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Multidimensional local residues and holonomic D-modules

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Multidimensional local residues are fundamental objects in complex analysis and geometry. However, if the polar divisors of a meromorphic differential form are not in general position, the actual calculation of local residues is difficult in many cases. In this paper we study Grothendieck local residue from the viewpoint of D-modules. We mainly consider the case where the polar divisors are not in general position. We propose a new approach for calculating multidimensional local residues.

In the appendix we consider the zero-dimensional transversal complete intersection case. We present a simple method for computing residues for this case.

We use a computer algebra system Kan for Gröbner basis computation in Weyl algebra and a computer algebra system Risa/Asir for Gröbner basis computation, and primary decomposition in polynomial rings.

1. Algebraic local cohomologies

Let us recall some basic facts about algebraic local cohomology and holonomic D-modules. Let X be a complex manifold $\mathcal{O}_X$ the sheaf on X of holomorphic functions. Let Y a subvariety in X. Let $\mathcal{J}_Y$ be the sheaf of ideal of Y in X. The k-th algebraic local cohomology group supported in Y is defined as the inductive limit of extension groups

$$\mathcal{H}^k_{[Y]}(\mathcal{O}_X) = \lim_{\ell\to\infty} \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^\ell_Y; \mathcal{O}_X).$$

Note that for a hypersurface case, we have $\mathcal{H}^1_{[Y]}(\mathcal{O}_X) \simeq \mathcal{O}_X[*Y]/\mathcal{O}_X$, where $\mathcal{O}_X[*Y]$ stands for the sheaf of meromorphic functions on X with poles along Y.

Let $\mathcal{D}_X$ be the sheaf of rings on X of linear partial differential operators with holomorphic coefficients. Then $\mathcal{D}_X$ is coherent as a sheaf of rings. It is easy to see that the algebraic local cohomology group $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ is naturally endowed with a structure of left $\mathcal{D}_X$-module. In 1978, Kashiwara proved the following fundamental theorem.

Theorem (Kashiwara[12], Mebkhout[14])

(1) $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ *is coherent over* $\mathcal{D}_X$.
(2) $\mathcal{H}^k_{[Y]}(\mathcal{O}_X)$ *is a holonomic system.*

We refer to [11], [22] for the notion of a holonomic system.

Recently, one of the authors (T. Oaku ([19], [20])) constructed an algorithm to calculating Gröbner basis of an algebraic local cohomology group. His algorithm has been implemented in the computer algebra system Kan ([24]), developed by N. Takayama of Kobe University. The following computation was carried out by using Kan.

Example Let $f(x,y) = (x^2+y^2)^3 - 4x^2y^2$, $D = \{(x,y) \mid f(x,y) = 0\}$. Put

$$m = \left(\frac{1}{(x^2+y^2)^3 - 4x^2y^2} \ mod\ \mathcal{O}_X\right) \in \mathcal{H}^1_{[D]}(\mathcal{O}_X).$$

The generator m of the module $\mathcal{H}^1_{[D]}(\mathcal{O}_X)$ satisfies the following holonomic system.

$$\begin{cases} (x^3D_x - 2y^2xD_x + 2yx^2D_y - y^3D_y + 6x^2 - 6y^2)m = 0, \\ (-15y^2x^2D_x + 3y^4D_x + 3yx^3D_y - 15y^3xD_y - 72y^2x + 4x^2D_x \\ \qquad +8yxD_y + 24x)m = 0, \\ (-27y^3xD_x + 3x^4D_y + 15y^2x^2D_y - 15y^4D_y + 18yx^2 - 90y^3 \\ \qquad +8yxD_x + 4y^2D_y + 24y)m = 0, \\ (-x^6 - 3y^2x^4 - 3y^4x^2 - y^6 + 4y^2x^2)m = 0, \\ (-108y^5D_x + 60x^5D_y + 192y^2x^3D_y + 240y^4xD_y + 360yx^3 \\ \qquad +792y^3x + 16yx^2D_x - 208y^2xD_y - 384yx)m = 0, \\ (-972y^4D_x^2 - 216x^4D_y^2 - 756y^2x^2D_y^2 - 1512y^4D_y^2 - 8748y^2xD_x \\ \qquad +144x^2D_x^2 - 1296yx^2D_y - 18468y^3D_y + 432yxD_yD_x + 1152y^2D_y^2 \\ \qquad -1296x^2 - 53136y^2 + 3456xD_x + 9504yD_y + 16416)m = 0. \end{cases}$$

Moreover, these operators form a Gröbner basis of the annihilator ideal of the generator m.

