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The automorphism group of the Klein curve in the mapping class group of genus 3

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1 The main result and its proof

Let R be a compact Riemann surface of genus $g \geq 2$. Then $\mathrm{Aut}(R)$, the automorphism group of R, can be embedded into the mapping class group (for its definition, see [Bir, Ch. 4]) or the Teichmüller group Γ_g of genus g;

(1.1) $$\iota\colon \mathrm{Aut}(R) \hookrightarrow \Gamma_g \simeq \mathrm{Out}^+(\pi_1(R)) = \mathrm{Aut}^+(\pi_1(R))/\mathrm{Int}(\pi_1(R)).$$

Here, $\mathrm{Aut}^+(\pi_1(R))$ consists of the automorphisms of $\pi_1(R)$ inducing the trivial action on $H_2(\pi_1(R), \mathbb{Z}) \simeq \mathbb{Z}$.

Recall the Hurwitz theorem, which states that

(1.2) $$\#\mathrm{Aut}(R) \leq 84(g-1).$$

If the equality holds in (1.2), then R is called a Hurwitz Riemann surface and $\mathrm{Aut}(R)$ is called a Hurwitz group.

Let X be the Klein curve of genus 3 defined by the equation

$$x^3y + y^3z + z^3x = 0.$$

It is well known that X is a Hurwitz Riemann surface; $G := \mathrm{Aut}(X)$ is isomorphic to $PSL_2(\mathbb{F}_7)$ and has order 168.

Now let us forget about the Klein curve, and consider an orientable compact C^∞ surface X of genus 3. We define the canonical generators of $\pi_1(X, b)$ with base point b as in the figure below;

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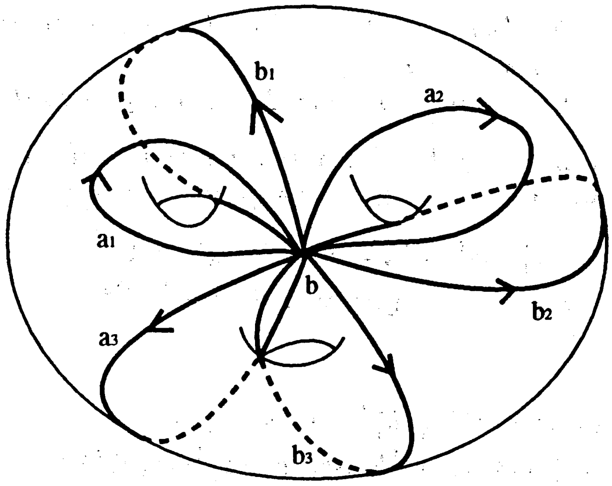


Figure 1

They satisfy the fundamental relation

$$(a_1b_1a_1^{-1}b_1^{-1})(a_2b_2a_2^{-1}b_2^{-1})(b_3a_3b_3^{-1}a_3^{-1}) = 1. \tag{1.3}$$

Let $\tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_7$ be the elements of $\mathrm{Aut}^+(\pi_1(X))$ defined by

$$\tilde{\varphi}_2(a_1) = a_2b_2^{-1}a_2^{-1}a_1^{-1}b_3^{-1}b_2 \qquad \tilde{\varphi}_2(b_1) = b_2^{-1}b_3b_1^{-1}a_2b_2a_2^{-1}$$
$$\tilde{\varphi}_2(a_2) = b_3^{-1}a_2^{-1} \qquad \tilde{\varphi}_2(b_2) = a_2b_3b_2^{-1}a_2^{-1}$$
$$\tilde{\varphi}_2(a_3) = a_2b_2^{-1}a_2^{-1}b_1^{-1}a_1^{-1}a_3a_2^{-1} \qquad \tilde{\varphi}_2(b_3) = a_2b_3a_2^{-1},$$

$$\tilde{\varphi}_3(a_1) = a_2b_3a_3^{-1}a_1a_2b_2a_2^{-1} \qquad \tilde{\varphi}_3(b_1) = a_2b_2^{-1}a_2^{-1}a_1^{-1}a_3a_1a_2b_2a_2^{-1}$$
$$\tilde{\varphi}_3(a_2) = a_3^{-1}a_1b_1a_1^{-1} \qquad \tilde{\varphi}_3(b_2) = a_1b_1^{-1}a_1^{-1}a_3a_2b_2^{-1}a_2^{-1}b_1a_1^{-1}$$
$$\tilde{\varphi}_3(a_3) = a_2b_2a_2b_2^{-1}a_2^{-1}b_1 \qquad \tilde{\varphi}_3(b_3) = a_1b_1^{-1}a_1^{-1}a_3a_2b_2^{-1}a_2^{-1}b_1,$$

