-function is a clue to connect the defining equation to the moduli space and the properties of the theta-function are well-known, but the concrete examples of such functions are not known so much.

On the Riemann's period matrix of the Riemann surface $y^2 = x^{2n+1} - 1$, the determinant of the above matrix of the general type n (see [7]) and the above matrix of the type $n = 3$ (see [8]) are done.

In this paper, we have the Riemann's period matrix of the Riemann surface $y^2 = x^{2n+1} - 1$.

1 Introduction

In §2, we have the well-known general theorems of Riemann's period matrix of hyper-elliptic functions.

In §3, we show the ways of getting the Riemann's period matrix of hyper-elliptic functions.

The first is to choose a base of the vector spaces of holomorphic 1-form on the Riemann surface X for $y^2 = x^{2n+1} - 1$. The second is to make a model of $y^2 = x^{2n+1} - 1$. As a

*Tokyo University of Agriculture and Technology
†Yamanashi University
‡Yokohama National University
§Yokohama National University

topological pace, X is a compact orientable 2-manifold, it's genus is n by Riemann-Roch theorem and it's branch points are $(2n+1)^{th}$ root of 1 and ∞. The third is to analyze the periods of homology (2n disjoint simple closed paths with all beginning and ending at the same base point).

Since the periods have to be decided so that Riemann's period relations hold, we determine five rules to get the correct periods from a polygon with $4n$ sides (one side each for the left and right sides of each path).

In §4, by using the above five rules and Cauchy integral formula, we get the Period matrix of the Riemann surface $y^2 = x^{2n+1} - 1$ by the values of the branch points.

2 General Theory of Hyper-Elliptic Function

2.1 Period Matrix and Quasi-Period Matrix

Let C be the following hyper-elliptic curve

$$C: \quad y^2 = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2n+1} x^{2n+1} \quad \lambda_i \in \mathbf{C} \quad (i = 0, 1, \cdots, 2n+1)$$

Since the genus of C is n, the vector space of holomorphic 1-forms is an n-vector space by Riemann-Roch theorem. In fact the following is such a base.

$$\omega_1 = \frac{1}{y}dx, \quad \omega_2 = \frac{x}{y}dx, \quad \omega_3 = \frac{x^2}{y}dx, \quad \cdots \quad , \omega_n = \frac{x^{n-1}}{y}dx$$

And also, homology group on the Riemann's surface for the hyperelliptic curve C has 2n generator $A_i, B_i \quad (i = 1, \cdots, n)$ which satisfy the following conditions.

1) $A_i \times A_j = B_i \times B_j = 0 \quad (i \neq j)$

2) $A_i \times B_j = \delta_{ij} \qquad (i, j = 1, \ldots, n)$

where $\times$ means intersection number. So we have a Riemann's period matrix $\Omega = (\Pi, \Pi')$

from the above.

Definition 2.1 (Riemann's period matrix) *For a base of holomorphic 1-forms $\omega_i (i = 1, \ldots, n)$ and a base of homology $A_i, B_i (i = 1, \ldots, n)$ on C. Riemann's period matrix Ω is given as follows,*

$$\textit{Period matrix:} \quad \Omega = (\Pi, \Pi')$$

$$\Pi = \begin{pmatrix} \pi_{11} & \dots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \dots & \pi_{nn} \end{pmatrix} \qquad \Pi' = \begin{pmatrix} \pi'_{11} & \dots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \dots & \pi'_{nn} \end{pmatrix}$$

$$\pi_{ij} = \int_{A_j} \omega_i \qquad \pi'_{ij} = \int_{B_i} \omega_j$$

Lemma 2.1 (Riemann's period relation 1) *Let X be a compact Riemann surface of genus n, with canonical dissection $X = X_0 \cup A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n$.*
For any holomorphic 1-form $\omega_i, \omega_j \quad (i, j = 1, \ldots, n)$, the periods satisfy the following equation.

$$\sum_{k=1}^{n} [\int_{A_k} \omega_i \int_{B_k} \omega_j - \int_{B_k} \omega_i \int_{A_k} \omega_j] = 0$$

Proof. Since X_0 is simply connected, there is a holomorphic function f on X_0 such that $\omega_i = df$ namely, $f(z) = \int_{z_0}^{z} \omega_i$ then $f\omega_i$ is a closed 1-form, so by Green's theorem

$$\begin{aligned} 0 &= \int_{X_0} d(f\omega_j) \\ &= \int_{\partial X_0} f\omega_j \\ &= \sum_{k=1}^{n} [-\int_{A_k^+} f\omega_j + \int_{A_k^-} f\omega_j - \int_{B_k^+} f\omega_j + \int_{B_k^-} f\omega_j] \\ &= \sum_{k=1}^{n} \int_{A_k} [(f \text{ on } A_k^-) - (f \text{ on } A_k^+)] \, \omega_j + \sum_{k=1}^{n} \int_{B_k} [(f \text{ on } B_k^-) - (f \text{ on } B_k^+)] \, \omega_j \end{aligned}$$

As df has no discontinuity on A_k or B_k, f on A_k^+ must differ from f on A_k^- by constant, and likewise for B_k+, B_k-. But the path A_k lead from A_k^- to A_k^+ and the path A_k lead from B_k^+ to B_k^-.
Thus

