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ON AUTOMORPHISMS OF GENERALIZED CUNTZ ALGEBRAS

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1. X-APERIODICITY

Definition 1.1. *Let X be a full right Hilbert $B-$bimodule of finite type. The C^*-algebra B is called to be X-aperiodic if for a non-zero positive element b of B, there exists $\{x_{i,j}\} \subset X, j=1,2,\ldots,m_i, i=1,2,\ldots,l$ such that*

$$\sum_{i(1),\ldots,i(l)}^{m_1,\ldots,m_l} < x_{i(l),l}, \cdots < x_{i(2),2} < x_{i(1),1,}, bx_{i(1),1,} >_B x_{i(2),2,} >_B \ldots x_{i(l),l,} >_B \tag{1.1}$$

is invertible.

Note that, by functional calculus, the above equations (1.1) may be equal to an identity operator for the definition of X-aperiodicity. It is defined in [3] that B is X-simple if any non-zero X-invariant ideal J of B (i.e. $< x, Jy >_B \subset J$ for $x,y \in X$) must be the whole space B. It is clear that X-aperiodicity implies X-simplicity. Let α be an automorphism of B and its associated imprimitivity Hilbert $B-$bimodule ${}_\alpha B$ is B as a vector space with

$$a \cdot x \cdot b = \alpha(a)xb, \qquad {}_B< x,\ y >= \alpha^{-1}(xy^*), \quad < x,\ y >_B = x^*y \tag{1.2}$$

for $a,b \in B$ and $x,y \in {}_\alpha B$. We note that the unital C^*-algebra B is ${}_\alpha B-$aperodic if and only if B is simple (see Theorem 1.3). The notion of X-simple is related with its irreducible adjacent matrix in the case that B is finite abelian. The one of X-aperiodic is just related with its aperiodic adjacent matrix as follows.

Let X be full right Hilbert B-bimodule with finite dimensional abelian C^*-algebra B. Let Σ be a finite set such that $C(\Sigma) = B$ and $\{p_\tau\}_{\tau\in\Sigma}$ be all minimal projections of $C(\Sigma)$. as in [5] . We denote a matrix M by $(a_{\sigma,\tau})_{\sigma.\tau\in\Sigma}$ where $a_{\sigma,\tau} = \dim_{\mathbb{C}} p_\sigma X p_\tau$. Let $\{\xi_{\sigma,\tau,l} \in X : \sigma,\tau \in \Sigma$ with $a_{\sigma,\tau} \geq 1, 1 \leq l \leq a_{\sigma,\tau}\}$ be a basis of vector space X :

$$\begin{cases} p_{\sigma'}\xi_{\sigma,\tau,l} = \delta_{\sigma',\sigma}\xi_{\sigma,\tau,l}, \\ \xi_{\sigma,\tau,l}p_{\tau'} = \delta_{\tau,\tau'}\xi_{\sigma,\tau,l}, \\ < \xi_{\sigma,\tau,l},\ \xi_{\sigma',\tau',l'} >_B = \delta_{\sigma',\sigma}\delta_{\tau,\tau'}\delta_{l,l'}p_\tau. \end{cases} \tag{1.3}$$

We note that $\{\xi_{\sigma,\tau,l}\}_{\sigma,\tau,l}$ is right $B-$basis. We set $\xi_\sigma := \sum_{\tau,l}\xi_{\sigma,\tau,l}$, then

$$< \xi_\sigma,\ \xi_{\sigma'} >_B = \delta_{\sigma,\sigma'}\sum_\tau a_{\sigma,\tau}p_\tau. \tag{1.4}$$

Therefore we have

$$\sum_{\sigma(1)} < \xi_{\sigma(1)},\ p_\sigma\xi_{\sigma(1)} >_B = \sum_\tau a_{\sigma,\tau}p_\tau. \tag{1.5}$$

Proposition 1.2. *Let X be as above. The finite dimensional abelian C^*-algebra B is X-aperiodic if and only if the matrix M is aperiodic (i.e. there exists integer m such that $M^m(\sigma,\tau) > 0$ for all $\sigma,\tau \in \Sigma$ where $M^m(\sigma,\tau)$ is (σ,τ)-component of the matrix M^m).*

Proof. By (1.5), we have

$$\sum_{\sigma(1),\ldots,\sigma(m)} < \xi_{\sigma(m)}, \cdots < \xi_{\sigma(2)}, < \xi_{\sigma(1)}, p_\sigma \xi_{\sigma(1)} >_B \xi_{\sigma(2)} >_B \cdots \xi_{\sigma(m)} >_B$$
$$= \sum_\tau M^m(\sigma,\tau)\, p_\tau.$$

If M is aperiodic, then

$$\sum_{\sigma(1),\ldots,\sigma(m)} < \xi_{\sigma(m)}, \cdots < \xi_{\sigma(2)}, < \xi_{\sigma(1)}, p_\sigma \xi_{\sigma(1)} >_B \xi_{\sigma(2)} >_B \cdots \xi_{\sigma(m)} >_B$$

is invertible. Since B is finite dimensional, the C^*-algebra B is X-aperiodic.