$$\tilde{\varphi}_7(a_1) = b_1^{-1}a_1^{-1}a_3b_3^{-1}a_2^{-1} \qquad \tilde{\varphi}_7(b_1) = a_2b_3a_3^{-1}a_1a_2b_2a_2^{-1}b_3^{-1}a_2^{-1}$$
$$\tilde{\varphi}_7(a_2) = a_2b_2^{-1}a_2^{-1}a_1^{-1} \qquad \tilde{\varphi}_7(b_2) = a_1a_2b_2b_3a_3^{-1}$$
$$\tilde{\varphi}_7(a_3) = b_1^{-1}a_2b_2a_2^{-1}a_3^{-1}a_1b_1a_1^{-1} \qquad \tilde{\varphi}_7(b_3) = a_1a_2b_2a_3^{-1}a_1b_1a_1^{-1}.$$

Then, we have the following:

Theorem 1.1. (1) *The classes φ_i of $\tilde\varphi_i$ in* $\mathrm{Out}^+(\pi_1(X))$ *generate a subgroup* H *of* Γ_3, *which is isomorphic to* $PSL_2(\mathbb{F}_7)$.
(2) *Moreover, if X is the Klein curve, then H is conjugate to the image of ι.*

Proof. (1) First note that $H \neq \{1\}$, because the action of H on the homology group $H_1(X,\mathbb{Z})$ is not trivial. By direct computation using (1.3), we have

$$\begin{gathered}\tilde\varphi_2^2 = \tilde\varphi_3^3 = \tilde\varphi_7^7 = 1, \quad \tilde\varphi_2\tilde\varphi_3\tilde\varphi_7 = 1, \\ (\tilde\varphi_7\tilde\varphi_3\tilde\varphi_2)^4 = [\text{conjugation by } a_2b_2^{-1}a_2^{-1}b_1].\end{gathered} \tag{1.4}$$

For example,

$$\begin{aligned}\tilde\varphi_3^2\cdot b_3 =&(a_2^{-1}b_2a_2a_1a_3a_1^{-1}a_2^{-1}b_2^{-1}a_2)(a_3a_1^{-1}b_1^{-1}a_1)\\ &\times(a_1^{-1}b_1a_1a_3^{-1}a_2^{-1}b_2a_2b_1^{-1}a_1)(a_1^{-1}b_1a_1a_3^{-1})\\ &\times(b_1a_2^{-1}b_2^{-1}a_2b_2a_2)(a_2^{-1}b_3^{-1}a_3a_1^{-1}a_2^{-1}b_2^{-1}a_2)\\ &\times(a_2^{-1}b_2a_2a_1a_3^{-1}a_1^{-1}a_2^{-1}b_2^{-1}a_2)(a_2^{-1}b_2a_2a_1a_3^{-1}b_3a_2)\\ =&a_2^{-1}b_2a_2(a_1b_1a_2^{-1}b_2^{-1}a_2b_2b_3^{-1}a_3^{-1}b_3)a_2\\ =&a_2^{-1}b_2a_2b_1a_1a_3^{-1}a_2,\end{aligned}$$

hence

$$\begin{aligned}\tilde\varphi_3^3\cdot b_3 =&(a_3a_1^{-1}b_1^{-1}a_1)(a_1^{-1}b_1a_2^{-1}b_2^{-1}a_2a_3a_1^{-1}b_1^{-1}a_1)\\ &\times(a_1^{-1}b_1a_1a_3^{-1})(a_2^{-1}b_2a_2a_1a_3a_1^{-1}a_2^{-1}b_2^{-1}a_2)\\ &\times(a_2^{-1}b_2a_2a_1a_3^{-1}b_3a_2)(a_2^{-1}b_2^{-1}a_2^{-1}b_2a_2b_1^{-1})(a_1^{-1}b_1a_1a_3^{-1})\\ =&b_3.\end{aligned}$$