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \int_{A_k} [f(z^-) - f(z^+)] \, \omega_j + \sum_{k=1}^{n} \int_{B_k} [f(\zeta^-) - f(\zeta^+)] \, \omega_j \\ &= \sum_{k=1}^{n} \int_{A_k} (-\int_{z^-}^{z^+} df) \, \omega_j + \sum_{k=1}^{n} \int_{B_k} (-\int_{\zeta^-}^{\zeta^+} df) \, \omega_j \\ &= \sum_{k=1}^{n} \int_{A_k} (-\int_{z^-}^{t_2} \omega_i - \int_{t_4}^{t_3} \omega_i - \int_{t_3}^{z^+} \omega_i) \, \omega_j \\ &+ \sum_{k=1}^{n} \int_{B_i} (-\int_{\zeta^-}^{t_3} \omega_i - \int_{t_3}^{t_2} \omega_i - \int_{t_2}^{\zeta^+} \omega_i) \, \omega_j \end{aligned}$$

$$= \sum_{k=1}^{n} [\int_{A_k} (-\int_{B_k} \omega_i) \, \omega_j] + \sum_{k=1}^{n} [\int_{B_k} (+\int_{A_k} \omega_i) \, \omega_j]$$
$$= \sum_{k=1}^{n} [-(\int_{B_k} \omega_i)(\int_{A_k} \omega_j)] + \sum_{i=k}^{n} [(\int_{A_k} \omega_i)(\int_{B_k} \omega_j)]$$

which proves the lemma.

Lemma 2.2 (Riemann's period relation 2) *Let X be a compact Riemann surface of genus n, with canonical dissection $X = X_0 \cup A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_n$. For any holomorphic 1-forms ω_i $(i = 1, \ldots, n)$, the period satisfy the following equation*

$$i \sum_{k=1}^{n} [\int_{A_k} \omega_i \int_{B_k} \bar{\omega}_i - \int_{B_k} \omega_i \int_{A_k} \bar{\omega}_i] \; > 0$$

namely,

$$Im \sum_{k=1}^{n} (\int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i) > 0$$

Proof. Likewise proof of lemma 2.1.

$$\begin{aligned} -i \int_{X_0} d(\bar{f}\omega_i) &= -i \int_{\partial X_0} \bar{f}\omega_i \\ &= -\sum_{k=1}^{n} [-\int_{A_k^+} \bar{f}\omega_i + \int_{A_k^-} \bar{f}\omega_i - \int_{B_k^+} \bar{f}\omega_i + \int_{B_k^-} \bar{f}\omega_i] \\ &= -i \sum_{k=1}^{n} [\int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i - \int_{B_k} \bar{\omega}_i \int_{A_k} \omega_i] \end{aligned}$$

On the other hand, $d(\bar{f}\omega) = d\bar{f} \wedge df$. Wherever f is a local analytic coordinates, let $f = x + iy$ and x, y are real coordinates, then

$$d\bar{f} \wedge df = (dx - idy) \wedge (dx + idy) = 2idx \wedge dy$$

$$-i \sum_{k=1}^{n} [\int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i - \int_{B_k} \bar{\omega}_i \int_{A_k} \omega_i] = 2 \int_{X_0} dx \wedge dy > 0$$

So we have

$$Im \sum_{k=1}^{n} (\int_{A_k} \bar{\omega}_i \int_{B_k} \omega_i) > 0$$

which proves lemma 2.2. q.e.d.

Theorem 2.1 (Riemann) *For the Riemann's period matrix* $\Omega = (\Pi, \Pi')$ *of* C

$$\text{Modular matrix} \quad T = \Pi^{-1}\Pi' \quad \text{is symmetric matrix}$$

Proof. By lemma 2.1, for all $i, j \quad (i, j = 1, \cdots, n)$, we have

$$\sum_{k=1}^{n} [\ \pi_{ik}\pi'_{jk} - \pi'_{ik}\pi_{jk}\ ] = 0$$

$$\begin{pmatrix} \pi_{i1} & \cdots & \pi_{in} \end{pmatrix} \begin{pmatrix} \pi'_{j1} \\ \vdots \\ \pi'_{jn} \end{pmatrix} - \begin{pmatrix} \pi'_{i1} & \cdots & \pi'_{in} \end{pmatrix} \begin{pmatrix} \pi_{j1} \\ \vdots \\ \pi_{jn} \end{pmatrix} = 0$$

$$\begin{pmatrix} \pi_{i1} & \cdots & \pi_{in} \end{pmatrix}^t \begin{pmatrix} \pi'_{j1} & \cdots & \pi'_{jn} \end{pmatrix} - \begin{pmatrix} \pi'_{i1} & \cdots & \pi'_{in} \end{pmatrix}^t \begin{pmatrix} \pi_{j1} & \cdots & \pi_{jn} \end{pmatrix} = 0$$

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix}^t \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix} - \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix}^t \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} = 0$$

$$\Pi^t\Pi' = \Pi'^t\Pi \qquad\qquad (*)$$

Here, let $^*\omega_1, \cdots, {}^*\omega_n$ be another base of holomorphic 1-forms which differ from $\omega_1, \cdots, \omega_n$ such that

$$\begin{pmatrix} ^*\omega_1 \\ \vdots \\ ^*\omega_n \end{pmatrix} = \Lambda \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

then the period matrix $^*\Omega = ({}^*\Pi, {}^*\Pi')$ for a base $^*\omega_1, \cdots, {}^*\omega_n$ is the following:

$$^*\pi_{ij} = \int_{A_j} {}^*\omega_i = \int_{A_j} \sum_{l=1}^{n} \lambda_{il}\omega_l = \sum_{l=1}^{n} \lambda_{il} \int_{A_j} \omega_l = \sum_{l=1}^{n} \lambda_{il}\pi_{lj}$$

$$^*\pi'_{ij} = \int_{B_j} {}^*\omega_i = \int_{B_j} \sum_{l=1}^{n} \lambda_{il}\omega_l = \sum_{l=1}^{n} \lambda_{il} \int_{B_j} \omega_l = \sum_{l=1}^{n} \lambda_{il}\pi'_{lj}$$

