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Life-span of Classical Solutions to Nonlinear Wave Equations in Four Space Dimensions

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Abstract In this paper we prove that in four-space-dimensional case, L. Hörmander's estimate $\widetilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-1}\}$ ($A > 0$, constant) can be improved by $\widetilde{T}(\varepsilon) \geq \exp\{A\varepsilon^{-2}\}$ on the lower bound of the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to the Cauchy problem with small initial data $(u, u_t)(0, x) = \varepsilon(\phi(x), \psi(x))$ for nonlinear wave equations of the form $\Box u = F(u, Du, D_xDu)$, where $F(\hat{\lambda}) = O(|\hat{\lambda}|^2)$ in a neighbourhood of $\hat{\lambda} = 0$.

1 Introduction

Consider the Cauchy problem for fully nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_xDu), \tag{1.1}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x), \tag{1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.3}$$

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is the wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \qquad D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \tag{1.4}$$

$\phi, \psi \in C_0^\infty(I\!R^n)$ and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), i, j = 0, 1, \cdots, n, i + j \geq 1). \tag{1.5}$$

Suppose that in a neighbourhood of $\hat{\lambda} = 0$, the nonlinear term $F = F(\hat{\lambda})$ in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{1.6}$$

where α is an integer ≥ 1.

For all integers n, α with $n \geq 1$ and $\alpha \geq 1$, the lower bound of the life-span of classical solutions to (1.1)-(1.2) was studied by S. Klainerman [1]-[2], [5], [7], J. L. Shatah [3], S. Klainerman & G. Ponce [4], F. John [6], F. John & S. Klainerman [8], M. Kovalyov [9], L. Hörmander [10], Li Ta-tsien & Yu Xin [11] etc. for the special case

$$F = F(Du, D_x Du),$$

and by D. Christodoulou [12], Li Ta-tsien & Chen Yun-mei [13], L. Hörmander [14], H. Lindblad [15], Li Ta-tsien & Yu Xin [16], Li Ta-tsien & Zhou Yi [17], Li Ta-tsien, Yu Xin & Zhou Yi [18]-[20], Li Ta-tsien & Zhou Yi [21]-[23] etc. for the general case

$$F = F(u, Du, D_x Du).$$

A summary of all the results mentioned above can be found in Li Ta-tsien & Chen Yu-mei [24]. All these lower bounds, except in the case $n = 4$, $\alpha = 1$ and $\partial_u^2 F(0,0,0) \neq 0$, have been known to be sharp.

Thus, in the case $n=4$, $\alpha=1$ and $\partial_u^2 F(0,0,0)\neq 0$, it is natural to ask if the lower bound of the life-span

$$\widetilde{T}(\varepsilon)\geq \exp\{A\varepsilon^{-1}\}, \tag{1.7}$$

where A is a positive constant independent of ε, originally obtained by L. Hörmander [14] and then, by means of the global iteration method, by Li Ta-tsien & Yu Xin [16], is sharp or not. In this paper, as a joint work with Zhou Yi, we shall prove that (1.7) can be improved by

$$\widetilde{T}(\varepsilon)\geq \exp\{A\varepsilon^{-2}\}, \tag{1.8}$$

where A is a positive constant independent of ε (see Li Ta-tsien & Zhou Yi [35]-[36]).$\square$

2 Motivation

By L. Hörmander [14] and Li Ta-tsien & Zhou Yi [17], if there is no u^2 term in the Taylor expansion of F, i.e., $\partial_u^2 F(0,0,0)=0$, then in four space dimensions Cauchy problem (1.1)-(1.2) always admits a unique global classical solution on $t\geq 0$, provided that $\varepsilon>0$ is suitably small.

In order to illustrate the motivation of expecting estimate (1.8), as the 'worst' case of equation (1.1) we consider the equation

$$\square u = u^2 \tag{2.1}$$

which can be regarded as a special case ($p=2$) of the following equation

$$\square u = |u|^p \qquad (p>1). \tag{2.2}$$

When $n=3$, F. John [25] proved that if $p>1+\sqrt{2}$, then for $\varepsilon>0$ suitably small, Cauchy problem (2.2) and (1.2) admits a unique global solution on $t\geq 0$;

while, if $1 < p < 1+\sqrt{2}$, then any nontrivial solution with compact support to equation (2.2) must blow up in a finite time. Thus, in the case $n = 3$ the critical value of p is equal to $p_0(3) = 1+\sqrt{2}$. In general, as suggested and studied by W. Strauss [26], R. T. Glassey [27]-[28], T. C. Sideris [29], J. Schaeffer [30] etc., in n space dimensions, $p_0(n)$, the critical value of p, should be the positive root of the following quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{2.3}$$

In particular, we have $p_0(4) = 2$. Hence, equation (2.1) corresponds to the critical value of p in four space dimensions.

