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Korovkin-type theorems in C^*-algebras.

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abstract

Korovkin の定理:
$C_r[a,b]$ の正値線型写像の列 $\{\Phi_n\}$ が
$\Phi_n x^j \to x^j \ (n\to\infty) \quad j=0,1,2,$ ならば $\Phi_n u \to u \ (u\in C_r[a,b])$.
更に Korovkin は上の定理は $\{1,x,x^2\}$ の代わりに Tschebyshev system でも成立する事を示した。このような集合は Korovkin 集合と呼ばれるが、これについては多くの研究結果がある。一般 choquet 境界を用いて C^*-algebra の場合にも拡張された。
Priestley(1976) は C^*-algebra $\mathcal{A}\ni 1$ における正値写像 $\Phi_n(1)\leq 1$ について次の結果を得た。$\{,\}$ は Jordan product である。
$\{a\in\mathcal{A}:\Phi_n(a)\to a, \Phi_n(a^2)\to a^2\ \Phi_n(\{a^*,a\})\to\{a^*,a\}\}$ は J^*-subalgebra.
定義。C^*-algebra $\mathcal{A}\ni 1$ 上の線型写像 Φ が $\Phi(a)^*\Phi(a)\leq\Phi(a^*a)\ (a\in\mathcal{A})$ をみたしているとき Schwarz map といわれる。
Robertson は Priestley の研究を次のように発展させた。
$\{\Phi_n\}$ を Schwarz map の列で $\Phi_n(1)\leq 1$ とする。
$K:=\{a\in\mathcal{A}:\Phi_n(x)\to x\ (n\to\infty)\quad (x=a,a^*a,aa^*)\}$ は C^*-algebra
これから $C(X)\supseteq M$ が X の点を分離すれば $M\cup\{|h|^2:h\in M\}$ は Korovkin 集合 になる。
Limaye, Namboodiri は次のように拡張した。
$D:=\{a\in\mathcal{A}:\Phi_n(a)\to a,\Phi_n(a^*a)\to a^*a\}$ は norm closed subalgebra である。
彼らは正値写像は、Jordan product $\{,\}$ について Schwarz 写像になる事から 正値写像 $\Phi_n(1)\leq 1$ について次の結果を得た。
$C:=\{a\in\mathcal{A}:\Phi_n(a)\to a,\ \Phi_n(\{a^*,a\})\to\{a^*,a\}\}$ は J^*-subalgebra.
これらの定理は 可環である場合に知られている Korovkin集合 についての結果をほとんど含んでいない。この講演(論文)の目標は非可環のばあいにそれを

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実現することである。そのために上記の定理の証明を明確にすることから始める。

1. Introduction

The Korovkin theorem [6] says that an arbitrary sequence $\{\Phi_n\}$ of positive linear maps on $C_r([a,b])$, the Banach algebra of continuous real valued functions on $[a,b]$, strongly converges to the identity map if $\Phi_n u \to u (n \to \infty)$ for $u(t) = 1, t, t^2$. Moreover Korovkin showed that this result holds for any Tshebyshev system $\{f_0, f_1, f_2\}$ of order 2 instead of $\{1, t, t^2\}$. Here $\{f_0, f_1, f_2\}$ is called Tshebyshev system of order 2 if $a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) = 0$ has at most 2 zeros in this interval. A subset K of $C_r([a,b])$ is called a *Korovkin set,* provided an arbitrary sequence $\{\Phi_n\}$ of positive linear maps on $C_r([a,b])$ strongly converges to the identity map if $\Phi_n u \to u(n \to \infty)$ for every $u \in K$. A lot of Korovkin sets are known. (H. Watanabe, T. Nishishiraho)

Definition Let $1 \in M \subseteq C(X)$ and set $S(M) = \{l \in M^* : l(1) = 1 = \|l\|\}$. It is clear $\hat{x} \in S(M)$ for every $x \in X$. The *Choquet boundary* B_M is $\{x : \hat{x} \quad isaextremepointofS(M)\}$.

Wulbert has shown that if $1 \in M \subseteq C(X)$ and if $B_M = X$, then M is a Korovkin set.

Definition For a normed space E and for its subspace M, the *generalized Choquet boundary* is

$$B_M = \{l \in extS(E^*) : l|_M \in extS(M^*)\}.$$

Recently Operators of the property like I are investigated by S.Takahashi, Izuchi, Takagi, S.Watanabe.