From (1.4) we obtain

$$\varphi_2^2 = \varphi_3^3 = \varphi_7^7 = \varphi_2\varphi_3\varphi_7 = (\varphi_7\varphi_3\varphi_2)^4 = 1 \tag{1.5}$$

in $\mathrm{Out}^+(\pi_1(X))$. Since (1.5) is the presentation of $PSL_2(\mathbb{F}_7)$ (see [CM, p. 96]), there is a surjective map

$$PSL_2(\mathbb{F}_7) \twoheadrightarrow H.$$

The group $PSL_2(\mathbb{F}_7)$ is simple, and the map is an isomorphism.
(2) To see that H is the automorphism group of a Riemann surface, it is enough to recall the Nielsen realization problem, which was positively solved in [Ker]:

Theorem of Kerckhoff . *For any finite subgroup G of Γ_g, there is a compact Riemann surface R of genus g such that*

$$G \subset \mathrm{Aut}(R) \subset \Gamma_g. \qquad \square$$

This theorem shows that there exists a Riemann surface R of genus 3 with $H \subset \mathrm{Aut}(R)$. On the other hand, $\#\mathrm{Aut}(R) \leq 168 = \#H$ by the Hurwitz inequality. Consequently $H = \mathrm{Aut}(R)$. It is classically known that the Klein curve is the unique compact Riemann surface of genus 3 such that $\mathrm{Aut}(R) \simeq PSL_2(\mathbb{F}_7)$. Thus we have proved Theorem 1.1. □

2 $\pi_1(X)$ as a subgroup of the triangle group of type $(2, 3, 7)$

In this section, we give a more elementary proof of Theorem 1.1. The outline is as follows: Let T be the triangle group with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$ defined below, and N its normal subgroup. Then, T (resp. N) has a fundamental domain Δ (resp. Λ) in the Poincaré unit disk. As was shown in [Kle], the Klein curve X can be realized by gluing the boundaries of Λ. The elements of T act on Λ, hence on X. This action induces an isomorphism $T/N \simeq \mathrm{Aut}(X)$. Moreover, N is isomorphic to $\pi_1(X)$. Because T acts on N by conjugation, T/N can be embedded in $\mathrm{Out}^+(\pi_1(X))$. In this way, we obtain the map ι in (1.1). First, we compute the elements of N corresponding to the generators of $\pi_1(X)$. Using this identification, we show that $\iota(T/N) = H$, which is equivalent to Theorem 1.1.

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ be the generators of $PSL_2(\mathbb{F}_7)$. Then $S^2 = T^7 = (ST^{-1})^3 = 1$. For the triangle group

$$T := \langle \gamma_2, \gamma_3, \gamma_7 | \gamma_2^2 = \gamma_3^3 = \gamma_7^7 = 1, \gamma_2\gamma_3\gamma_7 = 1 \rangle,$$

we define a group homomorphism

$$\varphi \colon T \to PSL_2(\mathbb{F}_7)$$

by $\varphi(\gamma_2) = S, \varphi(\gamma_3) = ST^{-1}, \varphi(\gamma_7) = T$. Clearly φ is surjective. The map φ gives an exact sequence

$$(2.1) \qquad 1 \to N \to T \to PSL_2(\mathbb{F}_7) \to 1,$$

where $N := \ker\varphi \simeq \pi_1(X)$ is the kernel of φ. Hence we have $G \simeq T/N$. For any element $\hat{\alpha}$ of N, we shall denote by α the loop with base point b representing $\hat{\alpha}$. First, we give the elements of N corresponding to the canonical generators of $\pi_1(X,b)$. Note that, for two elements $\hat{\alpha}, \hat{\beta} \in N$, their product $\hat{\alpha}\hat{\beta} \in N$ corresponds to the loop $\beta\alpha$.

