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MOURRE THEORY FOR TIME–PERIODIC SYSTEMS

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We consider the following Schrödinger equation with time-dependent Hamiltonian on $\mathbb{R}^{\nu}$,

$$i\frac{\partial}{\partial t}u(t,x)=H(t)u(t,x), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^{\nu}, \tag{1}$$

$$H(t)=-\Delta_{x}+V(t), \tag{2}$$

where $V(t)$ is a multiplicative operator by a function $V(t,x)$ which is periodic in t with period 2π:

$$V(t+2\pi,x)=V(t,x). \tag{3}$$

As is well-known, with some suitable conditions on $V(t,x)$, $H(t)$ generates a unique unitary propagator $\{U_{1}(t,s)\}_{-\infty<t,s<\infty}$. For $H_{0}=-\Delta_{x}$, the associated unitary propagator is denoted by $U_{0}(t,s)=e^{-i(t-s)H_{0}}$. A traditional way to study the temporal asymptotics as $t\rightarrow\pm\infty$ of $U_{1}(t,s)$ is to introduce an operator $K=-i\frac{d}{dt}+H(t)$ on $\mathbb{T}\times\mathbb{R}^{\nu}$, where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, and to investigate the asymptotic behavior of $e^{-i\sigma K}$. They are mutually related through the following formula

$$(e^{-i\sigma K}f)(t,x)=(U_{1}(t,t-\sigma)f(t-\sigma,\cdot))(x), \tag{4}$$

for $f\in\mathbb{H}=L^{2}(\mathbb{T}\times\mathbb{R}^{\nu})$. Let

$$K_{0}=-i\frac{d}{dt}+H_{0}. \tag{5}$$

Definition 1 (conjugate operator).

$$A=\frac{1}{2}(L_{D}\cdot x+x\cdot L_{D}) \tag{6}$$

where $D_{x}=\frac{1}{i}\nabla_{x}$ *and* $L_{D}=(L_{j})_{1\leq j\leq\nu}$ *with* $L_{j}=D_{x_{j}}<D_{x}>^{-2}$.

The following assumption is imposed on $V(t)$.

Assumption 1. *Let V be the operator of multiplication by the function $V(t,x)$ on $\mathbb{H}$. We assume that*

(i) V, $[V,A]$ *are extended to K_{0}-compact operators.*

(ii) $[[V,A],A]$ *is extended to a K_{0}-bounded operator.*

We denote the extension of the form $[K, A]$ as $[K, A]^0$.

Theorem 1. *Suppose Assumption* 1 *is satisfied. For* $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, *let* $d(\lambda, \mathbb{Z})$ *denote the distance from* λ *to* $\mathbb{Z}$. *Then, Eigenvalues of* K(*the set of which are denoted by* $\sigma_{pp}(K)$) *are discrete with possible accumulation points in* $\mathbb{Z}$. *If* $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, *for each* $\epsilon > 0$ *there exists* $0 < \delta < d(\lambda, Z)$ *such that*

(7) $$f(K)i[K, A]^0 f(K) \geq \left(\frac{2d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} - \epsilon\right) f(K)^2$$

for all $f \in C_0^\infty([\lambda - \delta, \lambda + \delta])$.

Let $\mathfrak{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$.

Theorem 2. *Suppose* $\alpha > 1/2$.

(i) *For each closed interval* $I \subset \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$ *the following inequalities hold:*

(8) $$\sup_{Imz \neq 0, Rez \in I} \| < x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty.$$

(ii) *There exist the norm limits in* $\mathfrak{B}(\mathbb{H})$.

$$\lim_{Imz \to 0, Rez \in I} < x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha}.$$

$< x >^{-\alpha} (K - \lambda \mp i0)^{-1} < x >^{-\alpha}$
is Hölder continuous with respect to $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$.

Next we proceed to the propagation estimates. We need the following stronger assumption on the potential.

Assumption 2. *There exists* $\delta_0 > 0$ *such that*

(9) $$V(t, \cdot) \in C(\mathbb{T}; C^\infty(\mathbb{R}^\nu)), \quad |\partial_x^\alpha V(t, x)| \leq C_\alpha < x >^{-\delta_0 - |\alpha|}, \quad {}^\forall \alpha.$$

Theorem 3. *Suppose Assumption* 2 *is satisfied. Let* $E \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, *and* $\epsilon > 0$ *be given. Then there exists a small open interval* I *containing* E *such that for any* $f \in C_0^\infty(I)$ *and* $s' > s > 0$,

(10)
$$\|\chi\left(\frac{|x|^2}{4\sigma^2} - \frac{d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} < -\epsilon\right) e^{-i\sigma K} f(K) < x >^{-s'} \|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad as \quad \sigma \to \infty$$

where $\chi(x < a)$ *denotes the characteristic function of the interval* $(-\infty, a)$.