The Korovkin theorem was extended to non-commutative C^*-algebras. (H. Choda– M.Echigo, S.Takahashi, J. Fujii)

Let $\mathcal{A}$ be a C^*-algebra with an identity 1. A positive linear map Φ on $\mathcal{A}$ is called a *Schwarz map* if it satisfies $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ *for every* $a \in \mathcal{A}$. It is well-known that if $\mathcal{A}$ is commutative then every contractive positive linear map is a Schwarz map. Robertson [11] has proved that, for a sequence $\{\Phi_n\}$ of Schwarz maps, the set $\{a \in \mathcal{A} : \Phi_n(x) \to x(n \to \infty)$ for $x = a, a^*a, aa^*\}$ is a C^*-subalgebra. As a corollary he also stated that for a sequence $\{\Phi_n\}$ of contractive positive linear maps on the commutative C^*-algebra $C(X)$ of continuous complex valued functions on a compact Hausdorff space X, the set $\{u \in C(X) : \Phi_n(u) \to u, \Phi_n(|u|^2) \to |u|^2\}$ is a C^*-subalgebra. By identifying $C_r(X)$ with the subalgebra of $C(X)$, the Stone-Weierstrass theorem shows that this contains the Korovkin theorem.

Let us recall that if $\mathcal{B}$ is a C^*-subalgebra of $C(X)$ and if for any point $x \in X$ there is a $f \in \mathcal{B}$ such that $f(x) \neq 0$ and if $\mathcal{B}$ separates X, then $\mathcal{B} = C(X)$.

Limaye and Namboodiri[7] have shown that for a sequence $\{\Phi_n\}$ of Schwarz maps and a $*$-homomorphism Φ, the set $\{a \in \mathcal{A} : \Phi_n(a) \to \Phi(a), \Phi_n(a^*a) \to$

$\Phi(a^*a)\}$ is a closed (not necessarily $*$-closed) subalgebra and that $\{a \in \mathcal{A} : \Phi_n(x) \to \Phi(x)$ for $x = a, a^*a, aa^*\}$, the intersection of this subalgebra and its adjoint, is a C^*-subalgebra. By the Kadison theorem a contractive positive linear map Φ satisfies $\{\Phi(a)^*, \Phi(a)\} \leq \Phi(\{a^*, a\})$ for all $a \in \mathcal{A}$, where $\{,\}$ is the Jordan product, i.e., $\{x, y\} = xy + yx$.
Limaye and Namboodiri [8] have shown that, for a sequence $\{\Phi_n\}$ of positive linear maps and a $*$-homomorphism Φ, the set $\{a \in \mathcal{A} : \Phi_n(a) \to \Phi(a), \Phi_n(\{a^*, a\}) \to \Phi(\{a^*, a\})\}$ is a $*$-closed, norm closed subspace which is also closed with respect to the Jordan product.

A continuous real valued function $f(t)$ on $[0, \infty)$is called an *operator monotone function* if $f(a) \geq f(b)$ whenever $a \geq b \geq 0$, $a, b \in \mathcal{A}$. This function is characterized as follows: f is an operator function on $[0, \infty)$ if and only if f has an analytic extension $f(z)$ to the upper half plane such that $Im\ f(z) > 0$ for $Im\ z > 0$. Therefore if f is an operator function, then so are $f(\sqrt{t})^2$ and $f(1/t)^{-1}$. $t^p (0 < p \leq 1)$ and $\log(t + 1)$ are operator monotone functions. It is well-known that an operator monotone function is increasing and concave.

The aim of this paper is to give estimates of the norms related to schwarz maps and to extend Korovkin-type theorems by using operator monotone functions. These estimates seem to be very useful for studying Korovkin-type theorems in a non-commutative C^*-algebra; for instace we will give a quite simple proofs for many results given above.

2. generalized Schwarz maps.

Let $\mathcal{A}$ be a C^*-algebra with a unit 1. A linear map Φ is called a *Schwarz map* if $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ for every $a \in \mathcal{A}$, and a positive linear map Ψ with $\Psi(1) \leq 1$ was called a *Jordan-Schwarz map* in [3], since it satisfies $\{\Psi(a)^*, \Psi(a)\} \leq \Psi(\{a^*, a\})$ as we mentioned in the previous section. To investigate two cases given above all at once and to extend them, we consider the following binary operation $\circ$ in $\mathcal{A}$:

$$\begin{cases} (\alpha x + \beta y) \circ z = \alpha x \circ z + \beta y \circ z \ (\alpha, \beta \in \mathbf{C}, x, y, z \in \mathcal{A}) \ ; \\ (x \circ y) \circ z = x \circ (y \circ z) \ ; \\ (x \circ y)^* = y^* \circ x^* \ ; \\ x^* \circ x \geq 0 \ ; \\ \textit{there is a real number } M \textit{ such that } ||x \circ y|| \leq M||x||\ ||y||. \end{cases}$$

One may regard this binary operation as the ordinary product or the Jordan produt.

Beckhoff [3] called a $*$-closed and norm-closed subspace of $\mathcal{A}$ which is also closed with respect to the Jordan product a *J^*-subalgebra* of $\mathcal{A}$.