Proposition 2.1. *Define $\hat{a}_i, \hat{b}_i \in N, i = 1,2,3$ by*

$$\begin{aligned}
\hat{a}_1 &= \gamma_7\gamma_3^{-1}\gamma_7^{-3}\gamma_2\gamma_7^2(\gamma_3\gamma_2\gamma_7)^4\gamma_7^{-2}\gamma_2\gamma_7^3\gamma_3\gamma_7^{-1}\\
\hat{b}_1 &= \gamma_7\gamma_3^{-1}\gamma_7^{-3}\gamma_2\gamma_7^2(\gamma_2\gamma_7\gamma_3)^4\gamma_7^{-2}\gamma_2\gamma_7^3\gamma_3\gamma_7^{-1}\\
\hat{a}_2 &= \gamma_2\gamma_7^{-4}\gamma_2\gamma_7^{-4}\gamma_2\gamma_7^2(\gamma_3\gamma_2\gamma_7)^4\gamma_7^{-2}\gamma_2\gamma_7^4\gamma_2\gamma_7^4\gamma_2\\
\hat{b}_2 &= \gamma_2\gamma_7^{-4}\gamma_2\gamma_7^{-4}\gamma_2\gamma_7^2(\gamma_2\gamma_7\gamma_3)^4\gamma_7^{-2}\gamma_2\gamma_7^4\gamma_2\gamma_7^4\gamma_2\\
\hat{a}_3 &= \gamma_3\gamma_7^{-2}\gamma_2\gamma_7^{-4}\gamma_3(\gamma_7^{-1}\gamma_2\gamma_3^{-1})^4\gamma_3^{-1}\gamma_7^4\gamma_2\gamma_7^2\gamma_3^{-1}\\
\hat{b}_3 &= \gamma_3\gamma_7^{-2}\gamma_2\gamma_7^{-4}\gamma_3(\gamma_2\ \gamma_7\ \gamma_3)^4\gamma_3^{-1}\gamma_7^4\gamma_2\gamma_7^2\gamma_3^{-1}.
\end{aligned}$$

Set $\hat{a}'_3 = \hat{a}_3^{-1}\hat{b}_3$, $\hat{b}'_3 = \hat{a}_3$. Then the elements $\hat{a}_1, \hat{a}_2, \hat{a}'_3, \hat{b}_1, \hat{b}_2, \hat{b}'_3$ are identified with the canonical generators of $\pi_1(X)$ and they satisfy the equation $[\hat{a}'_3, \hat{b}'_3]\ [\hat{a}_2, \hat{b}_2]\ [\hat{a}_1, \hat{b}_1] = 1$. Here $[\alpha, \beta] := \beta^{-1}\alpha^{-1}\beta\alpha$.

Proof. Let Δ (resp. Λ) be the fundamental domain of T (resp. N). Figure 2 below illustrates that Δ is a hyperbolic triangle with angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{7}$ and Λ is the union of 168 copies of Δ. By tracing paths, we can easily see that the elements $\hat{a}_i, \hat{b}_i$ in the figure can be written as above.

By gluing corresponding edges, we obtain the Riemann surface X. The elements $\hat{a}_i, \hat{b}_i$ are represented by the loops a_i, b_i in Figure 1. We can also check the fundamental relation by computation. □

The conjugation gives the canonical map

$$\tilde{\iota}\colon T \to \mathrm{Aut}^+(N).$$

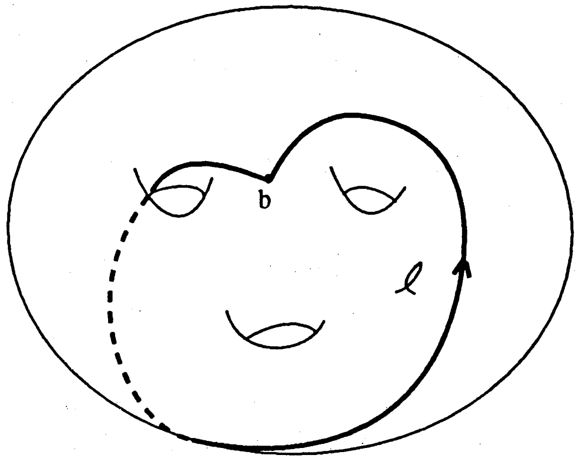
This induces the map ι in (1.1). We take $\tilde{\iota}(\gamma_2), \tilde{\iota}(\gamma_3), \tilde{\iota}(\gamma_7)$ as the generators of $\iota(T/N)$.

The following proposition finishes the direct proof of Theorem 1.1.