Conversely for $x = \sum_{\sigma,\tau,l} c_{\sigma,\tau,l}\xi_{\sigma,\tau,l} \in B, c_{\sigma,\tau,l} \in \mathbb{C}$, by (1.3) we have

$$< x, p_\sigma x >_B = \sum_{\tau,l,a_{\sigma,\tau}\neq 0} |c_{\sigma,\tau,l}|^2 p_\tau.$$

If the equation (1.1) holds, for $\sigma,\tau \in \Sigma$, there exists $\{\tau(i)\}_{i=1}^m \subset \Sigma$ such that $a_{\tau(i),\tau(i+1)} \neq 0$ for $i = 1,2,\ldots,m$, $\tau(1) = \sigma, \tau(m) = \tau$. Therefore $M^m(\sigma,\tau) > 0$ which implies that M is aperiodic. □

Let $\mathcal{F}_m(X)$ be a relative tensor product $\overbrace{X \otimes_B X \otimes_B \cdots \otimes_B X}^{m\text{-times}}$ for a full right Hilbert B-bimodule X and $\mathcal{F}_m$ is a C^*-subalgebra of $\mathcal{O}_X$ generated by

$$\{S_{x_1\otimes x_2\cdots\otimes x_m} S^*_{y_1\otimes y_2\cdots\otimes y_m} : x_1 \otimes x_2 \cdots \otimes x_m, \quad y_1 \otimes y_2 \cdots \otimes y_m \in \mathcal{F}_m(X)\}.$$

There exists a unital isomorphism $\psi_m : K_B(\mathcal{F}_m(X)_B) \longmapsto \mathcal{F}_m$ such that:

$$\psi_m(\theta_{x_1\otimes x_2\cdots\otimes x_m,\ y_1\otimes y_2\cdots\otimes y_m}) = S_{x_1\otimes x_2\cdots\otimes x_m} S^*_{y_1\otimes y_2\cdots\otimes y_m}$$

for finite rank operators $\theta_{x_1\otimes x_2\cdots\otimes x_m,\ y_1\otimes y_2\cdots\otimes y_m} \in K_B(\mathcal{F}_m(X)_B)$. Since X is of finite type, we have

$$\sum_{i=1}^n S_{u_i} S^*_{u_i} = 1 \quad \text{and} \quad \mathcal{F}_m \subset \mathcal{F}_{m+1}.$$

We set $\mathcal{F}_X := \overline{\cup_{m=1}^\infty \mathcal{F}_m}$. Moreover $\mathcal{F}_X$ is the fixed point algebra ${\mathcal{O}_X}^{\mathbb{T}}$ for the gauge action. We define a complete positive map $\sigma : \mathcal{O}_X \longmapsto \mathcal{O}_X$ by

$$\sigma(T) = \sum_{i=1}^n S_{u_i} T {S_{u_i}}^* \tag{1.6}$$

for $T \in \mathcal{O}_X$. In [3] Lemma 7.8, it is proved that the restriction of σ on $B' \cap \mathcal{O}_X$ is a unital isometric *-homomorphism and it does not depend on the choice of B-basis .Moreover $\sigma^m(T)$ commutes with $\mathcal{F}_m$ for $T \in B' \cap \mathcal{O}_X$. There is an isomorphism $\pi_m : \mathcal{F}_m \longmapsto (B \otimes M_n)_{P_m}$ such that, for $x \in \mathcal{F}_m$,

$$\begin{cases} x = \sum_{\substack{i(1),\ldots,i(m)\\ j(1),\ldots,j(m)}} S_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}} b_{i(1),\ldots,i(m),j(1),\ldots,j(m)} S^*_{u_{j(1)}\otimes u_{j(2)}\cdots\otimes u_{j(m)}} \\ \pi_m(x) = \big(b_{i(1),\ldots,i(m),j(1),\ldots,j(m)}\big) \in (B\otimes M_n)_{P_m} \end{cases} \tag{1.7}$$

where the projection P_m is

$$\big(< u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)},\quad u_{j(1)}\otimes u_{j(2)}\cdots\otimes u_{j(m)} >_B\big).$$

We note that if X is a Hilbert $B-$bimodule ([4]Definition 1.3), there exists a conditional expectation E_m from $\mathcal{F}_X$ onto $\mathcal{F}_m$ such that

$$\begin{cases} E_m = \lim_{k\to\infty} E_m^{m+k} \\ E_m^{m+k}(\theta_{x_1\otimes y_1,\ x_2\otimes y_2}) = \theta_{x_1 {}_B< y_1,\ y_2>, x_2} \end{cases} \tag{1.8}$$

for $x_1, x_2 \in \mathcal{F}_m(X), y_1, y_2 \in \mathcal{F}_k(X)$ ([4]Lemma 3.24, 3.25).

Theorem 1.3. *Let X be a full right Hilbert B-bimodule. The the C^*-algebra $\mathcal{F}_X$ is simple if and only if B is $X-$aperioddic.*

Proof. Let J be a non-zero closed ideal of of $\mathcal{F}_X$. Set $J_m := \mathcal{F}_m \cap J$ and $J = \overline{\cup_{m=1}^{\infty} J_m}$. Then for a non-zero element $x \in J_m$ for some m, there exists an element

$$\big(b_{i(1),\ldots,i(m),j(1),\ldots,j(m)}\big) \in (B\otimes M_n)_{P_m}$$

satisfying the relation (1.7). If necessary , consider x^*x instead of x, and we may assume that there is $(k(1),\ldots k(m))$ such that

$$b := b_{k(1),\ldots k(m),\ k(1),\ldots k(m)}$$

is a non-zero positive element of B. Suppose that B is X-aperiodic, and we choose the elements $\{x_{i,j}\}$ of X satisfying the relation (1.1) for b. We take $y_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)}$ of $\mathcal{F}_{m+l}$:

$$\begin{cases} y_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} =: \\ \sum_{k(1),\ldots,k(m)} S_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} S^*_{x_{i(1),1}\otimes\cdots\otimes x_{i(l),l}} S^*_{u_{k(1)}\otimes u_{k(m-1)}\cdots\otimes u_{k(m)}}. \end{cases} \tag{1.9}$$

Since $\pi_m(x)P_m = P_m\pi_m(x) = \pi_m(x)$, we compute

$$\begin{aligned} & y_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} b y^*_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} \\ =& S_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} < x_{i(1),1}\otimes\cdots\otimes x_{i(l),l},\ b x_{i(1),1}\otimes\cdots\otimes x_{i(l),l} >_B S^*_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} \end{aligned}$$

and

$$\begin{aligned} & \sum_{i(1),\ldots,i(l)} y_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} b y^*_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} \\ =& S_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} \\ & \times \sum_{i(1),\ldots,i(l)} < x_{i(1),1}\otimes\cdots\otimes x_{i(l),l},\ b x_{i(1),1}\otimes\cdots\otimes x_{i(l),l} >_B S^*_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}}. \end{aligned}$$