We call a linear subspace $\mathcal{B} \subseteq \mathcal{A}$ a $\circ$-*subalgebra* if $x \circ y \in \mathcal{B}$, whenever

$x, y \in \mathcal{B}$, and $\circ^*$*-subalgebra* if $\mathcal{B}$ is a $\circ$-subalgebra and $*$-closed.

If a $\circ^*$-subalgebra is complete, that is norm-closed, then it is called a *complete* $\circ^*$*-subalgebra* .

Definition. A linear map $\Phi : \mathcal{A} \to \mathcal{A}$ is called a *generalized Schwarz map* w.r.t. $\circ$ if Φ satisfies

$$\Phi(x^*) = \Phi(x)^* \text{ and } \Phi(x^*) \circ \Phi(x) \leq \Phi(x^* \circ x) \text{ for every } x \in \mathcal{A}.$$

We remark that a generalized Schwarz map Φ is not necessarily positive (that Φ is positive means $\Phi(a) \geq 0$ whenever $a \geq 0$).

Definition. A generalized Schwarz map Φ w.r.t. $\circ$ is called a *$*$-homomorphism* w.r.t. $\circ$ if $\Phi(x)^* \circ \Phi(x) = \Phi(x^* \circ x)$ *for* every $x \in \mathcal{A}$.

Let us note that if Φ is a $*$-homomorphism w.r.t $\circ$, then by a polarization

$$4x^* \circ y = \sum_{n=0}^{3} i^n (i^n x + y)^* \circ (i^n x + y),$$

we deduce $\Phi(x) \circ \Phi(y) = \Phi(x \circ y)$ for every $x, y \in \mathcal{A}$. It is clear that if $\circ$ is the original product in $\mathcal{A}$, then a $*$-homomorphism w.r.t. $\circ$ is a $*$-homomorphism in the ordinal sense, and that if $\circ$ is the Jordan product, then a $*$-homomorphism w.r.t. $\circ$ is a C^*-homomorphism in the ordinal sense. A bounded linear functional ϕ of $\mathcal{A}$ is called a *state* if ϕ is positive and $\phi(1) = 1$.

Theorem 2.1. *Let Φ be a generalized Schwarz map w.r.t. $\circ$ on $\mathcal{A}$. For $x, y \in \mathcal{A}$ set*

$X := \Phi(x^* \circ x) - \Phi(x)^* \circ \Phi(x) \geq 0,$
$Y := \Phi(y^* \circ y) - \Phi(y)^* \circ \Phi(y) \geq 0,$
$Z := \Phi(x^* \circ y) - \Phi(x)^* \circ \Phi(y).$
Then we have

$$|\phi(Z)| \leq \phi(X)^{\frac{1}{2}} \phi(Y)^{\frac{1}{2}} \tag{1}$$

for every state $\phi \in \mathcal{A}'$. Further we have

$$\frac{1}{2}||Z|| \leq ||X||^{\frac{1}{2}} ||Y||^{\frac{1}{2}} \tag{2}$$

Proof. For every complex number α, we have
$0 \leq \Phi((x + \alpha y)^* \circ (x + \alpha y)) - \Phi(x + \alpha y)^* \circ \Phi(x + \alpha y) = X + \alpha Z + \overline{\alpha} \overline{Z} + |\alpha|^2 Y,$
from which it follows that
$0 \leq \phi(X) + 2Re\ \alpha\phi(Z) + |\alpha|^2 \phi(Y)$ for every state $\phi \in \mathcal{A}'$. Thus we can easily

get (1). Since $\sup\{\phi(Z) : \phi \text{ is a state of } \mathcal{A}\}$ is the numerical radius $w(Z)$, from (1) we obtain $w(Z) \leq w(X)^{\frac{1}{2}} w(Y)^{\frac{1}{2}}$.
It is well-known that $\frac{1}{2}||a|| \leq w(a) \leq ||a||$ for every $a \in \mathcal{A}$.
Thus we obtain (2). □

From the inequality (2) we can easily prove results mentioned in the first section.

Proposition 2.2. *Let $\{\Phi_n\}$ be a sequence of generalized Schwarz maps w.r.t. $\circ$ on $\mathcal{A}$ with $||\Phi_n|| \leq 1$, and Φ a $*$-homomorphism w.r.t. $\circ$ on $\mathcal{A}$ with $||\Phi|| \leq 1$. Then the set $D := \{x \in \mathcal{A} : ||\Phi_n(x) - \Phi(x)|| \to 0,\ ||\Phi_n(x^* \circ x) - \Phi(x^* \circ x)|| \to 0$ as $n \to \infty\}$ is a complete $\circ$-subalgebra.*