Example(cf. [25]) Let $f(x,y) = x^6 - x^2y^3 - y^5$, $g(x,y) = y$.
Let m be the cohomology class associated to the meromorphic function $\frac{1}{fg}$:

$$m = \left[\frac{1}{fg}\right] \in \mathcal{H}^2_{[0,0]}(\mathcal{O}_X).$$

We have

$$\begin{cases} x^6m = 0, \\ ym = 0 \\ (xD_x + 6)m = 0. \end{cases}$$

However, the $\mathcal{D}_X$-module structure of the algebraic local cohomology group supported on the curve $f(x,y) = 0$ is complicated.

```
[1] % sm1
sm1

Kan/StackMachine1                              1991 April --- 1996.
                                         Release 2.970417 (c) N. Takayama
This software may be freely distributed as is with no warranty expressed.
Please address bug reports and advices to kan@math.s.kobe-u.ac.jp

Ready

sm1>dr.sm1:   9/26,1995 --- Version 4/17, 1997.
sm1>module1.sm1, 1994
sm1>(bfrest.sm1) run ;
bfrest.sm1 ... Kan/sm1 programs for D-modules
Version 970623  by T. Oaku and N. Takayama
See usages by   (indicial) usage ; (rest0) usage ; (rest-1) usage ;
sm1>(bspoly.sm1) run ;
sm1>(toasir.sm1) run ;

sm1>(x^6-x^2*y^3-y^5) [(x) (y)] 0 0 alc1 ;
(x^6-x^2*y^3-y^5) [(x) (y)] 0 0 alc1 ;
Computing an FW-Groebner basis . Completed.

sm1>::

[$-75*y*x^2*Dx-6*x^3*Dy-90*y^2*x*Dy+9*x^2*Dx-3*y^2*Dx+12*y*x*Dy-450*y*x+54*x$ ,

 $-9*x^3*Dx-15*y^2*x*Dx-12*y*x^2*Dy-18*y^3*Dy-54*x^2-90*y^2$ ,

 $375*y^3*x*Dx-18*x^4*Dy+30*y^2*x^2*Dy+450*y^4*Dy-54*y^2*x*Dx-54*y^3*Dy
 +2250*y^3-270*y^2$ ,

 $-21093750*y^2*x*Dy*Dx-1687500*y*x^2*Dy^2-25312500*y^3*Dy^2+421875*x^2*Dx^2
 +703125*y^2*Dx^2+3093750*y*x*Dy*Dx+3375000*y^2*Dy^2-101953125*y*x*Dx
 -6468750*x^2*Dy-274218750*y^2*Dy+13500000*x*Dx+32625000*y*Dy-611718750*y
 +63281250$ ,

 $-x^6+y^3*x^2+y^5$ ,

 $18*x^5*Dy+9*y^2*x^2*Dx+15*y^4*Dx-6*y^3*x*Dy$ ,

 $-379687500*x^4*Dy^2+263671875*y^3*Dx^2+28687500*y*x^2*Dy^2
 +329062500*y^3*Dy^2-14501953125*y^2*x*Dx+16031250*x^2*Dx^2
 -11250000*y^2*Dx^2-1160156250*y*x^2*Dy-17402343750*y^3*Dy+3656250*y*x*Dy*Dx
 -23625000*y^2*Dy^2+2274609375*y*x*Dx+100125000*x^2*Dy+5054062500*y^2*Dy
 -87011718750*y^2+57375000*x*Dx-203062500*y*Dy+15925781250*y-329062500$ ,

 $-455625000*y*x^3*Dy^3-263671875*y^3*Dx^3-303750000*y^3*Dy^2*Dx
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+8370000*x^3*Dy^3+24300000*y^2*x*Dy^3+14501953125*y^2*x*Dx^2
-16031250*x^2*Dx^3+11250000*y^2*Dx^3+17402343750*y^3*Dy*Dx
-3656250*y*x*Dy*Dx^2-1763437500*x^3*Dy^2-1392187500*y^2*x*Dy^2
-12555000*x^2*Dy^2*Dx+27810000*y^2*Dy^2*Dx-16740000*y*x*Dy^3
-2274609375*y*x*Dx^2+96468750*x^2*Dy*Dx-4948593750*y^2*Dy*Dx
+397575000*y*x*Dy^2-1160156250*x^2*Dx+101513671875*y^2*Dx-89437500*x*Dx^2
-7425000000*y*x*Dy+216146250*y*Dy*Dx-108810000*x*Dy^2-18141328125*y*Dx
+1213537500*x*Dy-6960937500*x+280057500*Dx$ ,

$-205031250000*y^2*x^2*Dy^4+98876953125*y^3*Dx^4+104414062500*y^3*Dy^2*Dx^2
+6743250000*y*x^2*Dy^4+7290000000*y^3*Dy^4-5438232421875*y^2*x*Dx^3
+6011718750*x^2*Dx^4-4218750000*y^2*Dx^4-6525878906250*y^3*Dy*Dx^2