$$^*\pi_{ij} = \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{in} \end{pmatrix} \begin{pmatrix} \pi_{1j} \\ \vdots \\ \pi_{nj} \end{pmatrix} \qquad {}^*\pi'_{ij} = \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{in} \end{pmatrix} \begin{pmatrix} \pi'_{1j} \\ \vdots \\ \pi'_{nj} \end{pmatrix}$$

$$\begin{pmatrix} {}^*\pi_{11} & \cdots & {}^*\pi_{1n} \\ \vdots & & \vdots \\ {}^*\pi_{n1} & \cdots & {}^*\pi_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} \quad : \quad {}^*\Pi = \Lambda\Pi$$

$$\begin{pmatrix} {}^*\pi'_{11} & \cdots & {}^*\pi'_{1n} \\ \vdots & & \vdots \\ {}^*\pi'_{n1} & \cdots & {}^*\pi'_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \pi'_{11} & \cdots & \pi'_{1n} \\ \vdots & & \vdots \\ \pi'_{n1} & \cdots & \pi'_{nn} \end{pmatrix} \quad ; \quad {}^*\Pi' = \Lambda\Pi'$$

$${}^*\Omega = ({}^*\Pi, {}^*\Pi') = (\Lambda\Pi, \Lambda\Pi') = \Lambda(\Pi, \Pi') = \Lambda\Omega$$

If $\Lambda = \Pi^{-1} \quad (det\Pi \neq 0)$, then

$${}^*\Omega = ({}^*\Pi, {}^*\Pi') = (I_n, \Pi^{-1}\Pi') = (I_n, T)$$

Using equation $(*)$ in lemma 2.2 for ${}^*\Omega$

$${}^*\Pi^t({}^*\Pi') = {}^*\Pi'^t({}^*\Pi)$$

By ${}^*\Pi = I_n$ (unit matrix) and ${}^*\Pi' = T$

$${}^tT = T$$

Theorem 2.1 means that there is a suitable base of holomorphic 1-forms so that A-period matrix Π is a unit matrix I_n and B-period matrix Π' is a symmetric matrix T.
Thus, $T = \Pi^{-1}\Pi$ is a symmetric matrix. q.e.d.

Theorem 2.2 (Riemann) *For the Riemann's period matrix $\Omega = (\Pi, \Pi')$ of hyper-elliptic curve C*

$$ImT = Im(\Pi^{-1}\Pi') \quad \textit{is a real symmetric matrix of positive definite}$$

Proof.

$$-i\overline{\Omega}J^t\Omega = -i\begin{pmatrix} \overline{\Pi} & \overline{\Pi'} \end{pmatrix} \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \begin{pmatrix} {}^t\Pi \\ {}^t\Pi' \end{pmatrix} = i(\overline{\Pi'}{}^t\Pi - \overline{\Pi}{}^t\Pi')$$

From the above equation, $-i\overline{\Omega}J^t\Omega$ is Hermite matrix. Making a following Hermite form for this matrix,

$$
\begin{aligned}
\vec{\lambda}(-i\bar{\Omega}J^t\Omega)^t\vec{\lambda} &= i(\bar{\lambda}_1\cdots\bar{\lambda}_n)(\overline{\Pi'}^t\Pi - \overline{\Pi}^t\Pi')^t(\lambda_1\cdots\lambda_n) \\
&= i\begin{pmatrix}\bar{\lambda}_1 & \cdots & \bar{\lambda}_n\end{pmatrix}\begin{pmatrix}\bar{\pi}'_{11} & \cdots & \bar{\pi}'_{1n}\\ \vdots & & \vdots \\ \bar{\pi}'_{ni} & \cdots & \bar{\pi}'_{nn}\end{pmatrix}\begin{pmatrix}\pi_{11} & \cdots & \pi_{1n}\\ \vdots & & \vdots \\ \pi_{ni} & \cdots & \pi_{nn}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} \\
&\quad -i\begin{pmatrix}\bar{\lambda}_1 & \cdots & \bar{\lambda}_n\end{pmatrix}\begin{pmatrix}\bar{\pi}_{11} & \cdots & \bar{\pi}_{1n}\\ \vdots & & \vdots \\ \bar{\pi}_{ni} & \cdots & \bar{\pi}_{nn}\end{pmatrix}\begin{pmatrix}\pi'_{11} & \cdots & \pi'_{1n}\\ \vdots & & \vdots \\ \pi'_{ni} & \cdots & \pi'_{nn}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} \\
&= i\sum_{k=1}^{n}\left[(\sum_{i=1}^{n}\lambda_i\pi_{ik})(\sum_{j=1}^{n}\overline{\lambda_j\pi'_{jk}}) - (\sum_{i=1}^{n}\lambda_i\pi'_{ik})(\sum_{j=1}^{n}\overline{\lambda_j\pi_{jk}})\right] \\
&= i\sum_{k=1}^{n}\left[(\sum_{i=1}^{n}\lambda_i\int_{A_k}\omega_i)(\sum_{j=1}^{n}\bar{\lambda}_j\int_{B_k}\bar{\omega}_j) - (\sum_{i=1}^{n}\lambda_i\int_{B_k}\omega_i)(\sum_{j=1}^{n}\bar{\lambda}_j\int_{A_k}\bar{\omega}_j)\right] \\
&= i\sum_{k=1}^{n}\left[\int_{A_k}(\sum_{i=1}^{n}\lambda_i\omega_i)\int_{B_k}\overline{(\sum_{i=1}^{n}\lambda_i\omega_i)} - \int_{B_k}(\sum_{i=1}^{n}\lambda_i\omega_i)\int_{A_k}\overline{(\sum_{i=1}^{n}\lambda_i\omega_i)}\right]
\end{aligned}
$$