Since there is a positive number $\lambda \in \mathbb{R}$ such that

$$\sum_{i(1),\ldots,i(l)} < x_{i(1),1}\otimes\cdots\otimes x_{i(l),l},\ b x_{i(1),1}\otimes\cdots\otimes x_{i(l),l} >_B \geq \lambda I,$$

we have

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{i(1),\ldots,i(l)\\ s(1),\ldots,s(m+l)}} y_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} b y^*_{i(1),\ldots,i(l),s(1),\ldots,s(m+l)} \\ \displaystyle\geq \lambda \sum_{s(1),\ldots,s(m+l)} S_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} S^*_{u_{s(1)}\otimes u_{s(2)}\cdots\otimes u_{s(m+l)}} = \lambda I \end{array}\right. \quad (1.10)$$

Thus J_{l+m} contains the above invertible element. We conclude that the ideal J is B.

Convesely we assume that $\mathcal{F}_X$ is simple. Since B is unital and X is full, by [3]Proposition15, there is a finite set $\{x_i\} \subset X$ such that

$$\sum_i < x_i,\ x_i >_B = I. \quad (1.11)$$

For a non-zero element $b \in B$, we consider a closed ideal $\overline{\cup_{m=1}^{\infty} \mathcal{F}_m b \mathcal{F}_m}$ of $\mathcal{F}_X$. we can choose a finite subset $\{f_i\}_{i=1}^l \subset \mathcal{F}_m$ with $\sum_{i=1}^l f_i^* b f_i = I$ for some m. The element f_i of $\mathcal{F}_m$ is of the form:

$$f_i = \sum_{k=1}^{nm} S_{z^i_{1,k}\otimes\cdots\otimes z^i_{m,k}} S^*_{y^i_{1,k}\otimes\cdots\otimes y^i_{m,k}}.$$

Since an operator inequality :

$$(\sum_{i=1}^l T_i)^* S (\sum_{i=1}^l T_i) \leq l(\sum_{i=1}^l T_i^* S T_i)$$

holds, we obtain

$$\begin{aligned} I &= \sum_{i=1}^l f_i^* b f_i \\ &\leq nm \sum_{i,k} S_{y^i_{1,k}\otimes\cdots\otimes y^i_{m,k}} < z^i_{1,k}\otimes\cdots\otimes z^i_{m,k},\ b z^i_{1,k}\otimes\cdots\otimes z^i_{m,k} >_B S^*_{y^i_{1,k}\otimes\cdots\otimes y^i_{m,k}}. \end{aligned}$$

By (1.11), we have

$$\begin{aligned} I = & \sum_{i(1),\ldots,i(m)} S^*_{x_{i(1)}\otimes\cdots\otimes x_{i(m)}} I S_{x_{i(1)}\otimes\cdots\otimes x_{i(m)}} \\ \leq & \ nm \sum_{i(1),\ldots,i(m)} \sum_{i,k} < x_{i(1)}\otimes\cdots\otimes x_{i(m)},\ y^i_{1,k}\otimes\cdots\otimes y^i_{m,k} >_B \\ & \times < z^i_{1,k}\otimes\cdots\otimes z^i_{m,k},\ b z^i_{1,k}\otimes\cdots\otimes z^i_{m,k} >_B \\ & \times < y^i_{1,k}\otimes\cdots\otimes y^i_{m,k},\ x_{i(1)}\otimes\cdots\otimes x_{i(m)} >_B, \end{aligned}$$

which implies that B is X-aperiodic. □

2. Automorphisms of $\mathcal{O}_X$

Let θ be an automorphism of B and U be an invertible $\mathbb{C}$-linear map on the right Hilbert $B-$ bimodule X satisfying

$$< Ux,\ Uy >_B = \theta(< x,\ y >_B), \qquad U(bxb') = \theta(b)(Ux)\theta(b') \tag{2.1}$$

for $x, y \in X$ and $b, b' \in B$. This invertible operator U induces an automorphism α_U of $\mathcal{O}_X$ such that

$$\alpha_U(S_x) = S_{Ux}$$

for $x \in X$ We note that if the right Hilbert B-bimodule X is ${}_{\mathbb{C}}\mathbb{C}^n_{\mathbb{C}}$, then the U is a unitary operator on $\mathbb{C}^n$ and the automorphism α_U is the same as defined in [2]. It is remarked that the U is a unitary operator in ${}_B\mathcal{L}_B(X_B)$ if θ is trivial. At first, we give some results related with problems whether the restriction $\alpha_U|_{\mathcal{F}_X}$ on $\mathcal{F}_X$ for α_U is inner or not.

Proposition 2.1. *Let X be a right Hilbert B-bimodule of finite type and U be as (2.1). If the automorphism $\alpha_U|_{\mathcal{F}_X}$ is inner, then the restricted automorphism $\alpha_U|_{B'\cap\mathcal{F}_X}$ on the relative commutant $B' \cap \mathcal{F}_X$ must be trivial.*

Proof. Let $\alpha_U|_{\mathcal{F}_X}$ be of the form:

$$\alpha_U|_{\mathcal{F}_X} = \mathrm{Ad}V$$

for some $V \in \mathcal{F}_X$. For a right $B-$basis $\{u_i\}$ and $x \in X$, we get

$$\begin{aligned}\sum_i (Uu_i) < Uu_i,\ x >_B &= \sum_i (Uu_i)\theta(< u_i,\ U^{-1}x >_B) \\ &= \sum_i U(u_i < u_i,\ U^{-1}x >_B) = UU^{-1}x = x.\end{aligned}$$