When $p = p_0(n)$ with $n = 2, 3$, Zhou Yi [31]-[32] proved that the life-span $\widetilde{T}(\varepsilon)$ of solutions to Cauchy problem (2.2) and (1.2) satisfies

$$\widetilde{T}(\varepsilon) \approx \exp\{A\varepsilon^{-p(p-1)}\}, \tag{2.4}$$

where A is a positive constant independent of ε; namely, there exist two positive constants A_1 and A_2 independent of ε, such that

$$\exp\{A_1\varepsilon^{-p(p-1)}\} \le \widetilde{T}(\varepsilon) \le \exp\{A_2\varepsilon^{-p(p-1)}\}.$$

We guess that (2.4) still holds in the case $n = 4$. If so, in four space dimensions the life-span of solutions to Cauchy problem (2.2) and (1.2) should satisfy

$$\widetilde{T}(\varepsilon) \approx \exp\{A\varepsilon^{-2}\}. \tag{2.5}$$

This consideration leads us to prove (1.8) for Cauchy problem (1.1)-(1.2) and to believe that this lower bound of the life-span should be sharp.□

3 Proof of the main result

The general framework which we shall use to prove (1.8) is still the global iteration method, suggested in Li Ta-tsien & Chen Yun-mei [13] and Li Ta-tsien & Yu Xin

[16].

First of all, just by differentiation, it suffices to consider the Cauchy problem for the following general kind of quasilinear wave equations:

$$\Box u = \sum_{i,j=1}^{n} b_{ij}(u, Du)u_{x_ix_j} + 2\sum_{j=1}^{n} a_{0j}(u, Du)u_{tx_j} + F_0(u, Du), \tag{3.1}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x), \tag{3.2}$$

where b_{ij}, a_{0j} $(i, j = 1, \cdots, n)$ and F_0 satisfy certain suitable assumptions. Without loss of generality, in what follows we may suppose that

$$\text{supp } \{\phi, \psi\} \subseteq \{x \mid |x| \leq 1\}. \tag{3.3}$$

The solution u to Cauchy problem (3.1)-(3.2) (in which $n = 4$) can be written as

$$u = w + u_\varepsilon, \tag{3.4}$$

where u_ε is the solution to the following Cauchy problem for the homogeneous wave equation:

$$\Box u_\varepsilon = 0, \tag{3.5}$$

$$t = 0 : u_\varepsilon = \varepsilon\phi(x), (u_\varepsilon)_t = \varepsilon\psi(x); \tag{3.6}$$

while w is the solution to the following Cauchy problem:

$$\Box w = \sum_{i,j=1}^{4} b_{ij}(u, Du)u_{x_ix_j} + 2\sum_{j=1}^{4} a_{0j}(u, Du)u_{tx_j} + F_0(u, Du), \tag{3.7}$$

$$t = 0 : w = 0, w_t = 0. \tag{3.8}$$

The iteration scheme is given after the preceding translation by

$$\begin{cases} \Box w = \sum_{i,j=1}^{4} b_{ij}(u_\varepsilon + \omega, D(u_\varepsilon + \omega))(u_\varepsilon + w)_{x_ix_j} \\ \quad +2\sum_{j=1}^{4} a_{0j}(u_\varepsilon + \omega, D(u_\varepsilon + \omega))(u_\varepsilon + w)_{tx_j} \\ \quad +F_0(u_\varepsilon + \omega, D(u_\varepsilon + \omega)), \\ t = 0 : w = 0, w_t = 0, \end{cases} \tag{3.9}$$

which defines a map

$$M : \omega \to w = M\omega. \square \tag{3.10}$$

For any integer $N \geq 0$, define

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \sum_{|k|\leq N} \|\chi(t,\cdot)\Gamma^k u(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)}, \tag{3.11}$$

where Γ denotes the following set of partial differential operators:

$$\Gamma = \{D, L, \Omega\} \triangleq (\Gamma_1, \cdots, \Gamma_\sigma), \tag{3.12}$$

in which

$$D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \tag{3.13}$$

$$L = (L_a, a = 0, 1, \cdots, n), \tag{3.14}$$

$$\Omega = (\Omega_{ij}, i, j = 1, \cdots, n) \tag{3.15}$$

with

$$\begin{cases} L_0 = t\partial_t + x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}, \\ L_i = t\partial_{x_i} + x_i\partial_t, \qquad i = 1, \cdots, n, \\ \Omega_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}, \qquad i, j = 1, \cdots, n, \end{cases} \tag{3.16}$$

