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光学的 Fredkin-Toffoli-Milburn ゲートについて

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Introduction

In order to construct an idealistic logical gate, Fredkin and Toffoli [1] proposed a logical conservative gate. Based on this logical gate, Milburn constructed a quantum logical gate [2] using a Mach - Zender interferometer with a Kerr medium. We call this gate a Fredkin - Toffoli - Milburn (FTM) gate in this paper.

The concept of channel is a fundamental tool to discuss the state change in several different fields [4, 5, 7]. The concept of quantum mutual entropy was formulated by Ohya [5, 6] measuring the amount of quantum information transmitted from an input system to an output system through a quantum channel.

In this paper, we construct a quantum channel for the FTM gate and discuss the information conservation by computing the quantum mutual entropy.

In section 1, we briefly explain quantum channel and the quantum mutual entropy. In section 2, we reformulate the FTM gate by means of a quantum channel. In section 3, we rigorously study information conservation through the FTM gate by the quantum mutual entropy.

1. Quantum channels and quantum mutual entropy

Let $(\mathbf{B}(\mathcal{H}_1), \mathfrak{S}(\mathcal{H}_1))$and $(\mathbf{B}(\mathcal{H}_2), \mathfrak{S}(\mathcal{H}_2))$ be input and output systems, respectively, where $\mathbf{B}(\mathcal{H}_k)$ is the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}_k$ and $\mathfrak{S}(\mathcal{H}_k)$ is the set of all density operators on $\mathcal{H}_k$ $(k = 1, 2)$. Quantum channel Λ^* is a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$.

(1) Λ^* is linear if $\Lambda^*(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\Lambda^*(\rho_1) + (1-\lambda)\Lambda^*(\rho_2)$ holds for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H}_1)$ and any $\lambda \in [0, 1]$.

(2) Λ^* is completely positive (C.P.) if Λ^* is linear and its dual $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ satisfies

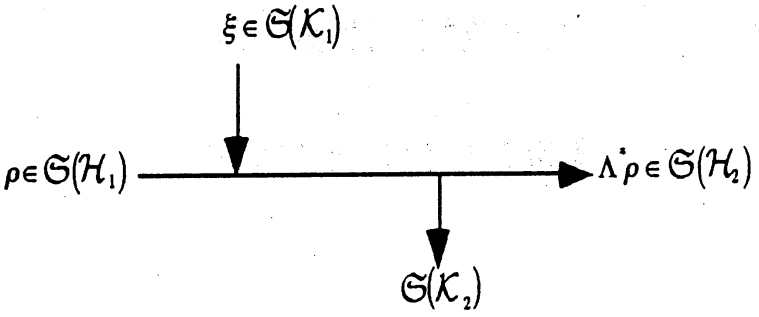
$$\sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j \geq 0$$

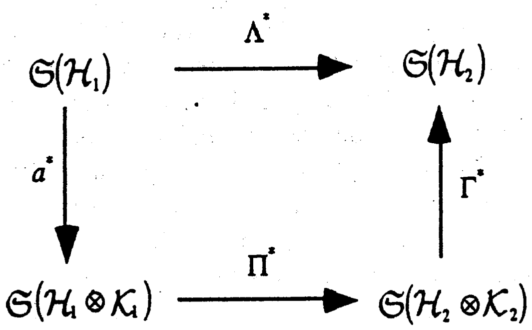
for any $n \in \mathbf{N}$, any $\{\overline{A}_i\} \subset \mathbf{B}(\mathcal{H}_2)$ and any $\{A_i\} \subset \mathbf{B}(\mathcal{H}_1)$, where the dual map Λ of Λ^* is defined by

$$tr\Lambda^*(\rho)B = tr\rho\Lambda(B), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathbf{B}(\mathcal{H}_2). \tag{1.1}$$

Almost all physical transformations are described by this mapping [4, 5, 7].We here explain how to mathematically construct a quantum channel describing quantum communication processes.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [5]: Let ρ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, ξ be a noise state in $\mathfrak{S}(\mathcal{K}_1)$.