Let $^*\omega_i, {}^*\bar{\omega}_i$ be $(\sum_{i=1}^{n}\lambda_i\omega_i)$, $\overline{(\sum_{i=1}^{n}\lambda_i\omega_i)}$ by lemma 2.2.

$$
\vec{\lambda}(-i\bar{\Omega}J^t\Omega)^t\vec{\lambda} = i\sum_{i=1}^{n}[\ \int_{A_k}{}^*\omega_i\int_{B_k}{}^*\bar{\omega}_i\int_{B_k}{}^*\omega_i\int_{A_k}{}^*\bar{\omega}_i\ ] > 0
$$

Likewise theorem2.1, if we choose a suitable base of holomorphic 1-forms, such that the period matrix $\Omega = (\Pi, \Pi')$ is (I_n, T). Thus we have

$$
-i\begin{pmatrix} I_n & \bar{T}\end{pmatrix}\begin{pmatrix} O & I_n \\ -I_n & O\end{pmatrix}\begin{pmatrix} I_n \\ {}^tT\end{pmatrix} = i(\bar{T} - T) = 2ImT > 0
$$

which proves lemma2.2 q.e.d.

3 Period Matrix of $y^2 = x^{2n+1} - 1$

3.1 The Way of Deciding Period Matrix of $y^2 = x^{2n+1} - 1$

3.1.1 STEP 1: A base of holomorphic 1-forms for $y^p = x^q - 1$

Lemma 3.1 *Let N be the dimension of vector space of holomorphic 1-form $\frac{x^b}{y^a}dx$ for $y^p = x^q - 1 \quad (p,q)=1$.*

$$N = \frac{(p-1)(q-1)}{2}$$

Proof. (a) $(x,y) = (\alpha_i, 0) \quad (i = 1,\dots,q)$
α_i is q-th root of 1. By $py^{p-1}dy = qx^{q-1}dx$, Order of dx's zero point in $y=0$ is $p-1$

$$\frac{x^b}{y^a}dx \quad \text{is holomorphic in (a)} \Longleftrightarrow 1 \le a \le p-1$$

(b) $(x,y) = (\infty, \infty)$
As we can view $y^p = x^q - 1$ as $y^p = x^q$ in (b), we can put x, y on t^{-p}, t^{-q} By $dx = -pt^{-p-1}dt$,

$$\frac{x^b}{y^a}dx = \frac{t^{-bp}}{t^{-aq}}(-pt^{-p-1})dt = -pt^{aq-bp-(p+1)}dt$$

$$\begin{aligned} \frac{x^b}{y^a}dx \quad \text{is holomorphic in (b)} \quad &\Longleftrightarrow \quad aq - bp - (p+1) \ge 0 \\ &\Longleftrightarrow \quad 0 \le b \le [\frac{aq-(p+1)}{p}] \end{aligned}$$

From (a),(b)

$$\begin{aligned} N &= \sum_{a=1}^{p-1} [\frac{aq-(p+1)}{p}] + 1 \\ &= \frac{1}{2}\sum_{a=1}^{p-1}[\frac{aq-(p+1)}{p}] + 1 + \frac{1}{2}\sum_{a=1}^{p-1}[\frac{(p-a)q-(p+1)}{p}] + 1 \\ &= \frac{1}{2}\sum_{a=1}^{p-1}(m-1) + 1 + \frac{1}{2}\sum_{a=1}^{p-1}(q-m-2) + 1 \quad aq = mp + r \quad (0 \le r \le p-1) \\ &= \frac{1}{2}\sum_{a=1}^{p-1}(q-1) = \frac{1}{2}(p-1)(q-1) \end{aligned}$$

which proves the lemma. q.e.d.

3.1.2 STEP2 : Riemann Surface of $y^2 = x^{2n+1} - 1$

At first, let think about Riemann Surface of $y^2 = x$ i.e. $y = \sqrt{x}$. We put the value of y for $x_1 = re^{i\theta}$, $x_2 = re^{i(\theta+2\pi)}$ on y_1, y_2. As we can get $y_2 = -y_1$ by easy calculation, $y = \sqrt{x}$ is one-to-two mapping from x-surface to y-surface i.e. 2-valued-function which is showed by the below figure.

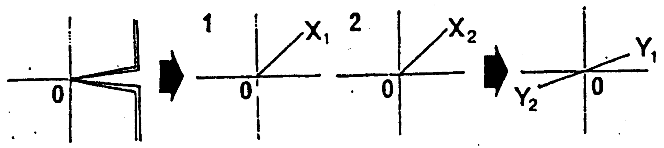

FIG 1

But $y = \sqrt{x}$ is not 2-valued-function in $x = 0$ which satisfy $y = \sqrt{x} = 0$.And As we can get $t = \sqrt{s}$ again by putting x, y on $1/s, 1/t$. $y = \sqrt{x}$ is not 2-valued-function in $x = \infty$ too. Thus $x = 0$ and $x = \infty$ are branch points of R_0. R_0 is the thing which joined up-side of x_1-surface to under-side of x_2-surface and under-side of x_1-surface to up-side of x_2-surface formally by cutting two surface x_1, x_2 along segment connecting two branch points $x = 0, \infty$.

Next let think about Riemann Surface R_n of $y^2 = x^{2n+1} - 1$ like $y = \sqrt{x}$. R_n has two sheets $y_1 = \sqrt{x^{2n+1} - 1}$ and $y_2 = -y_1 = -\sqrt{x^{2n+1} - 1}$. As we can get $t^2 = s^{2n+1}(1 - s^{2n+1})^{-1}$ by putting x, y on $1/s, 1/t$ in $y^2 = x^{2n+1} - 1$,branch points of R_n are $(2n+1)^{th}$-root of 1 $x = p_i$ $(i = 1, \cdots 2n+1)$ and ∞. To see the model of R_n ,we cut two sheets following so that the sheet can change by rounding each branch points one time.