+1371093750*y*x*Dy*Dx^3+3233250000*x^2*Dy^2*Dx^2-11520000000*y^2*Dy^2*Dx^2
-1962984375000*y*x^2*Dy^3-626484375000*y^3*Dy^3-437625000*y*x*Dy^3*Dx
-6331500000*y^2*Dy^4+852978515625*y*x*Dx^3-20250000000*x^2*Dy*Dx^2
+1825312500000*y^2*Dy*Dx^2+67668750000*y*x*Dy^2*Dx+37248750000*x^2*Dy^3
+285795000000*y^2*Dy^3+435058593750*x^2*Dx^2-43505859375000*y^2*Dx^2
+45562500000*x*Dx^3-1560058593750*y*x*Dy*Dx-84048750000*y*Dy*Dx^2-38
34984375000*x^2*Dy^2-9076640625000*y^2*Dy^2+5996250000*x*Dy^2*Dx
-81164250000*y*Dy^3+7652109375000*y*Dx^2+244251562500*x*Dy*Dx
+2257222500000*y*Dy^2+180175781250*x*Dx-73665000000*Dx^2-32455898437500*y*Dy
-218907000000*Dy^2+4175150625000*Dy-17191406250000$ ,

$-246037500000*y^3*x*Dy^5-98876953125*y^3*Dx^5-94921875000*y^3*Dy^2*Dx^3
-4556250000*y^3*Dy^4*Dx-1730160000*x^3*Dy^5+10497600000*y^2*x*Dy^5
+5438232421875*y^2*x*Dx^4-6011718750*x^2*Dx^5+4218750000*y^2*Dx^5
+6525878906250*y^3*Dy*Dx^3-1371093750*y*x*Dy*Dx^4-2656125000*x^2*Dy^2*Dx^3
+11115000000*y^2*Dy^2*Dx^3+626484375000*y^3*Dy^3*Dx+569250000*y*x*Dy^3*Dx^2
+1518750000*x^3*Dy^4-4159856250000*y^2*x*Dy^4+2595240000*x^2*Dy^4*Dx
+4615920000*y^2*Dy^4*Dx+3460320000*y*x*Dy^5-852978515625*y*x*Dx^4
+9808593750*x^2*Dy*Dx^3-1785761718750*y^2*Dy*Dx^3-62353125000*y*x*Dy^2*Dx^2
+29991375000*x^2*Dy^3*Dx-238561875000*y^2*Dy^3*Dx+158460300000*y*x*Dy^4
-435058593750*x^2*Dx^3+48944091796875*y^2*Dx^3-57585937500*x*Dx^4
+3039257812500*y*x*Dy*Dx^2+81057656250*y*Dy*Dx^3+524742187500*x^2*Dy^2*Dx
+10851679687500*y^2*Dy^2*Dx-8979750000*x*Dy^2*Dx^2-19032975000000*y*x*Dy^3
+64100610000*y*Dy^3*Dx+29412720000*x*Dy^4-8482939453125*y*Dx^3
-471557812500*x*Dy*Dx^2-2327422500000*y*Dy^2*Dx+587146950000*x*Dy^3
+428906250000*x*Dx^2+27292500000*Dx^3+47920429687500*y*Dy*Dx
-20893415625000*x*Dy^2+176690790000*Dy^2*Dx-5825612812500*Dy*Dx
+27365625000000*Dx$ ,

$36905625000000*y^4*Dy^6-12359619140625*y^3*Dx^6-10678710937500*y^3*Dy^2*Dx^4
-341718750000*y^3*Dy^4*Dx^2+478224000000*y*x^2*Dy^6-1880820000000*y^3*Dy^6
+679779052734375*y^2*x*Dx^5-751464843750*x^2*Dx^6+527343750000*y^2*Dx^6
+815734863281250*y^3*Dy*Dx^4-171386718750*y*x*Dy*Dx^5
-259875000000*x^2*Dy^2*Dx^4+1338750000000*y^2*Dy^2*Dx^4
+78310546875000*y^3*Dy^3*Dx^2+87609375000*y*x*Dy^3*Dx^3
+245754000000*x^2*Dy^4*Dx^2+390622500000*y^2*Dy^4*Dx^2
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 -3326062500000*y*x^2*Dy^5+962280000000000*y^3*Dy^5+8856000000*y*x*Dy^5*Dx
 -425088000000*y^2*Dy^6-106622314453125*y*x*Dx^5-79101562500*x^2*Dy*Dx^4
 -218276367187500*y^2*Dy*Dx^4-7129687500000*y*x*Dy^2*Dx^3
 +3969140625000*x^2*Dy^3*Dx^2-23622890625000*y^2*Dy^3*Dx^2
 +6100650000000*y*x*Dy^4*Dx+3591108000000*x^2*Dy^5-35450055000000*y^2*Dy^5
 -54382324218750*x^2*Dx^4+67977905273437 50*y^2*Dx^4-8701171875000*x*Dx^5
 +564807128906250*y*x*Dy*Dx^3+9758320312500*y*Dy*Dx^4
 +80384765625000*x^2*Dy^2*Dx^2+1578339843750000*y^2*Dy^2*Dx^2
 -1206843750000*x*Dy^2*Dx^3-68491406250000*y*x*Dy^3*Dx