Proof. Suppose $x \in D$. From the definition of $\circ$, it follows that
$0 \leq ||\Phi_n(x)^* \circ \Phi_n(x) - \Phi(x)^* \circ \Phi(x)||$
$\leq M||\Phi_n(x)^* - \Phi(x)^*||\ ||\Phi(x)|| + M||\Phi_n(x)^*||\ ||\Phi_n(x) - \Phi(x)|| \to 0.$
This and

$$\Phi_n(x^* \circ x) \to \Phi(x^* \circ x) = \Phi(x)^* \circ \Phi(x)$$

imply

$$||\Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x)|| \to 0 \quad (n \to \infty).$$

Thus for every $y \in \mathcal{A}$, in virtue of (2) we get

$$||\Phi_n(x^* \circ y) - \Phi_n(x)^* \circ \Phi_n(y)|| \to 0 \quad (n \to \infty),$$

which implies that

$$\Phi_n(x^* \circ y) \to \Phi(x)^* \circ \Phi(y) \text{ if } x \in D \text{ and } \Phi_n(y) \to \Phi(y).$$

From this one can see that $x \circ y \in D$ if $x, y \in D$. Since $\{\Phi_n\}$ is uniformly bounded, D is complete. □

Corollary 2.3. Under the above condition the set $D \cap D^*$ is a complete $\circ^*$-subalgebra.

Remark. Since every bounded linear functional on $\mathcal{A}$ is a linear combination of at most four states of $\mathcal{A}$, a sequence $\{a_n\}$ of $\mathcal{A}$ weakly converges to a if and only if $\phi(a_n) \to \phi(a)$ for every state ϕ. By using (1) we can see that
$D_1 := \{x \in \mathcal{A} : \Phi_n(x) \to \Phi(x)\ (w), \Phi_n(x^* \circ x) \to \Phi(x^* \circ x)\ (w)\}$

is a complete $\circ$-subalgebra, and hence that $D_1 \cap D_1^*$ is a complete $\circ^*$-subalgebra.

Proposition 2.2, Corollary 2.3 and Remark were proved in [7] [8] [11] when $\circ$ is the original product or the Jordan product in $\mathcal{A}$, but the above proof seems to be simple.

We denote the $\circ^*$-subalgebra of $\mathcal{A}$ generated by a subset S of $\mathcal{A}$ by $\mathcal{J}^*(S, \circ)$ or simply by $\mathcal{J}^*(S)$. We define the *Korovkin closure* $Kor_{\mathcal{A}}(S)$ of a subset $S \subseteq \mathcal{A}$ as follows : $Kor_{\mathcal{A}}(S)$ is the set of all $x \in \mathcal{A}$ such that for every sequence $\{\Phi_n\}$ of *positive* generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$, $\Phi_n x \to x$ whenever $\Phi_n a \to a$ for every $a \in S$. Here the convergence means convergence in the norm topology. From this definition the next follows :

Lemma 2.4. $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$ if $S \subseteq T$. $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$ if $S \subseteq Kor_{\mathcal{A}}(T)$.

Corollary 2.5. *For a subset $S \subseteq \mathcal{A}$, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_1), \textit{where } S_1 := S \cup \{x^* \circ x : x \in S\} \cup \{x \circ x^* : x \in S\}. \quad (3)$$

Proof. Fix a sequence $\{\Phi_n\}$ of positive generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_1$. We have only to show $\Phi_n(t) \to t$ for every $t \in \mathcal{J}^*(S)$. By Corollary 2.3, the set $\{x \in \mathcal{A} : \Phi_n(x) \to x, \Phi_n(x^* \circ x) \to x^* \circ x, \Phi_n(x \circ x^*) \to x \circ x^*\}$ is a $\circ^*$-subalgebra. Since it contains S, it contains $\mathcal{J}^*(S)$ too. Thus we have $\Phi_n(t) \to t$ for every $t \in \mathcal{J}^*(S)$. □

Theorem 2.6. *Let f be an operator monotone function on $[0, \infty)$ with $f(0) = 0$ and $\lim_{t\to\infty} f(x) = \infty$. Set $g = f^{-1}$. Then for a subset S of $\mathcal{A}$ we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2), \textit{where } S_2 := S \cup \{g(x^* \circ x) : x \in S\} \cup \{g(x \circ x^*) : x \in S\}. \quad (4)$$

Proof. Let $\{\Phi_n\}$ be a sequence of positive generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_2$. It was shown in [4] [5] that

$$\Phi_n(f(a)) \leq f(\Phi_n(a)) \textit{ for every } a \geq 0, \quad (5)$$

which implies

$$0 \leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \leq f(\Phi_n(g(x^* \circ x))) - \Phi_n(x)^* \circ \Phi_n(x)$$

for every x. From $\Phi_n(g(x^* \circ x)) \to g(x^* \circ x)$, it follows that $f(\Phi_n(g(x^* \circ x))) \to x^* \circ x$. Thus the right side of the above inequality converges to 0, from which it follows that

$$\lim \Phi_n(x^* \circ x) = \lim \Phi_n(x)^* \circ \Phi_n(x) = x^* \circ x^*.$$

Similarly we can get $\lim \Phi_n(x \circ x^*) = x \circ x^*$.