Hence $\{Uu_i\}$ is also right B-basis. Since σ on $B' \cap \mathcal{F}_X$ does not depend on the choice of B-basis, we have

$$\alpha_U\sigma|_{B'\cap\mathcal{F}_X} = \sigma\alpha_U|_{B'\cap\mathcal{F}_X}. \tag{2.2}$$

Since $\sigma^m(T)$ for $T \in B' \cap \mathcal{F}_X$ commutes with $\mathcal{F}_m$ and σ is isometric *-homomorphism, we get

$$\begin{aligned}&\|\sigma_U(T) - T\| \\ &= \lim_{m\to\infty} \|\sigma^m\alpha_U(T) - \sigma^m(T)\| \\ &= \lim_{m\to\infty} \|\alpha_U\sigma^m(T) - \sigma^m(T)\| \\ &= \lim_{m\to\infty} \|V\sigma^m(T)V^* - \sigma^m(T)\| = 0.\end{aligned}$$

We conclude that $\alpha_U(T) = T$ for $T \in B' \cap \mathcal{F}_X$. □

Next under the some restricted condition, we shall prove that α_U is inner on $\mathcal{F}_X$ if and only $Ux = \lambda uxu^*$ for some unitary u of $B, \lambda \in \mathbb{T}$ and all $x \in X$.

Lemma 2.2. *Let X be a full Hilbert B-bimodule with $Z(B) = \mathbb{C}$ and U be the invertible operator in (2.1). If the automorphism α_U is of the form:*

$$\alpha_U(T) = AdV(T)$$

for some $V \in \mathcal{F}_X$ and all $T \in \mathcal{F}_X$, then the automorphism θ of B is inner, i.e. $\theta = Adu$ for a unitary u in B.

Proof. Let E_m be an expectation as in (1.8). Then, for sufficient large m, the invertible $E_m(V)$ satisfies

$$\alpha_U(T)E_m(V) = E_m(V)T$$

for $T \in \mathcal{F}_m$. By [3] Lemma 1.6, this operator $E_m(V)$ is scalar multiple of a unitary $V_m \in \mathcal{F}_m$ such that $\alpha_U(T) = \mathrm{Ad}V_m(T)$ for $T \in \mathcal{F}_X$. We compute, for $T = S_{U^{-1}x_1 \otimes \cdots \otimes U^{-1}x_m} b S^*_{U^{-1}y_1 \otimes \cdots \otimes U^{-1}y_m}$,

$$\begin{aligned}
&S^*_{x_1 \otimes \cdots \otimes x_m} \alpha_U(T) S_{y_1 \otimes \cdots \otimes y_m} \\
=&S^*_{x_1 \otimes \cdots \otimes x_m} S_{x_1 \otimes \cdots \otimes x_m} \theta(b) S^*_{y_1 \otimes \cdots \otimes y_m} S_{y_1 \otimes \cdots \otimes y_m} \\
=& < x_1 \otimes \cdots \otimes x_m,\ x_1 \otimes \cdots \otimes x_m >_B \theta(b) < y_1 \otimes \cdots \otimes y_m,\ y_1 \otimes \cdots \otimes y_m >_B
\end{aligned}$$

and

$$\begin{aligned}
&S^*_{x_1 \otimes \cdots \otimes x_m} (V_m T V^*_m) S_{y_1 \otimes \cdots \otimes y_m} \\
=&\{S^*_{x_1 \otimes \cdots \otimes x_m} V_m S_{x_1 \otimes \cdots \otimes x_m}\} b \{S^*_{y_1 \otimes \cdots \otimes y_m} V^*_m S_{y_1 \otimes \cdots \otimes y_m}\}.
\end{aligned}$$

Since $\{S^*_{x_1 \otimes \cdots \otimes x_m} V_m S_{x_1 \otimes \cdots \otimes x_m}\}$ is an element of B, denoted by $d(x_1 \otimes \cdots \otimes x_m)$, we get

$$\begin{aligned}
&< x_1 \otimes \cdots \otimes x_m,\ x_1 \otimes \cdots \otimes x_m >_B \theta(b) < y_1 \otimes \cdots \otimes y_m,\ y_1 \otimes \cdots \otimes y_m >^*_B \\
=&d(x_1 \otimes \cdots \otimes x_m) b d(y_1 \otimes \cdots \otimes y_m)^*.
\end{aligned}$$

Since X is full , there exists a finite subset $\{z_i\}$ in X such that

$$\sum_i < z_i,\ z_i >_B = I.$$

Thus we get, for all $b \in B$,

$$\begin{aligned}
\theta(b) = &\sum_{\substack{i(1),\ldots,i(m) \\ j(1),\ldots,j(m)}} < z_{i(1)} \otimes \cdots \otimes z_{i(m)},\ z_{i(1)} \otimes \cdots \otimes z_{i(m)} >_B \theta(b) \\
&\times < z_{j(1)} \otimes \cdots \otimes z_{j(m)},\ z_{j(1)} \otimes \cdots \otimes z_{j(m)} >_B \\
=&ubu^*
\end{aligned}$$

where $u = \sum_{i(1),\ldots,i(m)} d(z_{i(1)} \otimes \cdots \otimes z_{i(m)})$. Therefore we conclude that the automorphism θ is implemented by the unitary u. □

If α_U is inner on $\mathcal{F}_X$, then by considering a perturbed operator U' on X by the unitary u such that $U'x = u^*(Ux)u$ for $x \in X$, we may assume that the invertible operator U is a unitary of ${}_B\mathcal{L}_B(x_B)$ and θ is trivial. The idea of the following lemma is borrowed from Cuntz [1]

Lemma 2.3. *Let U be a unitary of ${}_B\mathcal{L}_B(x_B)$. Then an operator W defined by:*

$$W = \sum_{i=1}^{n} S_{Uu_i} S^*_i \tag{2.3}$$

satisfies the statements:

1. W is independent of the choice for right B-basis $\{u_i\}$
2. W is a unitary operator of $B' \cap \mathcal{F}_1$ such that $\mathrm{Ad}W = \alpha_U$ on $\mathcal{F}_1$.