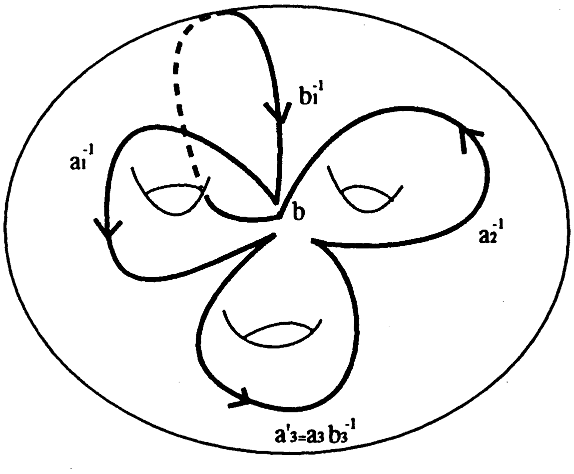
Proposition 2.2. *$\tilde{\iota}(\gamma_j)$ can be identified with $\tilde{\varphi}_j$ for $j = 1,2,3$.*

Proof. Set $\gamma_j \cdot \alpha := \tilde{\iota}(\gamma_j)(\alpha) = \gamma_j \alpha \gamma_j^{-1}$ for $\alpha \in N$. Then, for $\hat{a}_i, \hat{b}_i$ in Proposition 2.1, we can describe $\gamma_j \cdot \hat{a}_i, \gamma_j \cdot \hat{b}_i \in \Lambda$ as in the Figure 3.

We shall show that $\tilde{\varphi}_7(a_1)$ represents $\tilde{\iota}(\gamma_7)(\hat{a}_1)$. By gluing the edges of Λ, we get the following loop ℓ representing $\gamma_7 \cdot \hat{a}_1$.



We can check that ℓ is homotopic to the loop below, which is the loop $b_1^{-1} a_1^{-1} a_3 b_3^{-1} a_2^{-1}$.



The proofs for the other cases are similar and omitted. □

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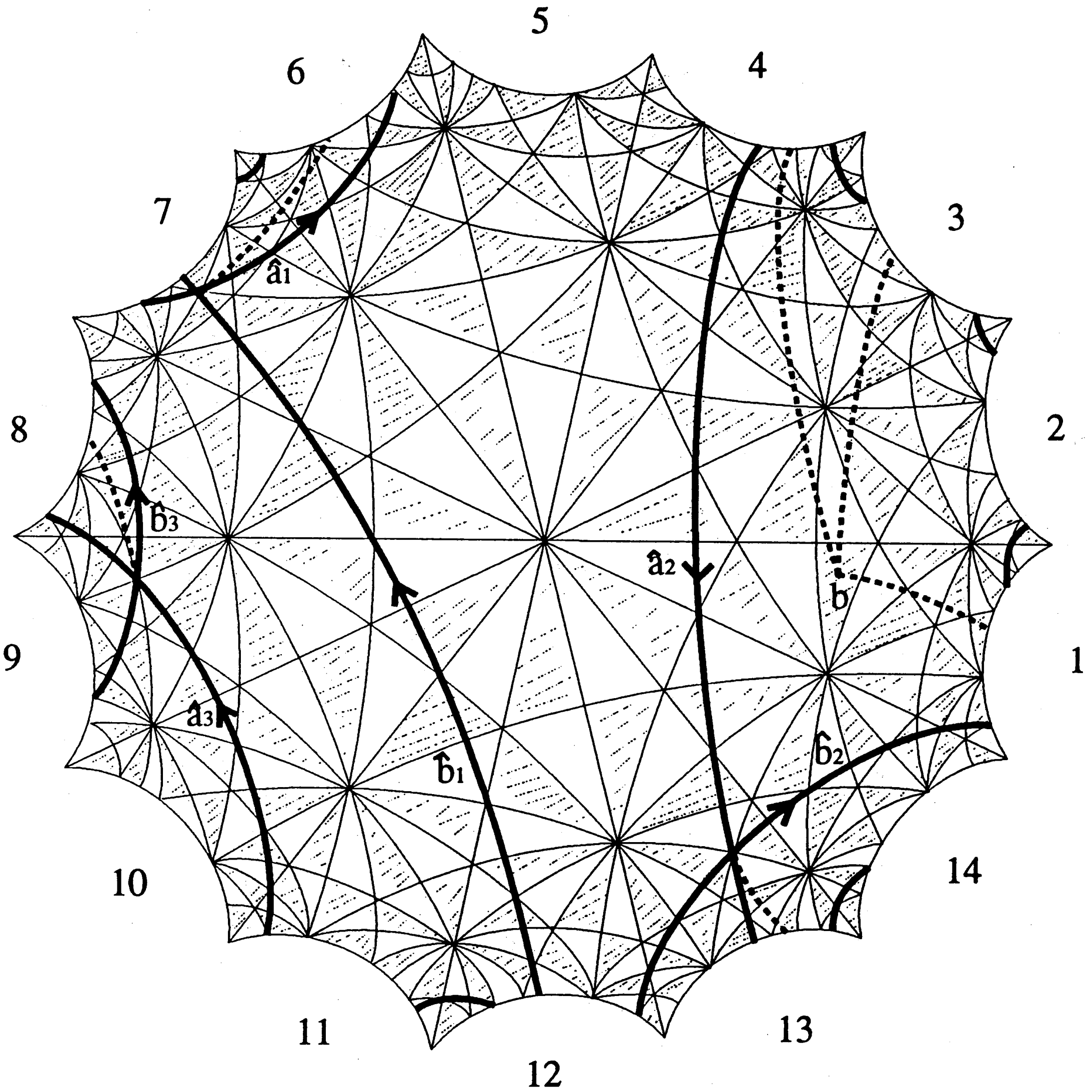