Thus we have shown that $\Phi_n(t) \to t$ for every t in S_1 which was given in Corollary 2.5, that is, we have shown $S_1 \subseteq Kor_{\mathcal{A}}(S_2)$. By (3) and Lemma 2.4 we have $J^*(S) \subseteq Kor_{\mathcal{A}}(S_1) \subseteq Kor_{\mathcal{A}}(S_2)$. Consequently we get (4). □

Theorem 2.7. *Let g be a function given in Theorem 2.6. For a finite set $S = \{s_1, \ldots, s_n\}$, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_3), \text{ where } S_3 = S \cup \{g(\sum_{i=1}^{n}(s_i^* \circ s_i + s_i \circ s_i^*))\}. \tag{6}$$

Proof. Let us take an arbitrary sequence Φ_n of positive generalized Schwarz maps w.r.t $\circ$ with $||\Phi_n|| \leq 1$ such that $\{\Phi_n(t)\} \to t$ for every $t \in S_3$.
For each i

$$\begin{aligned} 0 &\leq \Phi_n(s_i^* \circ s_i) - \Phi_n(s_i)^* \circ \Phi_n(s_i) \leq \sum_{j=1}^{n}\{\Phi_n(s_j^* \circ s_j) - \Phi_n(s_j)^* \circ \Phi_n(s_j)\} \\ &\leq \Phi_n(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)) - \sum_j \{\Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^*\} \\ &\leq f(\Phi_n(g(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)))) - \sum_j \{\Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^*\}. \end{aligned}$$

Since the right side converges to 0, $\Phi_n(s_i^* \circ s_i)$ converges to $s_i^* \circ s_i$. Similarly we can see that $\Phi_n(s_i \circ s_i^*)$ converges to $s_i \circ s_i^*$. Thus we have shown that $S_1 := S \cup \{s^* \circ s : s \in S\} \cup \{s \circ s^* : s \in S\} \subseteq Kor_{\mathcal{A}}(S_3)$.
By (3) and Lemma 2.4, we get (6). □

Theorem 2.8. *Under the same assumption as Theorem 2.6, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\}). \tag{7}$$

Proof. By substituting x for s_i in the inequalities of the proof of Theorem 2.7, we get

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x + x \circ x^*))) - \{\Phi_n(x)^* \circ \Phi_n(x) + \Phi_n(x) \circ \Phi_n(x)^*\}. \end{aligned}$$

Thus in the same fashion as Theorem 2.6 we can get (7). □

In the above three theorems we needed conditions $f(0) = 0,\ f(\infty) = \infty$ in order that $f^{-1} = g$ is defined on $[0, \infty)$and that (5) is valid for every positive map. However, when we consider the case of $1 \in S$, we can loose the condition $f(0) = 0$.

Theorem 2.9. *Suppose $1 \in S \subseteq \mathcal{A}$. Let f be an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0, f(\infty) = \infty$. Set $g = f^{-1}$. Then we have $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2)$, where $S_2 = S \cup \{g(x^* \circ x) | x \in S\} \cup \{g(x \circ x^*) | x \in S\}$.*

Proof. Let us take an arbitrary sequence $\{\Phi_n\}$ of positive generalized Schwarz maps w.r.t $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_2$. By (5) we get

$$\Phi_n(f(a) - f(0)1) \leq f(\phi_n(a)) - f(0)1 \ \ for\ every\ a \geq 0,$$

and hence

$$\Phi_n(a) = \Phi_n(f(g(a))) \leq f(\Phi_n(g(a))) - f(0)(1 - \Phi_n(1)).$$

From this, for every $x \in S$ we deduce

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x))) - f(0)(1 - \Phi_n(1)) - \Phi_n(x)^* \circ \Phi_n(x). \end{aligned}$$

Since the bigger side in the above converges to 0, we obtain that $\Phi_n(x^* \circ x) \to x^* \circ x$. Similarly we can get $\Phi_n(x \circ x^*) \to x \circ x^*$. By (3) we get $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2)$. □

In the same fashion as the above proof, we can easily extend Theorem 2.7 and Corollary 2.8 to the case of $1 \in S$ as follows :

Theorem 2.10. *Let $S = \{s_1, \ldots, s_n\}$ be a subset of $\mathcal{A}$ and include 1. Let f be an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$. $f(\infty) = \infty$. Set $g = f^{-1}$. Then we have $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_3)$, where $S_3 = S \cup \{g(\sum_{i=1}^n (s_i^* \circ s_i + s_i \circ s_i^*))\}$.*

Corollary 2.11. Under the same assumption as Theorem 2.9, we have $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\})$.