The above maps Γ^*, a^* are given as

$$\Gamma^*(\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \tag{1.2}$$

$$a^*(\sigma) = tr_{\mathcal{K}_2}\sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2), \tag{1.3}$$

The map Π^* is a certain channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ determined by physical properties of the device transmitting information. Hence the channel for the above process is given in [5] as

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \xi) = (a^* \circ \Pi^* \circ \Gamma^*)(\rho) \tag{1.4}$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the attenuation channel and the noisy quantum channel are constructed as follows:

(1) Attenuation channel Λ_0^* was formulated in[5] such as

$$\begin{aligned} \Lambda_0^*(\rho) &= tr_{\mathcal{K}_2}\Pi_0^*(\rho \otimes \xi_0) \\ &= tr_{\mathcal{K}_2}V_0(\rho \otimes |0\rangle\langle 0|)V_0^*, \end{aligned} \tag{1.5}$$

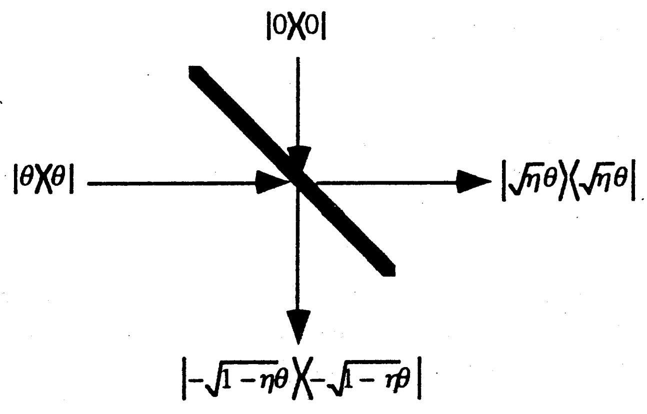
where $\xi_0 = |0\rangle\langle 0|$ is the vacuum state in $\mathfrak{S}(\mathcal{K}_1)$, V_0 is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V_0(|n_1\rangle \otimes |0\rangle) = \sum_j^{n_1} C_j^{n_1} |j\rangle \otimes |n_1 - j\rangle, \tag{1.6}$$

$$C_j^{n_1} = \sqrt{\frac{n_1!}{j!(n_1-j)!}\eta^j (1-\eta)^{n_1-j}} \tag{1.7}$$

where $|n_1\rangle$ is the n_1 photon number state vector in $\mathcal{H}_1$ and η is a transmission rate of the channel. For the coherent state $|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|$, the $\Pi_0^*(|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|)$ is obtained by

$$\Pi_0^*(|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|) = |\sqrt{\eta}\theta\rangle\langle\sqrt{\eta}\theta| \otimes \left|-\sqrt{1-\eta}\theta\right\rangle\left\langle-\sqrt{1-\eta}\theta\right|.$$



(2) Noisy quantum channel Λ^* with a noise state ξ is defined in [12] as

$$\begin{aligned}\Lambda^*(\rho) &= tr_{\mathcal{K}_2}\Pi^*(\rho\otimes\xi) \\ &= tr_{\mathcal{K}_2} V(\rho\otimes\xi)V^*,\end{aligned} \tag{1.8}$$