Moreover set $W_m := W\sigma(W)\ldots\sigma^{m-1}(W)$ and the W_m is a unitary operator of $B' \cap \mathcal{F}_m$ such that $\mathrm{Ad}W_m = \alpha_U$ on $\mathcal{F}_m$.

Proof. Let $\{v_j\}$ be another right B-basis for X. Then we have

$$u_i = \sum_j v_j < v_j,\ u_i >_B$$

and

$$\begin{aligned} W &= \sum_i S_{(U\sum_j v_j < v_j,\ u_i >_B)} S^*_{u_i} \\ &= \sum_{i,j} S_{Uv_j} < v_j,\ u_i >_B S^*_{u_i} \\ &= \sum_j S_{Uv_j} S^*_{v_j} \end{aligned}$$

Hence the operator W in $\mathcal{F}_1$ is independent of the choice for right B-basis. To show the unitarity of W, we compute

$$\begin{aligned} W^*W &= \sum_{i,j} S_{u_i} S^*_{Uu_i} S_{Uu_j} S^*_{u_j} \\ &= \sum_{i,j} S_{u_i} < Uu_i,\ Uu_j >_B S^*_{u_j} \\ &= \sum_{i,j} S_{u_i} < u_i,\ u_j >_B S^*_{u_j} = I \end{aligned}$$

and similarly we have

$$WW^* = \sum_i S_{Uu_i} S^*_{Uu_i} = I.$$

For $b \in B$, we calculate

$$\begin{aligned} bW &= \sum_i S_{Ubu_i} S^*_{u_i} \\ &= \sum_{i,j} S_{Uu_j < u_j,\ bu_i >_B} S^*_{u_i} \\ &= \sum_j S_{Uu_j} S^*_{\sum_i u_i < u_i,\ b^* u_j >_B} \\ &= \sum_j S_{Uu_j} S^*_{b^* u_j} = Wb. \end{aligned}$$

Therefore W is an element of $B' \cap \mathcal{F}_1$. Since

$$\begin{aligned} WS_x &= \sum_i S_{Uu_i} S^*_{u_i} S_x \\ &= \sum_i S_{Uu_i} < u_i,\ x >_B \\ &= S_{Ux} = \alpha_U(S_x) \end{aligned}$$

and $\mathcal{F}_1$ is generated by $\{S_x S_y^* : x, y \in X\}$, we obtain $\alpha_U = \mathrm{Ad}W$ on $\mathcal{F}_1$. Finally it is clear that W_m is a unitary of $B' \cap \mathcal{F}_m$ by the definition of W_m. Since $\{u_{i(1)} \otimes$

$\cdots \otimes u_{i(m)}\}$ is a right basis for $\overbrace{X \otimes_B \cdots \otimes_B X}^{m\text{-times}}$ and

$$W_m = \sum_{i(1),\dots,i(m)} S_{Uu_{i(1)}\otimes\cdots\otimes Uu_{i(m)}} S^*_{u_{i(1)}\otimes\cdots\otimes u_{i(m)}},$$

it follows from (2) that $\alpha_U = \mathrm{Ad}W_m$ on $\mathcal{F}_m$. □

Proposition 2.4. *Let X be a full Hilbert B-bimodule of finite type with a left inner product $_B< , >$ and the center $Z(B) = \mathbb{C}$. Then the automorphism $\alpha_U|_{\mathcal{F}_X}$ is inner if and only if $Ux = \lambda uxu^*$ for some unitary u of B some $\lambda \in \mathbb{C}, |\lambda| = 1$ and all $x \in X$ and the automorphism θ is of the form: $\theta = Adu$.*

Proof. The part of "if" is trivial.

The automorphism $\alpha_U|_{\mathcal{F}_X}$ is of the form: $\alpha_U|_{\mathcal{F}_X} = \mathrm{Ad}V$ for some unitary V in $\mathcal{F}_X$. By Lemma 2.2, we may assume that θ is trivial and U is a unitary of $_B\mathcal{L}_B(X_B)$ by perturbing U by a unitary u in B. It follows from Lemma 2.3 and $\alpha_U(\mathcal{F}_m) = \mathcal{F}_m$ that

$$\begin{aligned} E_m(V)T =& E_m(VT) = E_m(\alpha_U(T)V) \\ =& \alpha_U(T)E_m(V) = W_m T W_m^* E_m(V) \end{aligned}$$

for $T \in \mathcal{F}_m$ where E_m is the expectation in (1.8). By $Z(\mathcal{F}_m) \simeq Z(B)$ in [3]Lemma 16, the element $W_n^* E_m(V) \in Z(\mathcal{F}_m)$ is scalar λ_m. Since $\lim_{m\to\infty} E_m(T) = T$ for $T \in \mathcal{F}_X$, we have $\lim_{m\to\infty} |\lambda_m| = 1$ and

$$\begin{aligned} &\lim_{m\to\infty} \|\lambda_{m+1}^{-1}\lambda_m - W\| \\ =& \lim_{m\to\infty} \|\lambda_{m+1}^{-1}\lambda_m - \sigma^{m+1}(W)\| \\ =& \lim_{m\to\infty} \|\lambda_{m+1}^{-1}\lambda_m - W_m^* W_{m+1}\| \\ =& \lim_{m\to\infty} \|\lambda_m W_m - \lambda_{m+1} W_{m+1}\| = 0. \end{aligned}$$

Hence there exists $\lambda \in \mathbb{C}$ such that $W = \lambda I$. For $x \in X$, we obtain

$$\begin{aligned} \lambda S_x = W S_x =& \sum_i S_{Uu_i} S^*_{u_i} S_x \\ =& \sum_i S_{Uu_i} < u_i,\ x >_B = S_{Ux}. \end{aligned}$$

We conclude that $Ux = \lambda x$ for $x \in X$. □

Next we give some results related with problems whether the automorphism α_U on $\mathcal{O}_X$ is inner or not. The X-aperiodicity of B plays a crucial role in proving the outerness of its automorphism.