Remark. In the above theorems we studied not the universal Korovkin closures (the definiton is given below) but the Korovkin closures, that is, the case where $\Phi_n \to 1$ instead of $\Phi_n \to \Phi$. To get the same conclusions for Φ as theorems, we would have to assume that Φ is $*$-homomorphism w.r.t. $\circ$ and $*$-homomorphism in the ordinary sense because of $\Phi(g(a)) = g(\Phi(a))$; we thought it is a bit complicated assumption. If a binary operation $\circ$ is the ordinary product or the Jordan product, then $*$-homomorphism in the ordinary sense is a $*$-homomorphism w.r.t. $\circ$ too. Now we consider this case. Let us define the *universal Korovkin closure* $Kor^u_{\mathcal{A}}(S)$ of a subset $S \subseteq \mathcal{A}$ as follows : $Kor^u_{\mathcal{A}}(S)$ is the set of all $x \in \mathcal{A}$ such that for every $*$-homomorphism Φ and for every sequence $\{\Phi_n\}$ of *positive* generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$, $\Phi_n x \to \Phi x$ whenever $\Phi_n(a) \to \Phi(a)$ for every $a \in S$. When $\circ$ is the ordinary product or the Jordan product, it is not difficult to see that we can substitute $Kor^u_{\mathcal{A}}(S)$ for $Kor_{\mathcal{A}}$ in the above theorems.

At the end of this section we consider the case where $\circ$ is the ordinary product, and we extend the Robertson's theorem in a visible form :

Theorem 2.12. *Let $\{\Phi_n\}$ be a sequence of Schwarz maps and Φ a $*$-homomorphism, and let f be an operator monotone function on $[0,\infty)$ with $f(0) = 0$, $f(\infty) = \infty$. Set $g = f^{-1}$. Then the set $C := \{a \in \mathcal{A} : \Phi_n(x) \to \Phi(x)$ for $x = a,\ g(a^*a),\ g(aa^*)\}$ is a C^*-subalgebra.*

Proof. That $\Phi_n(a)$ converges to $\Phi(a)$ implies $\Phi_n(a)^*\Phi_n(a) \to \Phi(a)^*\Phi(a)$, and that $\Phi_n(g(a^*a))$ converges to $\Phi(g(a^*a))$ implies

$$f(\Phi_n(g(a^*a))) \to f(\Phi(g(a^*a))) = \Phi(a^*a) = \Phi(a)^*\Phi(a).$$

Thus we have $f(\Phi_n(g(a^*a))) - \Phi_n(a)^*\Phi_n(a) \to 0$. From (5) it follows that

$$0 \leq \Phi_n(a^*a) - \Phi_n(a)^*\Phi_n(a) \leq f(\Phi_n(g(a^*a))) - \Phi_n(a)^*\Phi_n(a).$$

Hence we get $\Phi_n(a^*a) \to \Phi(a^*a)$. Similary we can get $\Phi_n(aa^*) \to \Phi(aa^*)$. Thus $C \subseteq D \cap D^*$, where D is given in Proposition 2.2. Conversely, since $D \cap D^*$ is a C^*-subalgebra (Corollary 2.3), $D \cap D^* \subseteq C$. Consequently C is a C^*-subalgebra. □

3. **Korovkin sets in $C(X)$.**

Let X be a compact Hausdorff space and $C(X)$ a C^*-algebra of all complex valued continuous functions . Though we treat only complex algebras, the results which will be gotten for complex algebras in this section hold for real algebras too. Since a positive linear map Φ on $C(X)$ satisfies $|\Phi(fg)|^2 \leq \Phi(|f|^2)\Phi(|g|^2)$, Φ is a Schwarz map with respect to the ordinary product if $\Phi(1) \leq 1$. A subset S of $C(X)$ is called a *Korovkin set* if $K_{C(X)}(S) = C(X)$.

Here $K_{C(X)}$ is the set of every $x \in C(X)$ which satisfies that $\Phi_n(x) \to x$ for every sequence of Schwarz maps (*i.e*, $0 \leq \Phi_n, \Phi_n(1) \leq 1$) such that $\Phi_n(s) \to s$ for all $s \in S$. $C^*(S)$ stands for the C^*-aubalgebra generated by S.