$1 \leq p, q \leq +\infty$, $\chi(t,x)$ is the characteristic function of any given set in $\mathbb{R}_+ \times \mathbb{R}^n$, $k = (k_1, \cdots, k_\sigma)$ are multi-indices and $L^{p,q}(\mathbb{R}^n)$ is a function space, introduced by Li Ta-tsien & Yu Xin in [16], with the norm

$$\|f(\cdot)\|_{L^{p,q}(\mathbb{R}^n)} = \|f(r\xi)r^{\frac{n-1}{p}}\|_{L^p(0,+\infty;L^q(S^{n-1}))}, \tag{3.17}$$

where $r = |x|$ and $\xi \in S^{n-1}$, S^{n-1} being the unit sphere in $\mathbb{R}^n$. We write

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \begin{cases} \|u(t,\cdot)\|_{\Gamma,N,p,\chi}, & \text{if } p = q, \\ \|u(t,\cdot)\|_{\Gamma,N,p,q}, & \text{if } \chi \equiv 1, \\ \|u(t,\cdot)\|_{p,q,\chi}, & \text{if } N = 0. \square \end{cases} \tag{3.18}$$

The essential point for the global iteration method is to choose a suitable function space with which we shall work in the iteration. This space should simultaneously reflect the decay property and the energy estimate of solutions to the linear wave equation.

In the present situation, the space is chosen as follows:

$$X_{S,E,T} = \{\omega(t,x) \mid D_{S,T}(\omega) \le E, \partial_t^l \omega(0,x) = w_l^{(0)}(x)\ (l = 0,1,\cdots,S+1)\}, \tag{3.19}$$

where S is an integer ≥ 11, E and T are positive numbers and

$$\begin{aligned} D_{S,T}(\omega) = & \sum_{|i|=1}^{2} \sup_{0\le t\le T} \|D^i\omega(t,\cdot)\|_{\Gamma,S,2} + \sup_{0\le t\le T} \|\omega(t,\cdot)\|_{\Gamma,S,2,\chi_1} \\ & + \sup_{0\le t\le T}(1+t)\|\omega(t,\cdot)\|_{\Gamma,S,6,2,\chi_2} + \sup_{0\le t\le T}(1+t)^{-\frac{3}{2}}(\ln(2+t))^{-1}|||\omega(t,\cdot)||| \\ & + \sup_{\substack{0\le t\le T \\ x\in \mathbf{R}^4}} (1+t)^{\frac{3}{2}}(1+|t-|x||)^{\frac{1}{2}} \sum_{|k|\le S-3} |\Gamma^k\omega(t,x)|, \end{aligned} \tag{3.20}$$

in which χ_1 is the characteristic function of the set $\{(t,x) \mid |x| \le \frac{t+1}{2}\}$, $\chi_2 = 1-\chi_1$ and

$$|||\omega(t,\cdot)||| = \sum_{|k|\le S} \|(1+|t-|\cdot||)^{-\frac{1}{2}}\chi_2\Gamma^k\omega(t,\cdot)\|_{L^{1,2}(\mathbf{R}^4)}. \tag{3.21}$$

Moreover, $w_0^{(0)} = w_1^{(0)} = 0$ and $w_l^{(0)}(x)$ $(l = 2,\cdots,S+1)$ are the values of $\partial_t^l w(t,x)$ at $t = 0$ formally determined by Cauchy problem (3.7)-(3.8) with (3.4)-(3.6).

Endowed with the metric

$$\rho(\bar\omega,\bar{\bar\omega}) = D_{S,T}(\bar\omega - \bar{\bar\omega}), \qquad \forall\, \bar\omega,\bar{\bar\omega} \in X_{S,E,T}, \tag{3.22}$$

$X_{S,E,T}$ is a nonempty complete metric space, provided that $\varepsilon > 0$ is suitably small.