 +6137167500000*y*Dy^3*Dx^2-24440906250000*x^2*Dy^4+7868580468750000*y^2*Dy^4
 +726192000000*x*Dy^4*Dx-7111368000000*y*Dy^5-1164221191406250*y*Dx^4
 -92578710937500*x*Dy*Dx^3-292514062500000*y*Dy^2*Dx^2+53763975000000*x*Dy^3*Dx
 -177632527500000*y*Dy^4+129748535156250*x*Dx^3-3887929687500*Dx^4
 +829291992187500 0*y*Dy*Dx^2+116064843750000*x*Dy^2*Dx+17208720000000*Dy^2*Dx^2
 +2284779656250000 0*y*Dy^3-26120772000000*Dy^4-989171718750000*Dy*Dx^2
 -203966628750000*Dy^3+4953515625000000*Dx^2+1873196859375000 0*Dy^2$ ]
sm1>quit ;
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2. Residue and residual duality

Let X be a domain in $\mathbf{C}^n$. Let $f_1, f_2, ..., f_n$ be a regular sequence of holomorphic functions on X. Let $\mathcal{I}$ be the ideal generated by $f_1, f_2, ..., f_n$ over $\mathcal{O}_X$. Let us denote by

$$\left[\begin{array}{c} 1 \\ f_1 f_2 \cdots f_n \end{array}\right] \in \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X)$$

the Grothendieck residue symbol associated to the meromorphic function

$$\frac{1}{f_1 f_2 \cdots f_n}.$$

Let i be the canonical map

$$\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/I, \mathcal{O}_X) \longrightarrow \mathcal{H}^n_{[A]}(\mathcal{O}_X),$$

where $A = \{z \in X \mid f_j(z) = 0,\ j = 1, 2, ..., n\}$.

Set:

$$m = i(\left[\begin{array}{c} 1 \\ f_1 f_2 \cdots f_n \end{array}\right]) \in H^n_{[A]}(\mathcal{O}_X).$$

We assume that the common locus A consists of finite number of points $A_k, k = 1, 2, ..., N$. Corresponding to the decomposition of the algebraic local cohomology group $\mathcal{H}^n_{[A]}(\mathcal{O}_X)$

$$\mathcal{H}^n_{[A]}(\mathcal{O}_X) = \mathcal{H}^n_{[A_1]}(\mathcal{O}_X) \oplus \mathcal{H}^n_{[A_2]}(\mathcal{O}_X) \oplus \ldots \oplus \mathcal{H}^n_{[A_N]}(\mathcal{O}_X),$$

we have $m = m_1 + m_2 + \cdots + m_N$ with $m_k \in H^n_{[A_k]}(\mathcal{O}_X)$.

Let Ω_X be the sheaf on X of holomorphic differential n-forms. The canonical pairing

$$\Omega_X \times H^n_{[A_k]}(\mathcal{O}_X) \longrightarrow H^n_{[A_k]}(\Omega_X)$$

composed with

$$H^n_{[A_k]}(\Omega_X) \longrightarrow \mathbf{C}$$

defines the residue pairing at the point A_k. Put

$$Res_{A_k}\langle\phi(z), m\rangle = \frac{1}{(2\pi i)^n}\oint_{A_k}\phi(z)mdz.$$

We regard $Res_{A_k}\langle\ \cdot\ , m\rangle$ as a linear map

$$\Omega_X \ni \phi(z)dz \longrightarrow Res_{A_k}\langle\phi(z), m\rangle \in \mathbf{C}.$$

Note There exists $m_k \in \mathbf{N}$ and complex constants $c_{\beta,k}$ $(0 \leq |\beta| \leq m_k)$ such that for every $\phi dz \in \Omega_X$,

$$Res_{A_k}\langle\phi(z), m\rangle = \sum_{0\leq|\beta|\leq m_k} c_{\beta,k}\cdot((\frac{\partial}{\partial z})^\beta\phi)(A_k).$$

3. Main Theorems

Let X be a domain in $\mathbf{C}^n$. Let $f_1, f_2, ..., f_n$ be a regular sequence of holomorphic functions on X. Let $A = \{z \in X \mid f_j(z) = 0,\ j = 1, 2, ..., n\}$. Let us denote by m the residue class