Theorem 3.1. *Let f be an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, $f(\infty) = \infty$, and set $g = f^{-1}$. Then for a subset S of $C(X)$*

$$C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u|^2) : u \in S\}) \text{ if } f(0) = 0, \text{ or } 1 \in S.$$

Proof. This follows from Theorems 2.6, 2.9. □

Theorem 3.2. *Let f be an operator monotone function defined on $[0, \infty)$ with $f(0) \leq 0$, $f(\infty) = \infty$, and set $g = f^{-1}$. If a finite subset $S = \{u_1, \ldots, u_m\} \subseteq C(X)$ separates strongly the points of X, then $S \cup \{g(|u_1|^2 + \ldots + |u_m|^2)\}$ is a Korovkin set if $f(0) = 0$, or $1 \in S$.*

Proof. By Theorems 2.7, 2.10, we have $C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u_1|^2 + \ldots + |u_m|^2)\})$ if $f(0) = 0$ or $1 \in S$. From the Stone-Weierstrass theorem $C^*(S) = C(X)$ follows. □

In [9], the above theorem was shown in the case where $g(t) = t$. The forms of Korovkin sets given above include many Korovkin sets in Appendix C of [1].

4. Minimal norm ideals.

In this section we treat the minimal norm ideals of C^*-algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H. Korovkin-type theory in the minimal norm ideals was studied in [10], [3]. Let $\mathcal{I}$ be a minimal norm ideal with a symmetric norm $\|\ \|$, and $\mathcal{A}$ a C^*-algebra generated by every compact operator and 1. We use the notation introduced in the second section. But we assume that a binary operation $\circ$ defined on $\mathcal{A}$ satisfies

$$\|x \circ y\| \leq M \|x\| \ \|y\|$$

instead of $||x \circ y|| \leq M||x|| \ ||y||$.

Theorem 4.1. *Let Φ be a generalized Schwarz map w.r.t. $\circ$ on $\mathcal{A}$. For $x, y \in \mathcal{A}$ set $X := \Phi(x^* \circ x) - \Phi(x)^* \circ \Phi(x) \geq 0$, $Y := \Phi(y^* \circ y) - \Phi(y)^* \circ \Phi(y) \geq 0$, $Z := \Phi(x^* \circ y) - \Phi(x)^* \circ \Phi(y)$.*
Then we have

$$\frac{1}{2}\|Z\| \leq \|X\|^{\frac{1}{2}} \|Y\|^{\frac{1}{2}} \tag{8}$$

Proof. (1) implies $|(Zu,u)| \leq (Xu,u)^{\frac{1}{2}}(Yu,u)^{\frac{1}{2}}$ for every $u \in \mathcal{H}$. By using the polarization :

$$4(Zu,v) = (Z(u+v),u+v)-(Z(u-v),u-v)+i(Z(u+iv),u+iv)-i(Z(u-iv),u-iv),$$

we get,

$$\begin{aligned}4|(Zu,v)| &\leq (X(u+v),u+v)^{\frac{1}{2}}(Y(u+v),u+v)^{\frac{1}{2}} + (X(u-v),u-v)^{\frac{1}{2}}(Y(u-v),u-v)^{\frac{1}{2}} \\ &+(X(u+iv),u+iv)^{\frac{1}{2}}(Y(u+iv),u+iv)^{\frac{1}{2}} + (X(u-iv),u-iv)^{\frac{1}{2}}(Y(u-iv),u-iv)^{\frac{1}{2}}\end{aligned}$$

and hence in virtue of Schwarz inequality,

$$|(Zu,v)| \leq \{(Xu,u)+(Xv,v)\}^{\frac{1}{2}}\{(Yu,u)+(Yv,v)\}^{\frac{1}{2}}$$

for every $u, v \in \mathcal{H}$. Thus, for arbitrary orthonormal sets $\{u_i\}, \{v_i\}$

$$\begin{aligned}&|(Zu_i,v_i)| \leq \{(Xu_i,u_i)+(Xv_i,v_i)\}^{\frac{1}{2}}\{(Yu_i,u_i)+(Yv_i,v_i)^{\frac{1}{2}} \\ &\leq \tfrac{1}{2}t\{(Xu_i,u_i)+(Xv_i,v_i)\} + \tfrac{1}{2}\tfrac{1}{t}\{(Yu_i,u_i)+(Yv_i,v_i)\}\end{aligned}$$

for every $t > 0$, because of $2\sqrt{\lambda\mu} = \min\{t\lambda + \frac{1}{t}\mu : t > 0\}$. Now we consider this inequality to be an estimate of general terms of sequences. By taking the symmetric norm of these sequences, we get
$\|(Zu_i,v_i)\| \leq \frac{1}{2}t\|(Xu_i,u_i)+(Xv_i,v_i)\| + \frac{1}{2}\frac{1}{t}\|(Yu_i,u_i)+(Yv_i,v_i)\| \leq t\|X\| + \frac{1}{t}\|Y\|$.
Since

$$\sup\{\|(Zu_i,v_i)\| : \{u_i\},\{v_i\}\} = \|Z\| \quad ([12]),$$

we have $\|Z\| \leq (t\|X\| + \frac{1}{t}\|Y\|)$ for every $t > 0$, and hence $\|Z\| \leq 2\|X\|^{\frac{1}{2}}\|Y\|^{\frac{1}{2}}$.
□