Let $\widetilde{X}_{S,E,T}$ be the subset of $X_{S,E,T}$ composed of all elements $\omega \in X_{S,E,T}$ such that

$$\operatorname{supp} \omega \subseteq \{(t,x) \mid |x| \le t+1\}. \tag{3.23}$$

If we can show that for $\varepsilon > 0$ suitably small there exist $E = E(\varepsilon)$ and $T = T(\varepsilon)$ such that the map M has a unique fixed point $w \in \widetilde{X}_{S,E(\varepsilon),T(\varepsilon)}$, which implies that

$u = w + u_\varepsilon$ is the unique classical solution to Cauchy problem (3.1)-(3.2), then we get the following lower bound of the life-span

$$\widetilde{T}(\varepsilon) \geq T(\varepsilon).\square \tag{3.24}$$

Noting that the initial data in (3.9) are zero and the nonlinear term in (3.7) is quadratic with respect to $u = u_\varepsilon + w$, we have

Lemma 3.1. For $w = M\omega$, $\partial_t^l w(0,x)$ $(l = 0, 1, \cdots, S+2)$ are independent of $\omega \in \widetilde{X}_{S,E,T}$, and

$$\|w(0,\cdot)\|_{\Gamma,S+2,p,q} \leq C\varepsilon^2, \tag{3.25}$$

where $1 \leq p, q \leq +\infty$, C is a positive constant independent of ε and $\|w(0,\cdot)\|_{\Gamma,S+2,p,q}$ is the value of $\|w(t,\cdot)\|_{\Gamma,S+2,p,q}$ at $t = 0$.$\square$

According to the basic procedure of the global iteration method, in order to get the desired result, it suffices to show the following two lemmas.

Lemma 3.2. For any $\omega \in \widetilde{X}_{S,E,T}$, $w = M\omega$ satisfies

$$D_{S,T}(w) \leq C_1\{\varepsilon^2 + (R + \sqrt{R})(E + D_{S,T}(w))\}.\square \tag{3.26}$$

Lemma 3.3. Let $\bar{\omega}$, $\bar{\bar{\omega}} \in \widetilde{X}_{S,E,T}$. If $\bar{w} = M\bar{\omega}$ and $\bar{\bar{w}} = M\bar{\bar{\omega}}$ also satisfy $\bar{w}$, $\bar{\bar{w}} \in \widetilde{X}_{S,E,T}$, then

$$D_{S-1,T}(\bar{w} - \bar{\bar{w}}) \leq C_2(R + \sqrt{R})(D_{S-1,T}(\bar{w} - \bar{\bar{w}}) + D_{S-1,T}(\bar{\omega} - \bar{\bar{\omega}})).\square \tag{3.27}$$

In Lemma 3.2 and Lemma 3.3, C_1, C_2 are positive constants and

$$R = R(\varepsilon, E, T) = E\ln(2+T) + \varepsilon\sqrt{\ln(2+T)}. \tag{3.28}$$

Based on these two lemmas, the standard contraction mapping principle can be easily used to show that the existence time interval $[0, T(\varepsilon)]$ will be determined by

$$R(\varepsilon, E(\varepsilon), T(\varepsilon)) + \sqrt{R(\varepsilon, E(\varepsilon), T(\varepsilon))} \leq \frac{1}{C_0}, \tag{3.29}$$

where $C_0 = 3\max(C_1, C_2)$ and $E(\varepsilon) = C_0\varepsilon^2$. Obviously, if we take

$$T(\varepsilon) = \exp\{\bar{A}\varepsilon^{-2}\} - 2 \tag{3.30}$$

with $\bar{A} > 0$ suitably small, then (3.29) holds. This gives the desired estimate (1.8).$\square$

For the proof of Lemmas 3.2 and 3.3, we need some refined estimates on the solution to the Cauchy problem

$$\begin{cases} \Box u = F(t, x), \\ t = 0 : u = f(x), u_t = g(x) \end{cases} \tag{3.31}$$

in four space dimensions.

As mentioned above, L. Hörmander's estimate (1.7) was reproved by Li Tatsien & Yu Xin in [16]. The key tool in the proof is the following lemma.

Lemma 3.4. Suppose that $n \geq 3$. Let $u = u(t, x)$ be the solution to Cauchy problem (3.31). Then

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq\ & C\{\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^q(\mathbb{R}^n)} \\ & + \int_0^t [\|\chi_1 F(\tau, \cdot)\|_{L^q(\mathbb{R}^n)} + (1+\tau)^{-\frac{n-2}{2}} \|\chi_2 F(\tau, \cdot)\|_{L^{1,2}(\mathbb{R}^n)}] d\tau, \\ & \forall\, t \geq 0, \end{aligned} \tag{3.32}$$

where $\frac{1}{q} = \frac{1}{2} + \frac{1}{n}$, χ_1 is the characteristic function of the set $\{(t, x) \mid |x| \leq \frac{1+t}{2}\}$, $\chi_2 = 1 - \chi_1$ and C is a positive constant.$\square$

Inequality (3.32), established in Li Ta-tsien & Yu Xin [16], may be regarded as an improved form of Von Wahl's inequality (cf. [33]).

Based on a Sobolev embedding theorem in the radial direction, the idea of the proof of Lemma 3.4 can be applied to get the following two lemmas.