Figure 2: Fundamental domain Λ of N ([Kle, p. 126])

Glue 1=6, 7=12, 2=11, 3=8, 5=10, 4=13, 9=14 in this order. Each loop is connected to the base point b by dotted path.

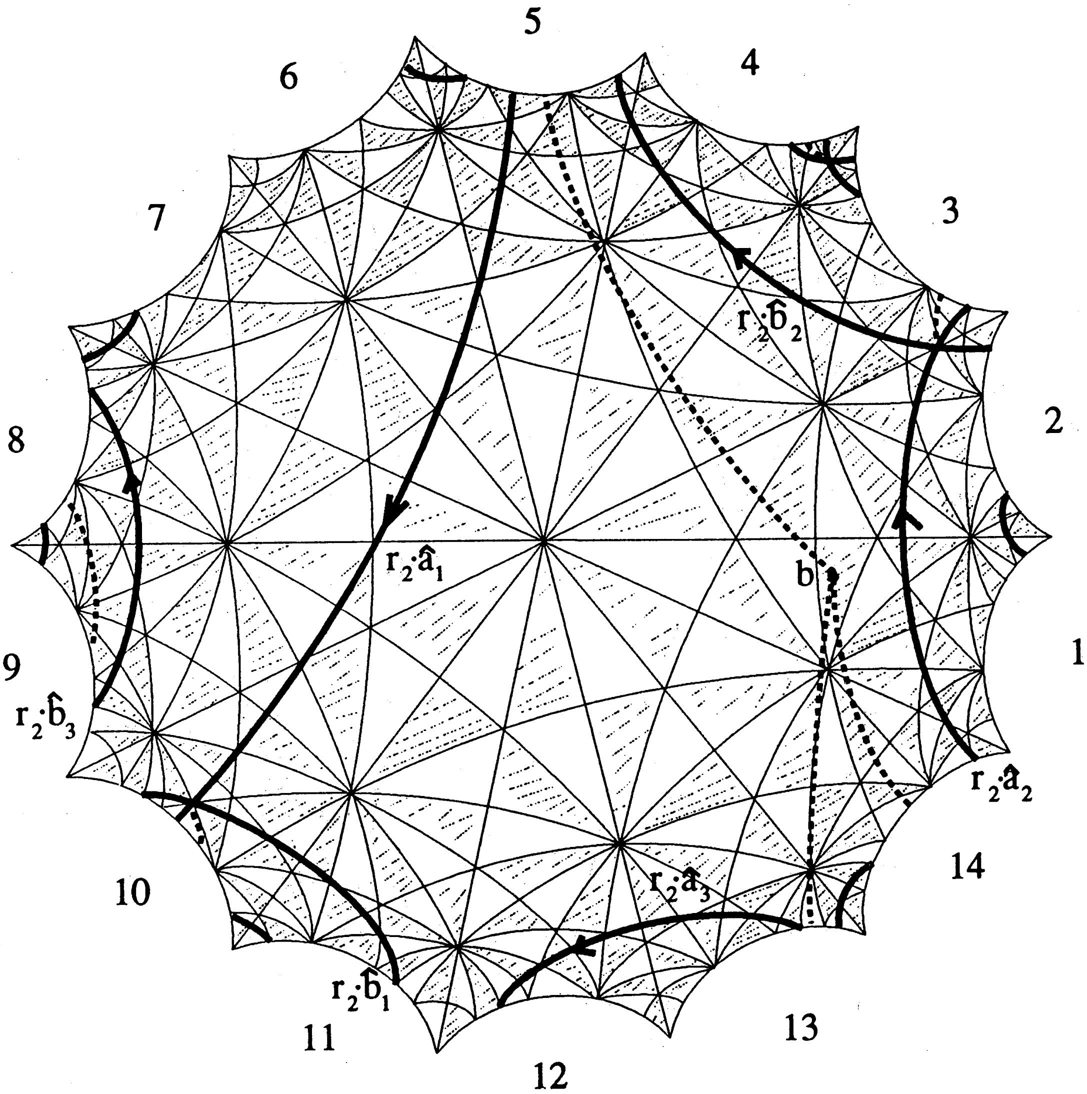

Figure 3-(i): Action of γ_2