$$m = i(\left[\begin{array}{c} 1 \\ f_1 f_2 \cdots f_n \end{array}\right]) \in H^n_{[A]}(\mathcal{O}_X).$$

We assume that the common locus A consists of finite number of points $A_k, k = 1, 2, ..., N$. We have $m = m_1 + m_2 + \cdots + m_N$ with $m_k \in H^n_{[A_k]}(\mathcal{O}_X)$.

The following theorem asserts that the cohomology class m can be characterized as a solution of linear partial differential equations up to constant factor.

Theorem A *Let $\mathcal{J} = \{P \in \mathcal{D}_X \mid Pm = 0\}$ be the annihilator ideal of m. Then at each point A_k, we have*

$$\{u \mid Pu = 0, u \in H^n_{[A_k]}(\mathcal{O}_X), P \in \mathcal{J}\} = \{cm_k \mid c \in \mathbf{C}\}.$$

Proof

Put $\mathcal{M}_k = \mathcal{H}^n_{[A_k]}(\mathcal{O}_X)$. We have $m_k \in \mathcal{M}_k$. Since the algebraic local cohomology group $\mathcal{M}_k$ is simple as a $\mathcal{D}_X$-module, we have $\mathcal{D}_X m_k = \mathcal{M}_k$.

Hence we have

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{J},\mathcal{M}_k) &= \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X m,\mathcal{M}_k)\\ &= \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X m_k,\mathcal{M}_k)\\ &= \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_k,\mathcal{M}_k).\end{aligned}$$

The claim follows from the fact that

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_k,\mathcal{M}_k) = \mathbf{C}|_{A_k}.$$

q.e.d.

Let us recall the fact that Ω_X is naturally endowed with a structure of a right $\mathcal{D}_X$-module. The right action of $P = \sum a_\alpha(z)(\frac{\partial}{\partial z})^\alpha$ is described explicitly as follows.

$$(\phi(z)dz)P = (P^*\phi)(z)dz,$$

where $P^* = \sum(-\frac{\partial}{\partial z})^\alpha a_\alpha(z)$ is the formal adjoint of $P \in \mathcal{D}_X$.

If $P \in \mathcal{J}$, then we have

$$Res_{A_k}\langle (P^*\psi(z))dz, m\rangle = Res_{A_k}\langle (\psi(z)dz, Pm\rangle = 0.$$

Furthermore, we have the following Theorem, which is a direct consequence of a result of Kashiwara[11].

Theorem B *Let $\mathcal{J}$ be the annihilator ideal of m. Then, we have*

$$\begin{aligned}&\{\phi(z)dz \in \Omega_X \mid Res_{A_k}\langle \phi(z)dz, m\rangle = 0, \forall k = 1,2,...,N\}\\ &= \{(P^*\psi)(z)dz \mid P \in \mathcal{J}, \psi(z)dz \in \Omega_X\}.\end{aligned}$$

Note that, for the case of one variable, Theorem B provides a new theoretical foundation of the Horowitz-Ostrogradski algorithm ([10]) for the integration of rational functions.

4. Examples

Example Let $X = \{(x,y) \mid x,y \in \mathbf{C}\}$, $f(x,y) = y^2$, $g(x,y) = y - x^2$.

The multiplicity of intersection of these two curves at the origin is equal to 4. The cohomology class

$$m = [\frac{1}{y^2(y-x^2)}] \in \mathcal{H}^2_{[0,0]}(\mathcal{O}_X)$$

satisfies the following system of linear partial differential equations.

$$\begin{cases} y^2 m = 0, \\ (y - x^2)m = 0, \\ (xD_x + 2yD_y + 6)m = 0. \end{cases}$$