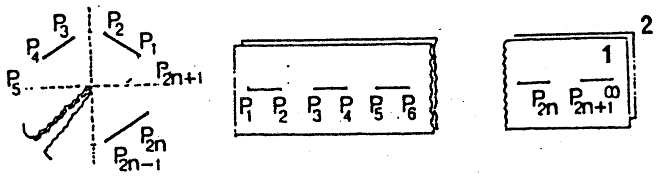

FIG 2

let join $x_1 - surface$ to $x_2 - surface$ like $y = \sqrt{x}$.
This is Riemann surface R_n of $y^2 = x^{2n+1} - 1$. Its genus g is the dimension of the vector space of holomorphic 1-forms by Riemann-Roch theorem. Therefore, by lemma 3.1.1 $g = 2\{(2n+1) - 1\}/2 = n$.

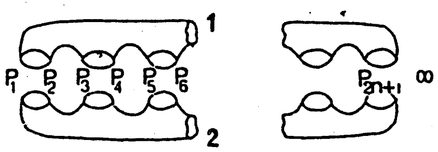

FIG 3

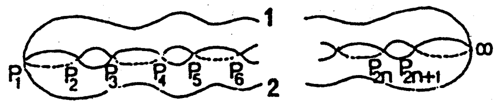

FIG 4

3.1.3 STEP 3: Five Rules of Deciding Period Matrix

We decide a base of homology for R_n which is made in Step 2. The following figure shows it.

$A_i \times B_i = 1 \qquad (i = 1, \cdots, n)$

$A_i \times A_j = B_i \times B_j = 0 \qquad (i \neq j)$

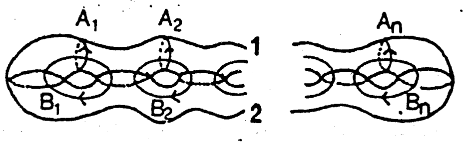

FIG 5

Here, let's put the cross points of A_i, B_i on T_i $(i =, \cdots, n)$ and we get together T_i to branch point ∞ from same direction.

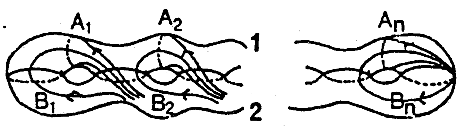

FIG 6

Next, we cut and open R_n along the base of homology A_i, B_i. It becomes a sheet of simple connected domain namely, $4n$-polynomial which has 4g-side $A_i^+, A_i^-, B_i^+, B_i^-$ $(i = 1, \cdots, n)$ provided thatA_i^+, B_i^+ shows right side of A_i, B_i and A_i^-, B_i^- shows left side A_i, B_i. At last, we have to write difference of two sheet of complex surface x_1, x_2 and arrangement of branch points in $4n$-polynomial.

On the above mentions, we decide the periods of holomorphic 1-forms for A_i, B_i by

On the above mentions, we decide the periods of holomorphic 1-forms for A_i, B_i by

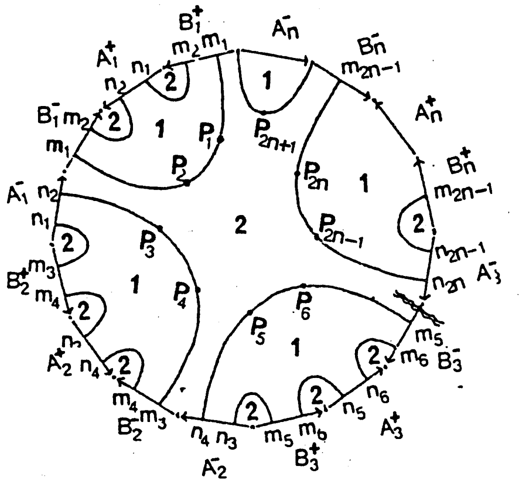

FIG 7

making some simple closed path which pass branch points. At this time, Cauchy's integral theorem plays the leading role. But the periods of holomorphic 1-forms for A_i, B_i must be decided uniquely so that they may satisfy Riemann's period relations i.e., Theorem 2.1 and Theorem 2.2. Therefore, to realize this object, we state the following five rules.

In making a simple closed path which include the line A_i, B_i

- **Rule 1**: *A simple closed path must include right side A_i^-, B_i^-.*
- **Rule 2**: *A simple closed path must include even branch point.*
- **Rule 3**: *The sign of holomorphic 1-form in the path which get out from starting points of A_i^-, B_i^- must be same sign in the path which get into end points of A_i, B_i.*
- **Rule 4**: *The sign of holomorphic 1-form in the path which get out from a branch point must be different from sign in the path which get into same branch point.*
- **Rule 5**: *The sign of holomorphic 1-form in the path which connect two branch points must be unchangeable.*

Remark 1: As the side of $4n$-polynomial -B_i- are enclosed by A_i^+ and A_i^-. We can look on the path of B_i^- as the path which connect a point m_i on A_i^+ and B_i^+ by Cauchy's integral theorem. Thus we decide the periods of B_i for this new path by using above five rules.

Remark 2: As the side of $4n$-polynomial: A_i are enclosed by B_i^+ and B_j^- $(i \neq j)$. We cannot choose a common point from the points on B_i^+ and B_j^-. But at this case, we can decide value of period by making simple closed path which some pair of the side of 4n-polynomial: $A_i^+, A_i^-, B_i^+, B_i^-$.