Theorem 2.5. *Let X be a full right Hilbert B-bimodule of finite type and C^*-algebra B is X-aperiodic. The automorphism α_U of $\mathcal{O}_X$ induced by the invertible operator U satisfying (2.1) is not inner if $B' \cap \mathcal{F}_X$ is not trivial and the restricted automorphism $\alpha_U|_{B'\cap\mathcal{F}_X}$ on $B' \cap \mathcal{F}_X$ is not trivial.*

Proof. Suppose that there is a unitary V in $\mathcal{O}_X$ such that

$$\alpha_U(T)V = VT$$

for $T \in \mathcal{O}_X$. By taking a consideration of a Fourier expansion $\{V_m\}_{m\in\mathbb{Z}}$ of V with respect to the gauge action, we have

$$\alpha_U(T)V_m = V_mT \tag{2.4}$$

for $T \in \mathcal{F}_X$ and $\alpha_t(V_m) = e^{-imt}V_m$. Its proof is divided into three cases:

(i) there is a positive integer m with $V_m \neq 0$
(ii) there is a negative integer $-m$ with $V_{-m} \neq 0$
(iii) $V_m = 0$ for all m except $m = 0$.

In the case (i), $V_m^*V_m$ and $V_mV_m^*$ are non-zero elements of $Z(\mathcal{F}_X)$. Since $\mathcal{F}_X$ is simple by Theorem 1.3 , $V_m^*V_m$ and $V_mV_m^*$ must be non-zero scalars. Hence we may assume that V_m is a unitary. The unitary V_m is of the form:

$$V_m = \sum_{i(1),\dots,i(m)} S_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}}\left\{S^*_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}}V_m\right\}$$

$$\in \sum_{i(1),\dots,i(m)} S_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}}\mathcal{F}_X.$$

Since $\sigma^m(T)$ for $T \in B' \cap \mathcal{F}_X$ commutes with $\mathcal{F}_m$ and $\mathcal{F}_X = \overline{\cup_{m=1}^\infty \mathcal{F}_m}$, for $\varepsilon > 0$, there is an integer $l_0 \in \mathbb{N}$ such that for $l > l_0$

$$\begin{aligned}&\|V_m\sigma^l(T) - \sigma^{l+m}(T)V_m\|\\ =&\|V_m\sigma^l(T) - \sum_{i(1),\dots,i(m)} S_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}}\sigma^l(T)S^*_{u_{i(1)}\otimes u_{i(2)}\cdots\otimes u_{i(m)}}V_m\| \leq \varepsilon\end{aligned}$$

for $T \in B' \cap \mathcal{F}_X$. By (2.2), we have for $l > l_0$

$$\begin{aligned}&\|\alpha_U(T) - \sigma^m(T)\|\\ =&\|\sigma^l(\alpha_U(T)) - \sigma^{l+m}(T)\|\\ =&\|\alpha_U(\sigma^l(T))V_m - \sigma^{l+m}(T)V_m\|\\ =&\|V_m\sigma^l(T) - \sigma^{l+m}(T)V_m\| \leq \varepsilon.\end{aligned}$$

Therefore we obtain $\alpha_U = \sigma^m$ on $B' \cap \mathcal{F}_X$. By the assumption:$B' \cap \mathcal{F}_X \neq \mathbb{C}$, take a non-scalar $T_0 \in B' \cap \mathcal{F}_X$. Since

$$T_0 = \alpha_U^{-1}\alpha_U(T_0) = \alpha_U^{-1}\sigma^m(T_0),$$

the operator T_0 commutes with $\mathcal{F}_m$. By an iteration, the operator T_0 is an element of $Z(\mathcal{F}_X)$. Since $\mathcal{F}_X$ is simple, the operator T_0 must be a scalar, which is a contradiction.

In the case (ii), the relation $\alpha_U(T)V_{-m} = V_{-m}T$ in (2.4) is equivalent to $\alpha_U^{-1}(T)V_{-m}^* = V_{-m}^*T$. Hence ,by the same way as in the case (i), we get $\alpha_U^{-1} = \sigma^m$ on $B' \cap \mathcal{F}_X$ and we get a contradiction similarly.

In the case (iii), It follows from Proposition 2.1 that α_U is not inner. □

We apply Theorem 2.5 to Cuntz-Krieger algebras.

Proposition 2.6 (Cuntz-Krieger Algebra). *Let X be a full right Hilbert $B-$bimodule of finite type* ($\dim_{\mathbb{C}} X > 1$) *and the finite dimensional abelian C^*-algebra B is X-aperiodic. The invertible operator U on X and the automorphism θ of B satisfy the relation (2.1). Then α_U is inner if and only if the operator U is of the form:*

$$Ux = uxu^* \qquad (x \in X) \tag{2.5}$$

for some $u \in B$ and the automorphism θ is trivial.

Proof. The part of "if" is clear.