It is easy to see that the annihilator ideal $\mathcal{J}$ of m is generated by these three operators:

$$\mathcal{J} = \langle y^2, y - x^2, xD_x + 2yD_y + 6 \rangle.$$

Put $m = \sum a_{\alpha,\beta}[\frac{1}{x^\alpha y^\beta}]$. Since

$$(xD_x + 2yD_y + 6)\frac{1}{x^\alpha y^\beta} = (-\alpha - 2\beta + 6)\frac{1}{x^\alpha y^\beta},$$

we have $m = [\frac{a_{2,2}}{x^2y^2} + \frac{a_{4,1}}{x^4 y}]$. The second equation $(y - x^2)m = 0$ implies

$$m = \text{const}[\frac{1}{x^2y^2} + \frac{1}{x^4y}].$$

Let $\mathcal{I} = \langle y^2, y - x^2 \rangle$ be the ideal generated by y^2 and $y - x^2$ over the ring $\mathcal{O}_X$. Then the quotient space $\Omega_X/\Omega_X\mathcal{I}$ is a 4-dimensional vector space.

Put

$$K = \{\phi(x,y)dx \wedge dy \mid Res_{[0,0]}(\phi(x,y)dx \wedge dy, m) = 0\}.$$

Obviously we have $\Omega\mathcal{I} \subset K$.

Since $P^* = -xD_x - 2yD_y + 3$, we have $P^*1 = 3, P^*x = 2x, P^*x^2 = x^2, P^*x^3 = 0$. Therefore, the differential forms $dx \wedge dy, xdx \wedge dy$ and $x^2dx \wedge dy$ belong to K and the differential form $x^3dx \wedge dy$ gives a representative of a non-trivial element of Ω_X/K.

Example Take $f(x,y) = (x^2 + y^2)^2 + 3x^2y - y^3,\ g(x,y) = y - x^2$. Let $A = \{(x,y) \mid f(x,y) = g(x,y) = 0\}$. Then

$$A = \{(0,0)\} \cup \{(x,y) \mid y - x^2 = 0,\ y^2 + y + 4 = 0\}.$$

Put

$$m = [\frac{1}{fg}] \in \mathcal{H}^2_{[A]}(\mathcal{O}_X).$$

Let $\mathcal{J} \subset \mathcal{D}_X$ be the annihilating ideal of m. Then

$$\{Q_1,\ Q_2,\ P_1,\ P_2,\ P_3\}$$

is an involutory base of the ideal $\mathcal{J}$, where

$$\begin{aligned} Q_1 &= -x^2 + y, \\ Q_2 &= -y^4 - y^3 - 4y^2, \\ P_1 &= x(y^2 + y + 4)D_x + 2y(y^2 + y + 4)D_y + 10y^2 + 8y + 24, \\ P_2 &= y(y^2 + y + 4)D_x + 2xy(y^2 + y + 4)D_y + 2x(4y^2 + 3y + 8), \\ P_3 &= -y(y^2 + y + 4)D_x^2 + 6y(y^2 + y + 4)D_y + 24y^2 + 18y + 48. \end{aligned}$$

Since

$$\begin{aligned} P_1 &= (xD_x + 2yD_y + 6)(y^2 + y + 4), \\ P_2 &= (yD_x + 2xyD_y + 4x)(y^2 + y + 4), \\ P_3 &= (-yD_x^2 + 6yD_y + 12)(y^2 + y + 4), \end{aligned}$$

hold, the annihilator ideal $\mathcal{J}$ of the cohomology class m is generated by $y - x^2, y^2 + y + 4$ over $\mathcal{D}_X$ at $\{(x,y) \mid y - x^2 = 0,\ y^2 + y + 4 = 0\}$.

5. Appendix

Let $f_1, f_2, ..., f_n \in \mathbf{C}[z_1, z_2, ..., z_n]$ be a regular sequence of polynomials. Let $A = \{z \in \mathbf{C}^n \mid f_1(z) = f_2(z) = \cdots = f_n(z) = 0\}$ be the common locus of $f_1, f_2, ..., f_n$. We assume that $f_1, f_2, ..., f_n$ are in general position, i.e., the Jacobian determinant

$$Jac = \frac{\partial(f_1, f_2, ..., f_n)}{\partial(z_1, z_2, ..., z_n)}$$

does not vanish at any point $A_k \in A$. Then we have

$$Res_{A_k}\langle \phi(z), m\rangle = \frac{\phi(A_j)}{Jac(A_k)}.$$

By rewriting the above relation, we get

$$Jac(A_k) \cdot Res_{A_k}\langle \phi(z), m\rangle - \phi(A_j).$$