By using above five rules and two remarks, we decided periods of holomorphic 1-form. A base of holomorphic 1-forms for $y^2 = x^{2n+1} - 1$ is following by lemma 3.1.1.

$$\omega_1 = \frac{1}{y}dx, \quad \omega_2 = \frac{x}{y}dx, \quad \omega_3 = \frac{x^2}{y}dx, \quad \cdots \quad , \omega_n = \frac{x^{n-1}}{y}dx$$

Namely,

$$\omega_i = \frac{x^{i-1}}{y}dx = \frac{x^{i-1}}{\sqrt{x^{2n+1}-1}}dx \quad (i = 1, \cdots, n)$$

And let stand for ω_i on x_1-surface, ω_i on x_2-surface by ω_{i1}, ω_{i2}.

$$\omega_{i1} = \omega_i = \frac{x^{i-1}}{\sqrt{x^{2n+1}-1}}dx, \quad \omega_{i2} = -\omega_i = \frac{-x^{i-1}}{\sqrt{x^{2n+1}-1}}dx$$

3.2 Period matrix of $y^2 = x^{2n+1} - 1$

3.2.1 Calculation

1. A - PERIOD MATRIX

(1) For $j = 1, 2, \cdots, n-1$

By Remark 2, we make a simple closed path which start from ∞ in start point of A_j^-, pass the sides $A_{j+1}^+ B_{j+1}^+ A_{j+1}^- B_{j+1}^-, A_{j+2}^+ B_{j+2}^+ A_{j+2}^- B_{j+2}^-, \cdots, A_n^+ B_n^+ A_n^- B_n^-$, arrive at ∞ in start point of B_n^- and start from its ∞, pass the branch points $P_1, P_2, \cdots, P_{2j-1}, P_{2j}$, come back ∞ in end point of A_j^-. As the sum of the integrate values on sides of $4n$ polynomial is 0 at this time, the integrate path of A_j is following.

$$\infty \xrightarrow{-} P_1 \xrightarrow{+} P_2 \xrightarrow{-} P_3 \xrightarrow{+} P_4 \quad \cdots \quad P_{2j-1} \xrightarrow{+} P_{2j} \xrightarrow{-} \infty$$

(2) For $j = n$

We make the closed simple path which include A_n^- and pass the branch points $P_1, P_2, P_3, P_4, \cdots, P_{2n-1}, P_{2n}$. Thus the integrate path of A_n is the following.

$$\infty \xrightarrow{-} P_1 \xrightarrow{+} P_2 \xrightarrow{-} P_3 \xrightarrow{+} P_4 \cdots P_{2n+1} \xrightarrow{+} P_{2n} \xrightarrow{-} \infty$$

We get period π_{ij} by the above integrate path

(1) For $j = 1, 2, \cdots, n-1$

$$\begin{aligned}
\pi_{ij} &= \int_{A_j} \omega_i = \int_{A_{j+1}^+ B_{j+1}^+ A_{j+1}^- B_{j+1}^-} \omega_i + \int_{A_{j+2}^+ B_{j+2}^+ A_{j+2}^- B_{j+2}^-} \omega_i + \cdots + \int_{A_n B_n A_n B_n} \omega_i \\
&+ \int_{\infty}^{P_1} \omega_{i2} + \int_{P_1}^{P_2} \omega_{i1} + \int_{P_2}^{P_3} \omega_{i2} + \int_{P_3}^{P_4} \omega_{i1} + \cdots + \int_{P_{2j-1}}^{P_{2j}} \omega_{i1} + \int_{P_{2j}}^{P_\infty} \omega_{i2} \\
&= \int_{\infty}^{P_1} -\omega_i + \int_{P_1}^{P_2} \omega_i + \int_{P_2}^{P_3} -\omega_i + \int_{P_3}^{P_4} \omega_i + \cdots + \int_{P_{2j-1}}^{P_{2j}} \omega_i + \int_{P_{2j}}^{P_\infty} -\omega_i \\
&= 2(P_1^i - P_2^i + P_3^i - P_4^i + \cdots + P_{2j-1}^i - P_{2j}^i) \int_1^{\infty} \omega_i
\end{aligned}$$

$$\pi_{ij} = 2 \sum_{l=1}^{j} (P_{2l-1}^i - P_{2l}^i) K_i$$

(2) For $j = n$

$$\pi_{in} = \int_{B_n} \omega_i = \int_{\infty}^{P_1} -\omega_i + \int_{P_1}^{P_2} \omega_i + \int_{P_2}^{P_3} -\omega_i + \int_{P_3}^{P_4} \omega_i + \cdots + \int_{P_{2n+1}}^{P_{2n}} \omega_i + \int_{P_{2n}}^{P_\infty} -\omega_i$$

$$\pi_{in} = 2 \sum_{l=1}^{n} (P_{2l-1}^i - P_{2l}^i) K_i$$

2. B - PERIOD MATRIX

(1) For $j = 1, 2, \cdots, n-1$

By remark 1, we can view the path of B_j as the new path which connect the point m_{2j}^+ on A_j^+ and the point m_{2j}^- on A_j^- . Therefore, we make a simple closed paths which enclosed the new path and pass the branch points P_{2j}, P_{2j+1}. At this time, the integrate path of B_j is following.

$$m_{2j}^+ \xrightarrow{+} P_{2j+1} \xrightarrow{-} P_{2j} \xrightarrow{+} n_{2j}^-$$

(2) For $j = n$

We make a closed simple path which include B_n^- and pass the branch points $P_1, P_2, P_3, P_4, \cdots, P_{2n}, P_{2n+1}$. Thus the integrate path of B_n is following.

$$\infty \xrightarrow{+} P_{2n+1} \xrightarrow{-} P_{2n} \xrightarrow{+} \infty$$