We suppose that $\alpha_U = \mathrm{Ad}V$ for some unitary $V \in \mathcal{O}_X$. Since $B' \cap \mathcal{F}_X$ is always not trivial, by Theorem 2.5 and its proof, we may assume that the unitary V is an element of $\mathcal{F}_X$ and $\alpha_U|_{B'\cap\mathcal{F}_X}$ is trivial. It can be shown that the operator U in (2.1) is of the form:

$$U\xi_{\sigma,\tau,l} = \sum_k c_{\sigma,\tau}(l,k)\xi_{\theta(\sigma),\theta(\tau),k} \tag{2.6}$$

where $\mathcal{C}_{\sigma,\tau} := \left(c_{\sigma,\tau}(l,k)\right)_{l,k}$ are unitary matrices and $\{\xi_{\sigma,\tau,l}\}$ is the basis for X. Moreover the automorphism θ satisfies a relation:

$$a_{\sigma,\tau} = a_{\theta(\sigma),\theta(\tau)}$$

where $a_{\sigma,\tau}$ is the entries of the matrix M above (1.3). By Lemma 2.2, the automorphism θ must be trivial. By considering element of $B' \cap \mathcal{F}_1$:

$$S_{\xi_{\sigma,\tau,l}} p_\tau S^*_{\xi_{\sigma,\tau,k}}$$

for $\sigma, \tau \in \Sigma$ and $1 \le l, k \le a_{\sigma,\tau}$, we have

$$\begin{aligned}
&S_{\xi_{\sigma,\tau,l}} p_\tau S^*_{\xi_{\sigma,\tau,k}}\\
=&\alpha_U(S_{\xi_{\sigma,\tau,l}} p_\tau S^*_{\xi_{\sigma,\tau,k}})\\
=&S_{U\xi_{\sigma,\tau,l}} p_\tau S^*_{U\xi_{\sigma,\tau,k}}\\
=&\sum_{l',k'} c_{\sigma.\tau}(l,l')\overline{c_{\sigma.\tau}(k,k')} S_{\xi_{\sigma,\tau,l'}} p_\tau S^*_{\xi_{\sigma,\tau,k'}}.
\end{aligned}$$

Hence a relation:

$$c_{\sigma.\tau}(l,l')\overline{c_{\sigma.\tau}(k,k')} = \delta(l,l')\delta(k,k')1.$$

holds for all $1 \le l, l', k, k' \le a_{\sigma,\tau}$. This implies that the matrices $\mathcal{C}_{\sigma.\tau}$ are scalar. Those scalar is denoted by $C_{\sigma.\tau}$ and $|C_{\sigma.\tau}| = 1$. Take elements of $B' \cap \mathcal{F}_m$:

$$S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}} p_\tau S^*_{\xi_{\sigma,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}$$

for the two paths $\sigma\sigma(1)\sigma(2)\ldots\sigma(m-1)\tau$ and $\sigma\tau(1)\tau(2)\ldots\tau(m-1)\tau$ between σ and τ, and we get

$$\begin{aligned}
&S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}} p_\tau S^*_{\xi_{\sigma,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}\\
=&\alpha_U(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}} p_\tau S^*_{\xi_{\sigma,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}})\\
=&C_{\sigma,\sigma(1)}C_{\sigma(1),\sigma(2)}\cdots C_{\sigma(m-1),\tau}\overline{C_{\sigma,\tau(1)}C_{\tau(1),\tau(2)}\cdots C_{\tau(m-1),\tau}}\\
&\qquad\times S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}} p_\tau S^*_{\xi_{\sigma,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}.
\end{aligned}$$

Therefore we have, for all $m \in \mathbb{N}$,

$$C_{\sigma,\sigma(1)}C_{\sigma(1),\sigma(2)}\cdots C_{\sigma(m-1),\tau} = C_{\sigma,\tau(1)}C_{\tau(1),\tau(2)}\cdots C_{\tau(m-1),\tau}. \tag{2.7}$$

Since the value of $C_{\sigma,\sigma(1)}C_{\sigma(1),\sigma(2)}\cdots C_{\sigma(m-1),\tau}$ depends only on the two end points σ, τ, it is denoted by $D^m(\sigma,\tau)$. Since B is $X-$aperiodic, there is a integer $m \in \mathbb{N}$ such that, for all $\sigma, \tau \in \Sigma$, a path of m-length connecting σ and τ exists. Fix $\tau_0 \in \Sigma$ and we have, by (2.7),

$$D^m(\sigma,\tau)D^m(\tau,\tau_0) = D^m(\sigma,\tau_0)D^m(\tau_0,\tau_0).$$

Set $d(\sigma) := D^m(\sigma,\tau_0)$ and $D^m(\sigma,\tau)$ is equal to $d(\sigma)d(\tau_0)\overline{d(\tau)}$. Then we compute, for two paths $\sigma\sigma(1)\dots\sigma(m-1)\tau$ and $\rho\tau(1)\dots\tau(m-1)\tau$,

$$\begin{aligned}&\alpha_U(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S^*_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}})\\ =&D^m(\sigma,\tau)\overline{D^m(\rho,\tau)}S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S^*_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}\\ =&d(\sigma)\overline{d(\rho)}S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S^*_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}.\end{aligned}$$

We set a unitary $u \in B$:

$$u := \sum_\sigma d(\sigma)p_\sigma.$$

Then all for two paths $\sigma\sigma(1)\dots\sigma(m-1)\tau$ and $\rho\tau(1)\dots\tau(m-1)\tau$,

$$\begin{aligned}&\alpha_U\big(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S^*_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}\big)\\ =&\mathrm{Ad}u\big(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S^*_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}\big).\end{aligned}$$

Hence the automorphism α_U satisfies $\alpha_U(T) = \mathrm{Ad}u(T)$ for $T \in \mathcal{F}_m$. Since

$$D^{km}(\sigma,\tau) = D^m(\sigma,\sigma(1))\dots D^m(\sigma(k-1),\tau) = d(\sigma)d(\tau_0)^k\overline{d(\tau)},$$

by the same argument as above, we get

$$\alpha_U(T) = \mathrm{Ad}u(T)$$

for $T \in \mathcal{F}_{km}$. Then $\alpha_U = \mathrm{Ad}u$ on $\mathcal{F}_X$. On the other hand, $\alpha_U = \mathrm{Ad}V$ on $\mathcal{O}_X$ for $V \in \mathcal{F}_X$. Since $\mathcal{F}_X$ is simple, we conclude that $V = \lambda u$ for a scalar $\lambda, |\lambda| = 1$. We compute

$$\begin{aligned}C_{\sigma,\tau}S_{\xi_{\sigma,\tau,l}} =&\alpha_U(S_{\xi_{\sigma,\tau,l}})\\ =&uS_{\xi_{\sigma,\tau,l}}u^*\\ =&d(\sigma)\overline{d(\tau)}S_{\xi_{\sigma,\tau,l}}.\end{aligned}$$