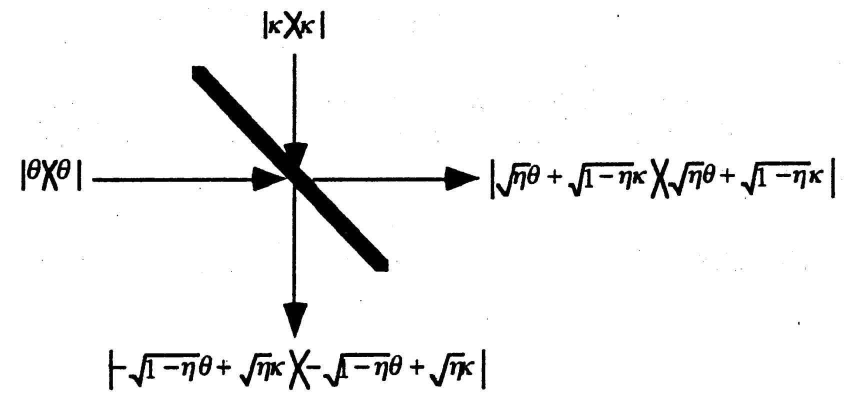
Here V is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_j^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1+m_1-j\rangle \tag{1.9}$$

$$\begin{aligned}C_j^{n_1,m_1} &= \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \\ &\quad \times \sqrt{\eta^{m_1-j+2r}(1-\eta)^{n_1+j-2r}},\end{aligned} \tag{1.10}$$

where $K = \min\{n_1, j\}$, $L = \max\{m_1 - j, 0\}$. For the coherent state $|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|$, the $\Pi^*(|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|)$ is obtained by

$$\Pi^*(|\theta\rangle\langle\theta|\otimes|0\rangle\langle 0|) = \left|\sqrt{\eta}\theta+\sqrt{1-\eta}\kappa\right\rangle\left\langle\sqrt{\eta}\theta+\sqrt{1-\eta}\right| \\ \otimes\left|-\sqrt{1-\eta}\theta+\sqrt{\eta}\kappa\right\rangle\left\langle-\sqrt{1-\eta}\theta+\sqrt{\eta}\kappa\right|.$$



A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropy of a state ρ was introduced by von Neumann [3] as

$$S(\rho)\equiv -\mathrm{tr}\rho\log\rho \tag{1.11}$$

If $\rho=\sum_k\lambda_k E_k$ is the Schatten decomposition [10] (i.e., λ_k is the eigenvalue of ρ and E_k is the one-dimensional projection associated with λ_k, this decomposition is not unique unless every eigenvalue is non-degenerated), then

$$S(\rho)=-\sum_k\lambda_k\log\lambda_k, \tag{1.12}$$

because $\{\lambda_k\}$ is a probability distribution. Therefore the von Neumann entropy contains the Shannon entropy [13] as a special case.

In order to define the quantum mutual entropy, we need a compound state [5, 6] corresponding to the joint distribution in classical systems. That is, the compound state of an input state and a channel Λ^*. is defined by

$$\sigma_E\equiv\sum_k\lambda_k E_k\otimes\Lambda^* E_k, \tag{1.13}$$

which expresses the correlation between the initial state ρ and the final state $\Lambda^*\rho$.

The mutual entropy $I(\rho;\Lambda^*)$ with respect to an input state ρ and a quantum channel Λ^* should satisfy the following conditions [5, 7]: (1) If a channel is trivial, i.e., $\Lambda^*=id$ (identical map), then $I(\rho;\Lambda^*)=S(\rho)$. (2) When system is classical, the quantum mutual entropy reduced to classical one. (3) Shannon's fundamental inequality $I(\rho;\Lambda^*)\le S(\rho)$ is satisfied. This mutual entropy for a state $\rho\in\mathfrak{S}(\mathcal{H}_1)$ and a channel Λ^* was given in [5] as follows:

$$I(\rho;\Lambda^*) \equiv \sup\{S(\sigma_E,\sigma_0); E=\{E_k\}\} \quad (1.14)$$

$$= \sup\left\{\sum_k \lambda_k S(\Lambda^* E_k, \Lambda^*\rho); E=\{E_k\}\right\}, \quad (1.15)$$

where the supremum is taken over all Schatten decompositions of ρ and $S(\Lambda^* E_k, \Lambda^*\rho)$ is the relative entropy [14] defined by