Lemma 3.5. Suppose that $n=4$. Let $u=u(t,x)$ be the solution to Cauchy problem (3.31) with

$$\begin{cases} \text{supp } \{f,g\} \subseteq \{x \mid |x| \leq 1\}, \\ \text{supp } F \subseteq \{(t,x) \mid |x| \leq t+1\}. \end{cases} \tag{3.33}$$

Then

$$\begin{aligned} \|\chi_1 u(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq\ & C(1+t)^{-1}\{\|f\|_{L^2(\mathbb{R}^4)} + \|g\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} \\ & + \int_0^t [(1+\tau)\|\chi_1 F(\tau,\cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} + \|\chi_2 F(\tau,\cdot)\|_{L^{1,2}(\mathbb{R}^4)}] d\tau\}, \\ & \forall\, t \geq 0, \end{aligned} \tag{3.34}$$

where χ_1 is the characteristic function of the set $\{(t,x) \mid |x| \leq \frac{t+1}{2}\}$, $\chi_2 = 1-\chi_1$ and C is a positive constant.□

Lemma 3.6. Suppose that $n=4$. Let $u=u(t,x)$ be the solution to Cauchy problem (3.31). Then

$$\begin{aligned} \|\chi_2 u(t,\cdot)\|_{L^{p,2}(\mathbb{R}^4)} \leq\ & C(1+t)^{-3(\frac{1}{2}-\frac{1}{p})}\{\|f\|_{\dot{H}^{S_0}(\mathbb{R}^4)} + \|g\|_{L^{\gamma}(\mathbb{R}^4)} \\ & + \int_0^t [\|\chi_1 F(\tau,\cdot)\|_{L^{\gamma}(\mathbb{R}^4)} + (1+\tau)^{-(1+S_0)}\|\chi_2 F(\tau,\cdot)\|_{L^{1,2}(\mathbb{R}^4)}] d\tau\}, \\ & \forall\, t \geq 0, \end{aligned} \tag{3.35}$$

where $p>2$, $S_0 = \frac{1}{2} - \frac{1}{p}$, $\frac{1}{\gamma} = \frac{1}{2} + \frac{1-S_0}{4}$, C is a positive constant and $\dot{H}^{S_0}(\mathbb{R}^4)$ stands for the homogeneous Sobolev space equipped with the norm

$$\|f\|_{\dot{H}^{S_0}(\mathbb{R}^4)} = \||\xi|^{S_0}\hat{f}(\xi)\|_{L^2(\mathbb{R}^4)}, \tag{3.36}$$

where $\hat{f}(\xi)$ is the Fourier transformation of $f(x)$.□

Moreover, noting that the initial data have compact support, a version of Huyghen's principle can be used to improve Hörmander's L^1-L^∞ estimate (see L. Hörmander [34]) and Lemma 3.4 as presented in Lemma 3.7 and Lemma 3.8 respectively.

Lemma 3.7. Under the assumptions of Lemma 3.5, we have

$$\begin{aligned} |u(t,x)| \;\le\; & C(1+t+|x|)^{-\frac{3}{2}}(1+|t-|x||)^{-\frac{3}{2}}\{\|u(0,\cdot)\|_{\Gamma,4,1} \\ & +\|u_t(0,\cdot)\|_{\Gamma,3,1} + (1+|t-|x||)\sup_{0\le\tau\le t}\|F(\tau,\cdot)\|_{\Gamma,3,1}\}, \\ & \forall\, t\ge 0, \forall\, x\in I\!R^4, \end{aligned} \tag{3.37}$$

where C is a positive constant.□

Lemma 3.8. Under the assumptions of Lemma 3.5, we have

$$\begin{aligned} & \|(1+|t-|\cdot||)^{-\frac{1}{2}}\chi_2 u(t,x)\|_{L^{1,2}(I\!R^4)} \\ & \le\; C(1+t)^{\frac{3}{2}}\{\|u(0,\cdot)\|_{\Gamma,4,1} + \|u_t(0,\cdot)\|_{\Gamma,3,1} \\ & \quad + \int_0^t [(1+\tau)^{\frac{1}{3}}\|\chi_1 F(\tau,\cdot)\|_{L^{\frac{3}{2}}(I\!R^4)} + (1+\tau)^{-1}\|\chi_2 F(\tau,\cdot)\|_{L^{1,2}(I\!R^4)}]d\tau\}, \\ & \quad \forall\, t\ge 0, \end{aligned} \tag{3.38}$$

where C is a positive constant.□

Lemmas 3.5-3.8 play an important role in the proof of Lemmas 3.2 and 3.3.□

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