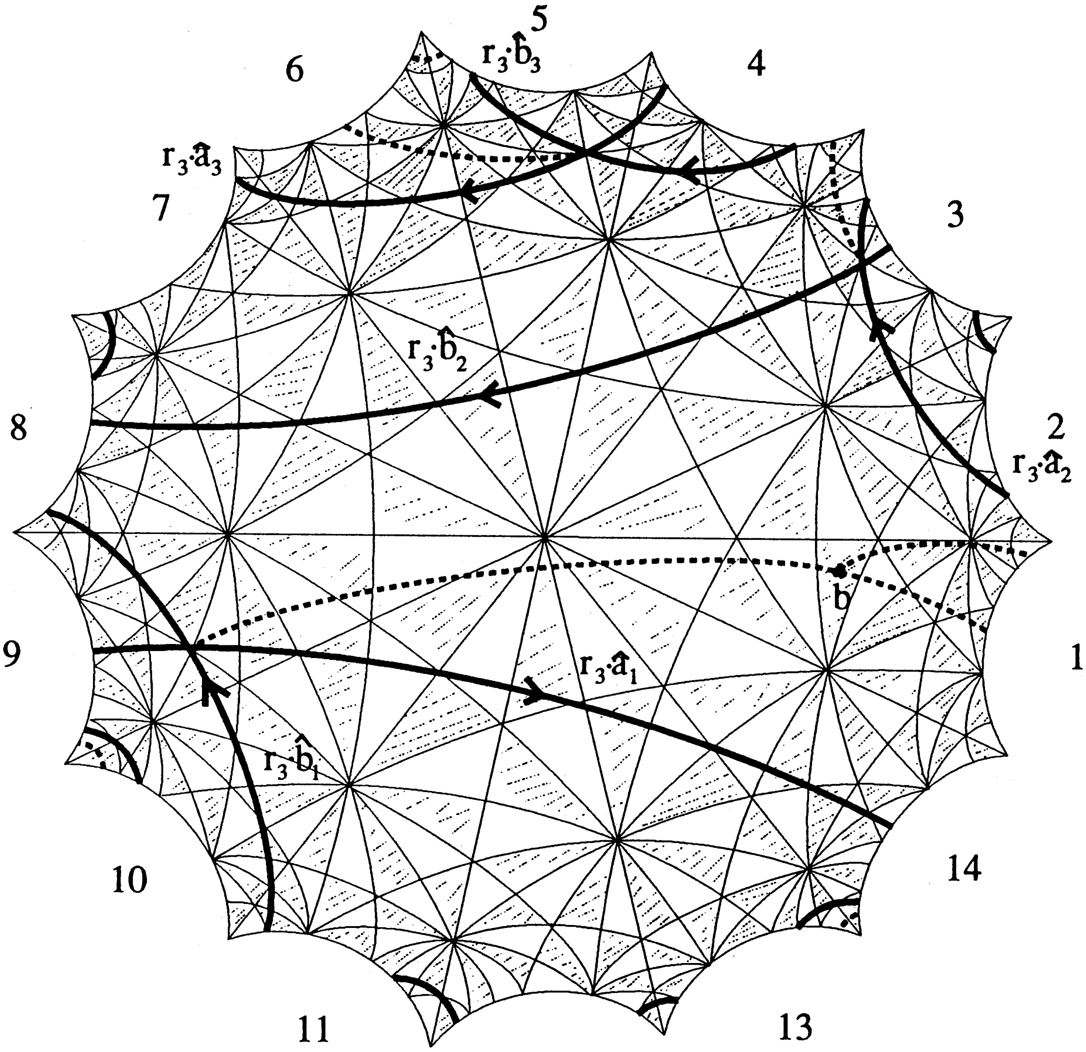

Figure 3-(ii): Action of γ_3

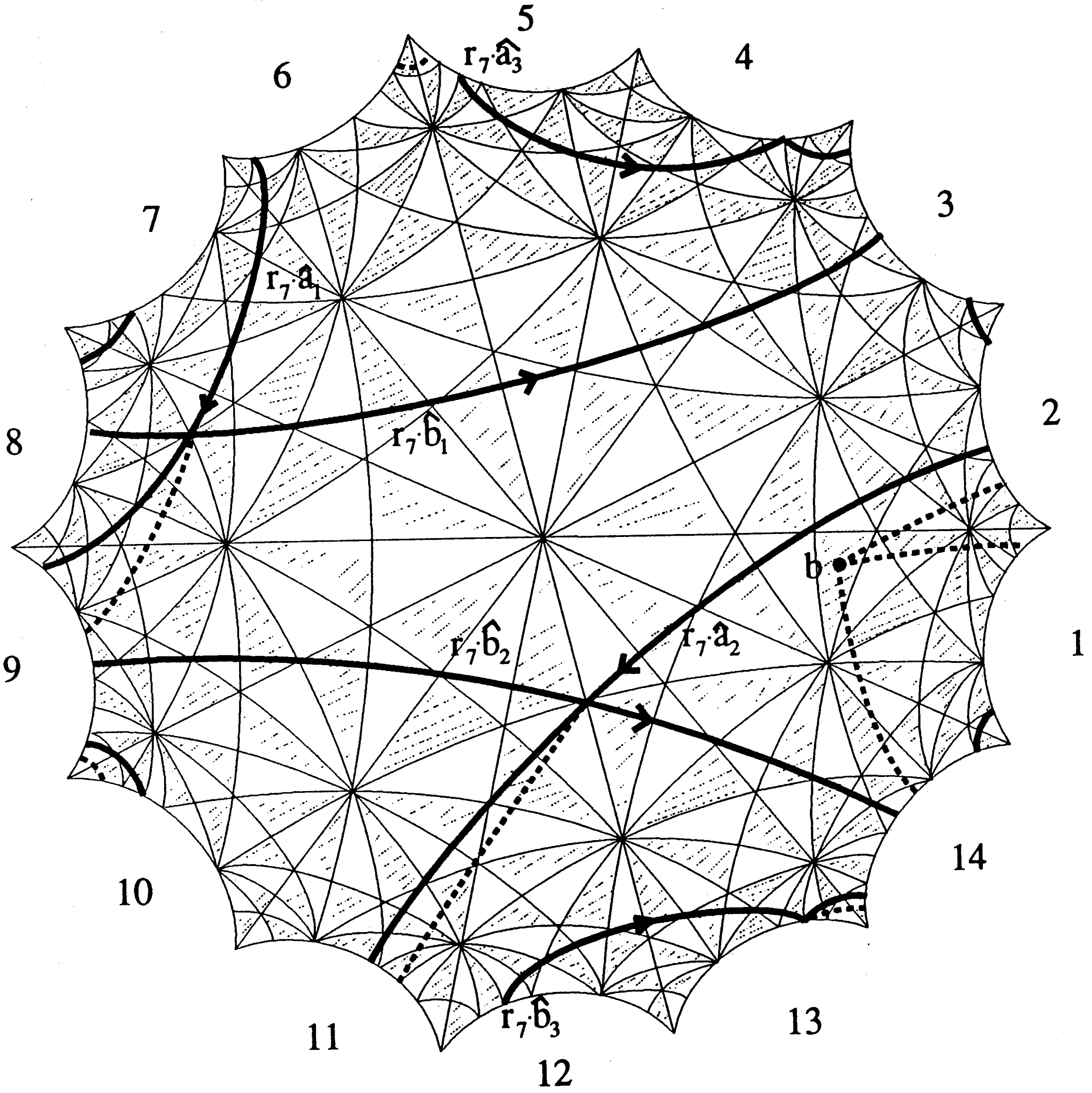

Figure 3-(iii): Action of γ_7