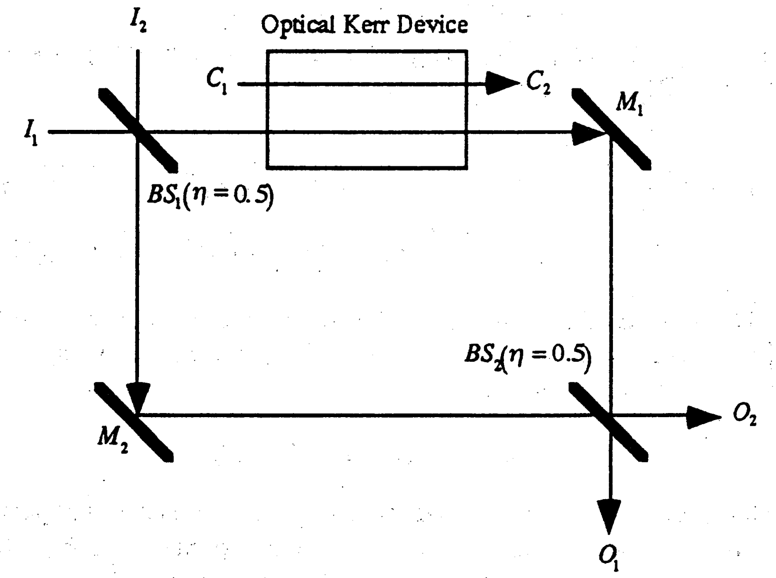
$$S(\Lambda^* E_k, \Lambda^*\rho) = tr\Lambda^* E_k (\log \Lambda^* E_k - \log \Lambda^*\rho). \quad (1.16)$$

This quantum mutual entropy contains other definitions of the mutual entropy for other channels like classical input and quantum output [8].

2. Quantum channel for Fredkin-Toffoli-Milburn gate

Fredkin and Toffoli [1] proposed a conservative gate, by which any logical gate is realized and it is shown to be a reversible gate in the sense that there is no loss of information. This gate was developed by Milburn [2] as a quantum gate with quantum input and output. We call this gate Fredkin-Toffoli-Milburn (FTM) gate here. In this section, we first formulate the FTM gate [2] by means of quantum channels and discuss the information conservation using the quantum mutual entropy in the next section.

The FTM gate is composed of two input gates I_1, I_2 and one control gate C. Two inputs come to the first beam splitter and one spliting input passes through the control gate made from an optical Kerr device, then two spliting inputs come in the second beam splitter and appear as two outputs (Fig.2.1). We construct quantum channels to express the beam splitters and the optical Kerr medium and discuss the works of the above gate, in particular, conservation of information.



(1) Beam splitters: (a) Let V_1 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_1 given by

$$V_1 (|n_1\rangle \otimes |n_2\rangle) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_1,n_2} |j\rangle \otimes |n_1 + n_2 - j\rangle \tag{2.1}$$

for any photon number state vectors $|n_1\rangle \otimes |n_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The quantum channel Π^*_{BS1} expressing the first beam splitter (beam splitter 1) is defined by

$$\Pi^*_{BS1} (\rho_1 \otimes \rho_2) \equiv V_1 (\rho_1 \otimes \rho_2) V_1^* \tag{2.2}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for an input state in two gates I_1 and I_2 given by the tensor product of two coherent states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi^*_{BS1}(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned} \Pi^*_{BS1}(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right\rangle \left\langle \sqrt{\eta_1}\theta_1 + \sqrt{1-\eta_1}\theta_2\right| \\ &\otimes \left|-\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right\rangle \left\langle -\sqrt{1-\eta_1}\theta_1 + \sqrt{\eta_1}\theta_2\right|. \end{aligned} \tag{2.3}$$