Theorem 4.2. *Let $\mathcal{I}$ be a minimal ideal of $\mathcal{B}(H)$ with the symmetric norm $\|\ \|$. Let $\{\Phi_n\}$ be a sequence of generalized Schwarz maps w.r.t $\circ$ on $\mathcal{I}$ with $\|\Phi_n\| \leq 1$, and Φ a $*$-homomorphism w.r.t $\circ$ on $\mathcal{I}$ with $\|\Phi_n\| \leq 1$. Then the set* $D = \{x \in \mathcal{I} : \|\Phi_n(x) - \Phi(x)\| \to 0,\ \|\Phi_n(x^* \circ x) - \Phi(x^* \circ x)\| \to 0\}$
is a $\|\ \|$-closed $\circ$-subalgebra of $\mathcal{I}$.

Proof. This theorem follows from (8) in the same way that Prposition 2.2 followed from (2). □

Corollary 4.3. Under the same condition as above, $D \cap D^*$ is a $\|\ \|$-closed $*$-subalgebra of $\mathcal{I}$.

Theorem 4.2 was proved in [10] in the case where $\mathcal{I}$ is the trace class, and other cases were described in [3]. But our proof is clearly simpler than the proof of [10].

We want to extend the above. But it is not easy, because $\|a_n - a\| \to 0$ does not necessarily imply $\|a_n^{\frac{1}{2}} - a^{\frac{1}{2}}\| \to 0$.

Therefore we could not get a theorem as Theorem 2.6. To get a slight extension we denote $\||A|^p\|^{\frac{1}{p}}$ by $\|A\|_p$ for $0 < p < \infty$.

Theorem 4.4. *Let $\mathcal{I}$ be a minimal norm ideal of $\mathcal{B}(H)$ with the symmetric norm $\|\ \ \|$. Let $\{\Phi_n\}$ be a sequence of generalized Schwarz maps w.r.t. $\circ$ on $\mathcal{I}$ with $\|\Phi_n\| \leq 1$. Let S be a subset of $\mathcal{I}$ and $\mathcal{I}_S^*$ a complete $\circ^*$-subalgebra generated by S. Then for an arbitrary integer m*
$\mathcal{I}_S^* \subseteq \{x \in \mathcal{I} : \|\Phi_n x - x\| \to 0$ *whenever* $\|\Phi_n(s) - s\| \to 0,$
$\|\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m\|_{\frac{1}{m}} \to 0$ *for all* $s \in S\}$ (9)

Proof. By the previous theorem we need only to show that if

$$\|\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m\|_{\frac{1}{m}} \to 0 \ (n \to \infty),$$

then

$$\|\Phi_n(s^* \circ s) - s^* \circ s\| \to 0 \text{ and } \|\Phi_n(s \circ s^*) - s \circ s^*\| \to 0.$$

By the definition of the norm we have

$$\||\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m|^{\frac{1}{m}}\| \to 0.$$

By the Ando theorem [2] :

$$\|x^{\frac{1}{m}} - y^{\frac{1}{m}}\| \leq \||x - y|^{\frac{1}{m}}\| \text{ for all } x, y \geq 0,$$

we obtain

$$\|\{\Phi_n((s^* \circ s + s \circ s^*)^m)\}^{\frac{1}{m}} - (s^* \circ s + s \circ s^*)\| \to 0.$$

Thus, from

$$\|\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^* - (s^* \circ s + s \circ s^*)\| \to 0,$$

it follows that

$$\|\{\Phi_n(s^* \circ s + s \circ s^*)^m\}^{\frac{1}{m}} - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^*\}\| \to 0.$$

Since

$$\begin{aligned} 0 &\leq \{\Phi_n(s^* \circ s) - \Phi_n(s^*) \circ \Phi_n(s)\} + \{\Phi_n(s \circ s^*) - \Phi_n(s) \circ \Phi_n(s^*)\} \\ &= \Phi_n(s^* \circ s + s \circ s^*) - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s^*)\} \\ &\leq \{\Phi_n(s^* \circ s + s \circ s^*)^m\}^{\frac{1}{m}} - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^*\}, \end{aligned}$$

we deduce $\|\{\Phi_n(s^* \circ s) - \Phi_n(s^*) \circ \Phi_n(s)\| \to 0$ and $\|\{\Phi_n(s \circ s^*) - \Phi_n(s) \circ \Phi_n(s)^*\| \to 0$. Here we used the fact that $0 \leq a \leq b$ generally implies $\|a\| \leq \|b\|$: in fact $0 \leq a \leq b$ implies that there is $c \in \mathcal{B}(\mathcal{H})$ such that $a = c^*bc$ and $||c|| \leq 1$. □

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