Finally we get $C_{\sigma,\tau} = d(\sigma)\overline{d(\tau)}$, and

$$U\xi_{\sigma,\tau,l} = u\xi_{\sigma,\tau,l}u^*$$

for all σ,τ,l. We conclude that $Ux = uxu^*$ for $x \in X$. □

When we consider the imprimitivity bimodule ${}_\alpha B$ defined in (1.2), The C^*-algebras $\mathcal{F}_{{}_\alpha B}$ and $\mathcal{O}_{{}_\alpha B}$ are isomorphic to B and the crossed product $B \rtimes_\alpha \mathbb{Z}$ respectively. Let U be an invertible operator defined by

$$Ub = \alpha(b)$$

for $b \in {}_\alpha B$. Then the automorphism α_U is inner in $\mathcal{O}_{{}_\alpha B} = B \rtimes_\alpha \mathbb{Z}$ with $\alpha_U = \mathrm{Ad}S_I^*$ where I is an identity of ${}_\alpha B$. Therefore, for our purpose, we need the assumption that the Hilbert B-bimodule X is not an imprimitivity bimodule.

Theorem 2.7. *Let X be a full self conjugate Hilbert B-bimodule of finite type and X is not similar to an imprimitivity Hilbert $B-$bimodule. The C^*-algebra B is X-aperiodic with $Z(B) = \mathbb{C}$. Then the automorphism α_U is inner on $\mathcal{O}_X$ if and only if*

$$Ux = uxu^*$$

for some unitary u in B and all $x \in X$ and the automorphism θ is implemented by u.

Proof. Since X is a self conjugate Hilbert B-bimodule with its conjugate Hilbert $B-$bimodule $\bar{X}$, There exists Jones projection e_X in ${}_B\mathcal{L}_B(X \otimes_B \overline{X}) = {}_B\mathcal{L}_B(X \otimes_B X) \simeq B' \cap \mathcal{F}_2$ such that

$$e_X(x \otimes \bar{x'}) = (\text{r-ind}[X])^{-1} \sum_i u_i \otimes \bar{u}_i \ {}_B < x,\ x' > \tag{2.8}$$

where $x \in X$ and $\bar{x'} \in \bar{X}$ and r-ind$[X]$ is a right index of X ([4]). Suppose that the projection e_X is an identity. The projection e_X induces the conditional expectation F from $\mathcal{L}_B(X_B)$ to B as follows:

$$F(T) = (\text{r-ind}[X])^{-1} \sum_i {}_B < Tu_i,\ u_i > \tag{2.9}$$

for $T \in \mathcal{L}_B(X_B)$ ([4]Proposition 3.2) and

$$e_X(T \otimes I)e_X = (F(T) \otimes I)e_X.$$

Therefore the fact $e_X = I$ leads us that the expectation $F = I$. Hence by (2.9), we have

$$x < y,\ z >_B = \theta_{x,y} z = F(\theta_{x,y})z$$
$$= (\text{r-ind}[X])^{-1} \sum_i {}_B < \theta_{x,y} u_i,\ u_i > = (\text{r-ind}[X])^{-1} {}_B < x,\ y > z.$$

Defining a new left inner product ${}_B < x,\ y >'$ on X by

$${}_B < x,\ y >' = (\text{r-ind}[X])^{-1} {}_B < x,\ y >,$$

the Hilbert B-bimodule X is similar to an imprimitivity Hilbert $B-$bimodule. This is a contradiction. Therefore $B' \cap \mathcal{F}_X$ contains non trivial projection $e_X \in B' \cap \mathcal{F}_2$. By the same proof as the cases (i) and (ii) in Theorem 2.5, we obtain that the automorphism α_U is of the form:

$$\alpha_U(T) = VTV^*$$

for some unitary $V \in \mathcal{F}_X$ and all $T \in \mathcal{O}_X$. By Proposition 2.4, we get

$$Ux = \lambda uxu^*$$

for some unitary $u \in B$ and $\lambda \in \mathbb{C}, |\lambda| = 1$. Since $\mathcal{F}_X$ is simple and

$$u^* V S_x V^* u = u^* \alpha_U(S_x) u = \lambda S_x$$

for $x \in X$, the element u^*V in $\mathcal{F}_X$ is contained in the center $Z(\mathcal{F}_X) = \mathbb{C}$. Hence $V = \gamma u$ for some $\gamma \in \mathbb{C}, |\gamma| = 1$. We finally obtain that

$$S_{Ux} = \alpha_U(S_x) = VS_xV^* = uS_xu^* = S_{uxu^*}$$

and

$$Ux = uxu^*$$

for $x \in X$. □

Professor T. Kajiwara teaches us the existence of Jones projection e_X for a bimodule X.

Example 2.8. *The Hilbert $B-$bimodule ${}_BA_B$, induced by a C^*-inclusion $(B \subset A, E)$ of finite index type with index $E > 1$ ([6]), is always full, self conjugate and not similar to an imprimitivity Hilbert B-bimodule. If the C^*-algebra B is simple, it is clear that $\mathcal{F}_{{}_BA_B}$ is simple. Therefore for $\varphi \in Aut(B, A) := \{\varphi \in Aut(A) :$*

$\varphi(B) = B\}$, α_φ *is inner if and only if* $\varphi(a) = uau^*$ *for* $a \in A$ *and some unitary* $u \in B$.

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