(b) Let V_2 be a mapping from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with transmission rate η_2 given by

$$V_2 (|n_1\rangle \otimes |n_2\rangle) \equiv \sum_{j=0}^{n_1+n_2} C_j^{n_2,n_1} |n_1 + n_2 - j\rangle \otimes |j\rangle \tag{2.4}$$

for any photon number state vectors $|n_1\rangle \otimes |n_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The quantum channel Π^*_{BS2} expressing the second beam splitter (beam splitter 2) is defined by

$$\Pi^*_{BS2} (\rho_1 \otimes \rho_2) \equiv V_2 (\rho_1 \otimes \rho_2) V_2^* \tag{2.5}$$

for any states $\rho_1 \otimes \rho_2 \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. In particular, for coherent input states $\rho_1 \otimes \rho_2 = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2|$, $\Pi^*_{BS2}(\rho_1 \otimes \rho_2)$ is written as

$$\begin{aligned} \Pi^*_{BS2}(\rho_1 \otimes \rho_2) &= \left|\sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right\rangle \left\langle \sqrt{\eta_2}\theta_1 - \sqrt{1-\eta_2}\theta_2\right| \\ &\otimes \left|\sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right\rangle \left\langle \sqrt{1-\eta_2}\theta_1 + \sqrt{\eta_2}\theta_2\right|. \end{aligned} \tag{2.6}$$

(2) Optical Kerr medium: The interaction Hamiltonian in the optical Kerr medium is given in [2] by the number operators N_1 and N_c for the input system 1 and the Kerr medium, respectively, such as

$$H_{int} = \hbar\chi (N_1 \otimes I_2 \otimes N_c), \tag{2.7}$$

where $\hbar$ is the Plank constant divided by 2π, χ is a constant proportional to the susceptibility of the medium and I_2 is the identity operator on $\mathcal{H}_2$. Let T be the passing time of a beam through the Kerr medium and put $\sqrt{F} = \hbar\chi T$, a parameter exhibiting the power of the Kerr effect. Then the unitary operator U_K describing the evolution for time T in the Kerr medium is given by

$$U_K = \exp\left(-i\sqrt{F}\left(N_1 \otimes I_2 \otimes N_c\right)\right). \tag{2.8}$$

We assume that an initial (input) state of the control gate is a number state $\xi = |n\rangle\langle n|$, a quantum channel Λ_K^* representing the optical Kerr effect is given by

$$\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi) \equiv U_K(\rho_1 \otimes \rho_2 \otimes \xi)U_K^* \tag{2.9}$$

for any state $\rho_1 \otimes \rho_2 \otimes \xi \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$. In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|$, $\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi)$ is denoted by

$$\begin{aligned}&\Lambda_K^*(\rho_1 \otimes \rho_2 \otimes \xi)\\ =\ & \left|\exp\left(-i\sqrt{F}n\right)\theta_1\right\rangle\left\langle\exp\left(-i\sqrt{F}n\right)\theta_1\right| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|,\end{aligned} \tag{2.10}$$

Using the above channels, the quantum channel for the whole FTM gate is constructed as follows: Let both one input and output gates be described by $\mathcal{H}_1$, another input and output gates be described by $\mathcal{H}_2$ and the control gate be done by $\mathcal{K}$, all of which are Fock spaces. For a total state $\rho_1 \otimes \rho_2 \otimes \xi$ of two input states and a control state, the quantum channels $\Lambda_{BS1}^*, \Lambda_{BS2}^*$ from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ are written by

$$\Lambda_{BSk}^*(\rho_1 \otimes \rho_2 \otimes \xi) = \Pi_{BSk}^*(\rho_1 \otimes \rho_2) \otimes \xi \quad (k = 1, 2) \tag{2.11}$$

Therefore, the whole quantum channel Λ_{FTM}^* of the FTM gate is defined by

$$\Lambda_{\mathrm{FTM}}^* \equiv \Lambda_{BS2}^* \circ \Lambda_K^* \circ \Lambda_{BS1}^*. \tag{2.12}$$