We get period π'_{ij} by the above integrate path

(1) For $j = 1, 2, \cdots, n-1$

$$\pi'_{ij} = \int_{B_j} \omega_i = \int_{m_{2j}^+}^{P_{2j+1}} \omega_{i1} + \int_{P_{2j+1}}^{P_{2j}} \omega_{i2} + \int_{P_{2j}}^{P_{2j+1}} \omega_{i1} = \int_{m_{2j}}^{P_{2j+1}} \omega_i + \int_{P_{2j+1}}^{P_{2j}} -\omega_i + \int_{P_{2j}}^{P_{2j+1}} \omega_i$$

$$= 2(P_{2j}^i - P_{2j+1}^i) \int_1^{\infty} \omega_i$$

$$\pi'_{ij} = 2(P_{2j}^i - P_{2j+1}^i) K_i$$

(2) For $j = n$

$$\pi'_{in} = \int_{B_n} \omega_i = \int_{\infty}^{P_{2n+1}} \omega_{i1} + \int_{P_{2n+1}}^{P_{2n}} \omega_{i2} + \int_{P_{2n}}^{\infty} -\omega_{i1}$$

$$\pi'_{in} = 2(P_{2n}^i - P_{2n+1}^i) K_i$$

where,

$$P_i = \exp(\frac{2\pi i}{2n+1})$$

$$K_i = \int_1^{\infty} \omega_i = \int_1^{\infty} \frac{1}{\sqrt{x^{2n+1} - 1}}$$

3.2.2 Result

1.PERIOD MATRIX

A - period matrix Π :

By the above calculation, $\pi_{ij} = 2 \sum_{l=1}^{j} (P_{2l-1}^i - P_{2l}^i) K_i \quad (i, j = 1, 2, \cdots, n)$

$$\Pi = 2 \begin{pmatrix} K_1 & & 0 \\ & K_2 & \\ & & \ddots \\ 0 & & & K_n \end{pmatrix} \begin{pmatrix} P_1 - P_2 & P_1 - P_2 + P_3 - P_4 & \cdots & \sum_{l=1}^{n}(P_{2l-1} - P_{2l}) \\ P_1^2 - P_2^2 & P_1^2 - P_2^2 + P_3^2 - P_4^2 & \cdots & \sum_{l=1}^{n}(P_{2l-1}^2 - P_{2l}^2) \\ \vdots & \vdots & & \vdots \\ P_1^n - P_2^n & P_1^n - P_2^n + P_3^n - P_4^n & \cdots & \sum_{l=1}^{n}(P_{2l-1}^n - P_{2l}^n) \end{pmatrix}$$

B - period matrix Π' :
By the above calculation, $\pi'_{ij} = 2(P^i_{2j} - P^i_{2j+1})K_i \quad (i,j = 1,2,\cdots,n)$

$$\Pi' = 2\begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{pmatrix}\begin{pmatrix} P_2 - P_3 & P_4 - P_5 & \cdots & P_{2n} - P_{2n+1} \\ P_2^2 - P_3^2 & P_4^2 - P_5^2 & \cdots & P_{2n}^2 - P_{2n+1}^2 \\ \vdots & \vdots & & \vdots \\ P_2^n - P_3^n & P_4^n - P_5^n & \cdots & P_{2n}^n - P_{2n+1}^n \end{pmatrix}$$

2. DETERMINANT

$$det\Pi = 2K_1K_2\cdots K_n(P - P^2)(P^2 - P^4)\cdots(P^n - P^{2n})H = 2(-1)^nKP^{\frac{n(n+1)(2n+1)}{6}}C$$

$$det\Pi' = 2K_1K_2\cdots K_n(P^2 - P^3)(P^4 - P^6)\cdots(P^{2n} - P^{3n})H = 2(-1)^nKP^{\frac{n(n+1)(n+2)}{3}}C$$

Let put C_k on $\prod_{l=1}^{k}(P^{2l} - 1)$, then H is following the Vandermonde determinant.

$$H = \begin{vmatrix} 1 & P^2 & P^4 & \cdots & P^{2n-2} \\ 1 & P^4 & P^8 & \cdots & P^{4n-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & P^{2n} & P^{4n} & \cdots & P^{2n^2-2n} \end{vmatrix} = P^{\frac{n(n-1)(n+1)}{3}}C_{n-1}C_{n-2}C_{n-3}\cdots C_2C_1$$

$$K = K_1K_2\cdots K_n \quad K_i = \int_1^{\infty}\omega_i \quad C = C_1C_2\cdots C_{n-1}C_n, \quad C_n = \prod_{l=1}^{n}(P^l - 1)$$

$$P^i = P_i \quad (i = 1,2,\cdots,n) \quad P = \exp(\frac{2\pi i}{2n+1})$$

3.2.3 Confirmation for lemma

In this section we show that the value of periods made by five rules satisfy Riemann's period relation 1.2.