Let us introduce a new indeterminant t. We see that the residues of $\dfrac{\phi(z)dz}{f_1 f_2 \cdots f_n}$ should satisfy

$$\begin{cases} Jac(z)t - \phi(z) = 0, \\ f_1(z) = f_2(z) = \cdots = f_n(z) = 0. \end{cases}$$

We arrive at the following method for computing the residues of $\left[\dfrac{\phi(z)dz}{f_1 f_2 \cdots f_n}\right]$.

- Set
$$I = \langle f_1(z), f_2(z), ..., f_n(z), Jac(z)t - \phi(z)\rangle \subset \mathbf{C}[z_1, z_2, ..., z_n, t].$$

- Compute the Gröbner basis of the ideal I with respect to pure lexicographic order $z \succ t$ and then perform the primary decomposition of the polynomial ideal I.

Note that the above method is a natural generalization of Trager-Lazard-Rioboo and Czichowski algorithm ([5],[13],[28]).

Example Let $f(x,y) = y - x^2$, $g(x,y) = y - x - 2$, $\phi(x,y) = 1$.

Put $h(x,y,t) = Jac(x,y) \cdot t - 1$, where $Jac(x,y) = -2x + 1$ is the jacobian determinant of f, g. Let

$$I = \langle f, g, h \rangle \subset K[x,y,t].$$

Then the Gröbner base of I with respect to the pure lexicographic ordering$x \succ y \succ t$ is

$$\{-9t^2 + 1, 2y + 9t - 5, 2x + 9t + 1\}.$$

The primary decomposition of this ideal is given by

$$\langle 3t + 1, y - 4, x - 2 \rangle, \ \langle 3t - 1, y - 1, x + 1 \rangle.$$

We thus get

$$Res_{(2,4)} \langle 1, [\frac{1}{fg}] \rangle = -\frac{1}{3}, \quad Res_{(-1,1)} \langle 1, [\frac{1}{fg}] \rangle = \frac{1}{3}.$$

Example Let $f(x,y) = (x^2 + y^2)^2 + 3x^2 y - y^3$, $g(x,y) = 3x^2 + 3y^2 - 1$, $\phi(x,y) = 1$. Put $h(x,y,t) = Jac(x,y) \cdot t - 1$, where $Jac(x,y)$ is the jacobian determinant of f, g. Let

$$I = \langle f, g, h \rangle \subset K[x,y,t].$$

Then the Gröbner base of I with respect to the pure lexicographic ordering$x \succ y \succ t$ is

$$\{8t^2 - 1, -36y^3 + 9y + 1, -x + 12ty^2 - 2ty - 2t\}.$$

References

[1] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Math. **146** (1970), Springer.

[2] B. Buchberger, *Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems*, Aequationes Math. **4** (1970), 374–383.

[3] R. Caccioppoli, *Residui di integrali doppi e intersezioni di curve analitiche*, Ann. Mat. Pura Appl. (4) **29** (1949), 1–14.

[4] E. Cattani, A. Dickenstein, B. Sturmfels, *Computing multidimensional residues*, Algorithms in Algebraic Geometry and Applications, (eds L. González-Vega and T. Recio) Progress in Math. **143** (1996), 135–164.

[5] G. Czichowski, *A note on Gröbner bases and integration of rational functions*, J. Symbolic Comput. **20** (1995), 163–167.

[6] A.M. Dickenstein and C. Serra, *Duality methods for the membership problem*, in Progress in Math. **94** Effective Methods in Algebraic Geometry, eds T. Mora and C. Traverso, Birkhaüser (1991), 89–103.

[7] M.F. Didon, *Note sur une formule de calcule intégral*, Ann. École Norm. Sup. (2) **2** (1873), 31–48.

[8] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley Interscience, 1978.

[9] A. Grothendieck, *Théorèmes de dualité pour les faisceaux algébriques cohérents*, Séminaire BOURBAKI **149** (1957).