In particular, for an initial state $\rho_1 \otimes \rho_2 \otimes \xi = |\theta_1\rangle\langle\theta_1| \otimes |\theta_2\rangle\langle\theta_2| \otimes |n\rangle\langle n|$, $\Lambda_{\mathrm{FTM}}^*(\rho_1 \otimes \rho_2 \otimes \xi)$ is obtained by

$$\begin{aligned}&\Lambda_{\mathrm{FTM}}^*(\rho_1 \otimes \rho_2 \otimes \xi)\\ =\ & |\mu\theta_1 + \nu\theta_2\rangle\langle\mu\theta_1 + \nu\theta_2| \otimes |\nu\theta_1 + \mu\theta_2\rangle\langle\nu\theta_1 + \mu\theta_2| \otimes |n\rangle\langle n|\end{aligned} \tag{2.13}$$

where

$$\mu = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}n\right) + 1\right\}, \tag{2.14}$$

$$\nu = \frac{1}{2}\left\{\exp\left(-i\sqrt{F}n\right) - 1\right\}. \tag{2.15}$$

3. Information change in optical Fredkin-Toffoli-Milburn gate

In this section, we examine information conservation in the FTM gate by computing the mutual entropy.

Although the control gate, hence the Hilbert space $\mathcal{K}$, is necessary to make the truth table, the original information is carried by the input states, so it is interesting to study conservation of the information from the input to the output. For this purpose, we need the quantum channel Λ^* describing the change of states from the input gate to the output gate, which is defined as

$$\Lambda^*(\rho_1 \otimes \rho_2) \equiv tr_{\mathcal{K}} \Lambda^*_{\mathrm{FT\,M}}(\rho_1 \otimes \rho_2 \otimes \xi) \tag{3.1}$$

for any input states $\rho_1 \otimes \rho_2$.

The total channel $\Lambda^*_{\mathrm{FT\,M}}$ is obviously unitarily implemented from the construction discussed in the previous section, but the channel Λ^* is not so as seen below:

When Λ^* is unitarily implemented, that is $\Lambda^*(\rho) = U\rho U^*$, $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with a certain unitary operator U, the dual Λ is written as $\Lambda(A) = U^*AU$ for any $A \in \mathbf{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Therefore for the CONS (complete orthonormal system) consisting of number vector states, namely, $\{|n_1\rangle\}$ in $\mathcal{H}_1$, $\{|n_2\rangle\}$ in $\mathcal{H}_2$, an equality

$$tr\Lambda(|n_1\rangle\langle k_1| \otimes |n_2\rangle\langle k_2|) = \delta_{n_1 k_1}\delta_{n_2 k_2}$$

should be satisfied. However the direct computation according to the definition of Λ^* implies the equality

$$\begin{aligned}
& tr\Lambda(|n_1\rangle\langle k_1| \otimes |n_2\rangle\langle k_2|) \\
= & \sum_{m_1}\sum_{m_2} tr\Lambda^*(|m_1\rangle\langle m_1| \otimes |m_2\rangle\langle m_2|)|n_1\rangle\langle k_1| \otimes |n_2\rangle \otimes \langle k_2| \\
= & \sum_{m_1}\sum_{m_2}\sum_{j=0}^{m_1+m_2}\sum_{j'=0}^{m_1+m_2} C_j^{m_1,m_2}\overline{C_{j'}^{m_1,m_2}} \exp(-i\sqrt{F}n(j-j')) \\
& \times \sum_{i=0}^{m_1+m_2}\sum_{i'=0}^{m_1+m_2} C_i^{m_1+m_2-j,j}\overline{C_{i'}^{m_1+m_2-j',j'}}\,\delta_{k_1,m_1+m_2-i}\delta_{k_2,i}\delta_{m_1+m_2-i',n_1}\delta_{i',n_2},
\end{aligned}$$

where $\sum_{m_j}|m_j\rangle\langle m_j| = I_j$, identity operator on $\mathcal{H}_j$ $(j = 1,2)$. The above equality is not zero if and only if

$$n_1 + n_2 = k_1 + k_2.$$

Thus Λ^* is not unitarily implemented.