$$\begin{aligned}
\sum_{k=1}^{n}[\int_{A_k}\omega_i\int_{B_k}\omega_j] &= \sum_{k=1}^{n}[\{2\sum_{l=1}^{k}(P_{2l-1}^i-P_{2l}^i)K_i\}\{2(P_{2k}^j-P_{2k+1}^j)K_i\}] \\
&= 4K_iK_j\sum_{k=1}^{n}[(1-P^j)P_{2k}^j\sum_{l=1}^{k}\frac{1-P^i}{P^i}P_{2l}^i] \\
&= 4K_iK_j\frac{(1-P^i)(1-P^j)}{P^i}\sum_{k=1}^{n}[P_{2k}^j\sum_{l=1}^{k}P_{2l}^i] \\
&= 4K_iK_j\frac{(1-P^i)(1-P^j)}{P^i}\sum_{k=1}^{n}[P_{2k}^jP_2^j\frac{1-P_{2k}^i}{1-P_2^j}] \\
&= 4K_iK_j\frac{1-P^j}{1+P^i}P_i\{\sum_{k=1}^{n}P_{2k}^j-\sum_{k=1}^{n}P_{2k}^{i+j}\} \\
&= 4K_iK_j\frac{1-P^j}{1+P^i}P^i\{P_2^i\frac{1-P_2^{ni}}{1-P_2^i}-P_2^{i+j}\frac{1-P_2^{n(i+j)}}{1-P_2^{i+j}}\} \\
&= -4K_iK_j\frac{1-P^j}{1+P^i}P^i\{\frac{P^j}{1+P^j}-\frac{P^{i+j}}{1+P^{i+j}}\} \\
&= -4K_iK_j\frac{P^{i+j}(1-P^i)(1-P^j)}{(1+P^{i+j})(1+P^i)(1+P^j)}
\end{aligned}$$

(symmetric expression for P^i, P^j)

$$\sum_{k=1}^{n}[\int_{A_k}\omega_i\int_{B_k}\omega_j]=\sum_{k=1}^{n}[\int_{A_k}\omega_j\int_{B_k}\omega_i]$$

which satisfy Riemann's period relation 1.

$$i\sum_{k=1}^{n}[\int_{A_k}\omega_i\int_{B_k}\bar{\omega}_i-\int_{B_k}\omega_i\int_{A_k}\bar{\omega}_i]>0$$

$$i\sum_{k=1}^{n}[\overline{\int_{A_k}\bar{\omega}_i\int_{B_k}\omega_i}-\int_{B_k}\omega_i\int_{A_k}\bar{\omega}_i]>0$$

$$Im\sum_{k=1}^{n}(\int_{A_i}\bar{\omega}_i\int_{B_i}\omega_i)>0$$

On the other hand, we have only to prove last inequality for Riemann's period relation 2.

$$\begin{aligned}
\sum_{k=1}^{n}[\int_{A_k}\bar{\omega}_i\int_{B_k}\omega_i] &= \sum_{k=1}^{n}[\{2\sum_{l=1}^{k}(\bar{P}^i_{2l-1}-\bar{P}^i_{2l})K_i\}\{2(P^i_{2k}-P^i_{2k+1})K_i\}] \\
&= 4K_i^2(1-\bar{P}^i)(1-P^i)\sum_{k=1}^{n}[\sum_{l=1}^{k}\frac{1}{\bar{P}^i}(\bar{P}^i_2)^l P^i_{2k}] \\
&= 4K_i^2\frac{(1-\bar{P}^i)(1-P^i)}{\bar{P}^i}\sum_{k=1}^{n}[\bar{P}^i_2\frac{1-\bar{P}^i_{2k}}{1-\bar{P}^i_2}P^i_{2k}] \\
&= 4K_i^2\frac{1-P^i}{1+\bar{P}^i}\bar{P}_i\sum_{k=1}^{n}[(1-\bar{P}^i_{2k})P^i_{2k}] \\
&= 4K_i^2\frac{1-P^j}{1+\bar{P}^i}\bar{P}^i\sum_{k=1}^{n}[P^i_{2k}-1] \\
&= 4K_i^2\frac{1-P^i}{1+\bar{P}^i}\bar{P}^i\{P^i_2\frac{1-P^i_{2n}}{1-P^i_2}-n\} \\
&= 4K_i^2\frac{1-P^i}{1+\bar{P}^i}\bar{P}^i\{\frac{-P^i}{1+P^i}-n\} \\
&= 4K_i^2\frac{(1-\bar{P}^i)}{(1+\bar{P}^i)(1+P^i)}\{P^i+n(1+P^i)\}
\end{aligned}$$

$$\sum_{k=1}^{n}[\int_{A_k}\bar{\omega}_i\int_{B_k}\omega_i] = \frac{4K_i^2(1-\bar{P}^i)\{n+(n+1)P^i\}}{|1+P^i|^2}$$

By putting

$$P^i = \exp(\frac{2\pi i}{2n+1})$$

$$Im\sum_{k=1}^{n}[\int_{A_k}\bar{\omega}_i\int_{B_k}\omega_i] = \frac{4K_i^2}{|1+P^i|^2}(2n+1)\sin(\frac{2\pi i}{2n+1}) > 0$$

which means that the period matrix for the homology in five rules satisfy Riemann's period relation. Thus we have the following theorem from the above result.

THEOREM **3.1** *Let X be the Riemann surface defined by $y^2 = x^{2n+1}-1$, then the period matrix of X is given by (Π,Π') of §3.2.2.*

References

[1] H.F.Baker. On a system of differential equations leading to periodic functions, Acta mathematica. 27(1902) 135-156

[2] H.F.Baker. An introduction to the Theory of Multiple Periodic functions. Cambridge. 1907

[3] K.Kawada. An Introduction to the Theory of Algebraic curve (in Japanese). Sibundo. 1964.

[4] Georg springer. Introduction to Riemann surfaces. Addison-Wesley. 1967.

[5] K.Watanabe. On pulurigenera of normal isolated singularities. Math. Ann. 250(1980), 65-94.

[6] David Mumford. Tata Lectures on Theta I. Birkhauser 1984

[7] S.Yamazaki. Y.Tashiro. H.Higuchi. Periods matrix of hyperelliptic Riemann surface Memoirs of the faculty of Liberal Art and Education of Yamaguchi Univ. 44(1993)10-13.

[8] Y.Tashiro, S.Yamazaki, H.Higuchi. Determinant of period matrices of hyperelliptic Riemann surfaces. Bull. Fac. Gen. Ed. Tokyo Univ. Agri. Tech. 31(1994)21-25