[10] E. Horowitz, *Algorithms for partial fraction decomposition and rational integration*, Proc. ACM Symposium on Symbolic and Algebraic Manipulation (SYMSAM'71), ed. S.R. Petrick (1971), 441–457.

[11] M. Kashiwara, *On the maximally overdetermined system of linear differenial equations,I*, Publ. RIMS, Kyoto Univ. **10** (1975), 563–579.

[12] M. Kashiwara, *On the holonomic systems of linear differntial eqauations,II*, Inventiones mathematicae **49** (1978), 121–135.

[13] D. Lazard and R. Rioboo, *Integration of rational functions: rational computation of the logarithmic part*, J. Symbolic Comput. **9** (1990), 113–115.

[14] Z. Mebkout, *Local cohomology of analytic spaces*, Publ. RIMS, Kyoto Univ. **12** Suppl. (1977), 247–256.

[15] Z. Mebkhout, *Dualité de Poincaré*, in Seminaire sur les Singularités (ed. Lê Dũng Tráng) Pub. Math. de l'Univ. Paris VII (1980), 139–182.

[16] Y. Nakamura and S. Tajima, *Residue theory and holonomic systems – the case of one variable*, preprint.

[17] M. Noro and T. Takeshima, *Risa/Asir–a computer algebra system*, in Proc. Internat. Symp. on Symbolic and Algebraic Computation, eds P.S. Wang, ACM New York (1992), 387–396 (ftp: endeavor.fujitsu.co.jp/pub/isis/asir).

[18] T. Oaku, *Computation of the characteristic variety and the singular locus of linear partial differential equations with polynomial coefficients*, Jpn. J. Indust. Appl. Math. **11** (1994), 485–497.

[19] T. Oaku, *Algorithms for b-functions, induced systems, and algebraic local cohomology of D-Modules*, Proc. Japan Acad. **72** (1996), 173–178.

[20] T. Oaku, *Algorithms for the b-functions, restrictions, and algebraic local cohomology groups of D-modules*, Adv. in Appl. Math. **19** (1997), 61–105.

[21] M. Passare, *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), 75–152.

[22] M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and pseudo-differential equations* Lecture Notes in Math. **287**, Springer-Verlag (1973), 265–529.

[23] T. Shimoyama and K. Yokoyama, *Localization and primary decomposition of polynomial ideals*, J. Symbolic Comput. **22** (1996), 247–277.

[24] N. Takayama, *Kan: A system for computation in algebraic analysis* (1991–), (http://www.math.s.kobe-u.ac.jp).

[25] S. Tajima, *A calculus of the tensor product of two holonomic systems with support on non-singular plane curves*, Proc. Japan Acad. **63**, Ser.A (1987), 390–391.

[26] S. Tajima, *Residual currents and tensor products of holonomic systems*, Sûrikaiseki Kenkyûshokôyûroku **725** (1990), 163–190.

[27] S. Tajima, *Grothendieck residue calculus and holonomic D-modules*, Proc. of the Fifth International Conference on Complex Analysis, Beijing, China, 1997, to appear.

[28] B.M. Trager, *Algebraic factoring and rational function integration*, Proc. of the 1976 ACM Symposium on Symbolic and Algebraic Computation, 219–226.