The next question is whether the information carried by two input states is preserved after passing through the whole gate, that is, whether the following equality is held or not for a certain class of input states $\rho = \rho_1 \otimes \rho_2$.

$$S(\rho) = S(\rho_1) + S(\rho_2) = I(\rho; \Lambda^*)$$

This equality means that all information carried by $\rho = \rho_1 \otimes \rho_2$ is completely transmitted to the output gates. If the channel Λ^* is unitarily implemented as $\Lambda^*_{\mathrm{FT\,M}}$, then the above equality is satisfied [10]. However, our Λ^* is not, so it is important to check the above equality.

Let us consider any state ρ_i given by

$$\rho_i = \lambda_i|0\rangle\langle 0| + (1-\lambda_i)|\theta_i\rangle\langle\theta_i| \text{ , } (i=1,2) \tag{3.2}$$

with $\lambda_i \in [0,1]$. Such a state is often used to send information expressed by two symbols 0 and 1. In order to compute quantum entropy and mutual entropy, we need the Schatten decomposition of $\rho = \rho_1 \otimes \rho_2$, which is uniquely given in [11] such that

$$\rho_i = \|\rho_i\| E_0^i + (1-\|\rho_i\|)E_1^i, (i=1,2) \tag{3.3}$$

where$\|\rho_i\|$ is one of the eigenvalues of ρ_i and E_0^i is its associated one dimensional projection;

$$\|\rho_i\| = \frac{1+\sqrt{1-4\lambda_i(1-\lambda_i)(1-\exp(-(|\theta_i|^2)))}}{2} \tag{3.4}$$

The Schatten decomposition of $\rho = \rho_1 \otimes \rho_2$ is written by

$$\rho = \sum_{j=0}^{1}\sum_{k=0}^{1} \mu_j^1 \mu_k^2 E_j^1 \otimes E_k^2,$$

where $\mu_0^i = \|\rho_i\|$ and $\mu_1^i = 1 - \|\rho_i\|$ $(i=1,2)$. Then von Neumann entropy of ρ becomes

$$S(\rho) = -\sum_{i=1}^{2}\sum_{j=0}^{1} \mu_j^i \log \mu_j^i.$$

We assume $\xi = |n\rangle\langle n|$ $(n \neq 0)$ and $\sqrt{F}n = (2m+1)\pi$ $(m = 0,1,2,\cdots)$. For the input state $\rho = \rho_1 \otimes \rho_2$, the output state $\Lambda^*\rho$ is given by

$$\Lambda^*\rho = \sigma_2 \otimes \sigma_1,$$

where $\sigma_i = \lambda_i|0\rangle\langle 0| + (1-\lambda_i)|-\theta_i\rangle\langle -\theta_i|$, $(i=1,2)$. Then von Neumann entropy of $\Lambda^*\rho$ is

$$S(\Lambda^*\rho) = S(\sigma_2) + S(\sigma_1) = S(\rho). \tag{3.5}$$

Since $\Lambda^*\left(E_j^1 \otimes E_k^2\right)$ is pure state, .$S\left(\Lambda^*\left(E_j^1 \otimes E_k^2\right)\right) = 0$ for each j,k. Thus the quantum mutual entropy is

$$\begin{aligned} I(\rho;\Lambda^*) &= S(\Lambda^*\rho) - \{\sum_{j=0}^{1}\sum_{k=0}^{1} \mu_j^1\mu_k^2 S\left(\Lambda^*\left(E_j^1 \otimes E_k^2\right)\right)\} \\ &= S(\Lambda^*\rho) = S(\rho). \end{aligned} \tag{3.6}$$

This equalities means that there does not exist the loss of information for the quantum channel of the FTM gate. Therefore the information is preserved for Λ^* through the FTM gate. From this result, the FTM gate is considered to be an idealistic logical gate for quantum computer. Along the line of our study for quantum computation, the notion of quantum complexity will be useful [9].

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