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Order preserving operator function via the inequality "$A\geq B\geq 0$ ensures $(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1+r}{p+r}}\geq(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p\geq 1$ and $r\geq 0$"

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1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in H$ and also an operator T is strictly positive (denoted by $T>0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem [12][10].

Theorem F (Furuta inequality) [4].

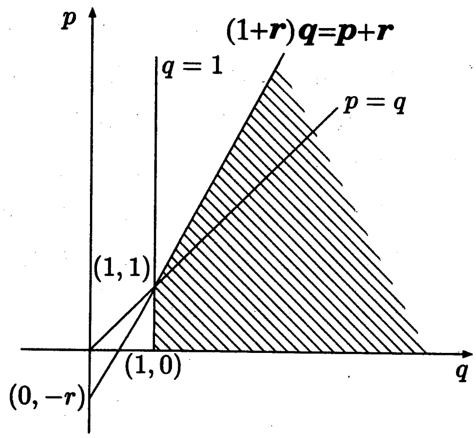
If $A\geq B\geq 0$, then for each $r\geq 0$

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}}\geq(B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}}\geq(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p\geq 0$ and $q\geq 1$ with $(1+r)q\geq p+r$.



Figure

We remark that Theorem F is essentially the same as the inequality made in its title and Theorem F yields the Löwner-Heinz theorem when we put $r=0$ in (i) or (ii) stated above: $A\geq B\geq 0$ ensures $A^{\alpha}\geq B^{\alpha}$ for any $\alpha\in[0,1]$. Alternative proofs of Theorem F are given in [2] [5] and [11] and also elementary one page proof in [6]. It was shown in [13] that the domain surrounded by p, q and r in the Figure is the best possible one for Theorem F. In [8] we established the following Theorem G as extensions of Theorem F.

Theorem G (Generalized Furuta inequality) [8]. *If $A\geq B\geq 0$ with $A>0$, then for each $t\in[0,1]$ and $p\geq 1$,*

$$F_{p,t}(A,B,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r\geq t$ and $s\geq 1$ and $F_{p,t}(A,A,r,s)\geq F_{p,t}(A,B,r,s)$, that is, for each $t\in[0,1]$ and $p\geq 1$,

(1.1) $$A^{1-t+r}\geq\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s\geq 1$ and $r\geq t$.

Recently a nice mean theoretic proof of Theorem G is shown in [3]. Ando-Hiai [1] established excellent log majorization results and proved the useful inequality equivalent to the main log majorization theorem as follows; If $A \geq B \geq 0$ with $A > 0$, then

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ and $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself and also extends results of [7].

Since now, many applications of Theorem F and Theorem G have been developed in the following branches by many authors.

APPLICATIONS OF THEOREM F

(A) OPERATOR INEQUALITIES

(1) Characterizations of operators satisfying $\log A \geq \log B$
(2) Generalizations of Ando's theorem
(3) Other order preserving operator inequalities
(4) Applications to the relative operator entropy
(5) Applications to Ando-Hiai log majorization
(6) Generalized Aluthge transformation

(B) NORM INEQUALITIES

(1) Several generalizations of Heinz-Kato theorem
(2) Generalizations of some theorems on norms
(3) An extension of Kosaki trace inequality and parallel results

(C) OPERATOR EQUATIONS

(1) Generalizations of Pedersen-Takesaki theorem and related results

Very recently the following result is obtained as an extension of Theorem G.

Theorem H [9]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1], q \geq 0$ and $p \geq \max\{q,t\}$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$. Moreover for each $t \in [0,1], q \in [t,1]$ and $p \geq q$, $G_{p,q,t}(A,A,r,s) \geq G_{p,q,t}(A,B,r,s)$, that is,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \tag{1.2}$$

holds for any $s \geq 1$ and $r \geq t$.

The proof in [8] of Theorem G is complicated and technical and also the proof in [3] is based on mean theoretic one. Here we show a simplified proof of Theorem H which is an extension form of Theorem G only using Theorem F and the following Lemma F.

Lemma F (Furuta lemma) [8]. *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Firstly we show a short proof of the inequality (1.2) of Theorem H. Secondly we show a proof of the monotonicity of the function $G_{p,q,t}(A, B, r, s)$ of Theorem H. Lastly we give three counterexamples and a conjecture related to Theorem G and Theorem H.

2 Results on inequalities

Theorem H-i [9]. *If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \tag{1.2}$$

holds for $s \geq 1$ and $r \geq t$.

Theorem H-i is proved as an immediate consequence of the following Theorem 1.

Theorem 1. *Let S and T be positive invertible operators on a Hilbert space such that $S^{\beta_0} \geq (S^{\frac{\beta_0}{2}}T^{\alpha_0}S^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then*

$$S^{\frac{\beta}{2}}T^{\alpha_0}S^{\frac{\beta}{2}} \geq (S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\alpha_0+\beta}{\alpha+\beta}} \tag{2.1}$$

holds for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

Proof of Theorem 1. Applying (ii) of Theorem F to the hypothesis $S^{\beta_0} \geq (S^{\frac{\beta_0}{2}}T^{\alpha_0}S^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$, we have

$$S^{(1+r_1)\beta_0} \geq \{S^{\frac{\beta_0 r_1}{2}}(S^{\frac{\beta_0}{2}}T^{\alpha_0}S^{\frac{\beta_0}{2}})^{\frac{\beta_0 p_1}{\alpha_0+\beta_0}}S^{\frac{\beta_0 r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \quad \text{for any } p_1 \geq 1 \text{ and } r_1 \geq 0. \tag{2.2}$$

Putting $p_1 = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1$ in (2.2), we have

$$S^{(1+r_1)\beta_0} \geq (S^{\frac{(1+r_1)\beta_0}{2}}T^{\alpha_0}S^{\frac{(1+r_1)\beta_0}{2}})^{\frac{(1+r_1)\beta_0}{\alpha_0+(1+r_1)\beta_0}}. \tag{2.3}$$

Put $\beta = (1+r_1)\beta_0 \geq \beta_0$ in (2.3). Then we have

$$S^{\beta} \geq (S^{\frac{\beta}{2}}T^{\alpha_0}S^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \quad \text{for } \beta \geq \beta_0. \tag{2.4}$$

(2.4) is equivalent to the following (2.5) by Lemma F

$$(2.5) \qquad T^{\alpha_0} \le (T^{\frac{\alpha_0}{2}} S^\beta T^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}} \qquad \text{for } \beta \ge \beta_0.$$

Again applying (i) of Theorem F to (2.5), we have

$$(2.6) \quad T^{(1+r_2)\alpha_0} \le \{T^{\frac{\alpha_0 r_2}{2}} (T^{\frac{\alpha_0}{2}} S^\beta T^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p_2}{\alpha_0+\beta}} T^{\frac{\alpha_0 r_2}{2}}\}^{\frac{1+r_2}{p_2+r_2}} \qquad \text{for any } p_2 \ge 1 \text{ and } r_2 \ge 0.$$

Putting $p_2 = \frac{\alpha_0+\beta}{\alpha_0} \ge 1$ in (2.6), we have

$$(2.7) \qquad T^{(1+r_2)\alpha_0} \le (T^{\frac{(1+r_2)\alpha_0}{2}} S^\beta T^{\frac{(1+r_2)\alpha_0}{2}})^{\frac{(1+r_2)\alpha_0}{(1+r_2)\alpha_0+\beta}}.$$

Put $\alpha = (1+r_2)\alpha_0 \ge \alpha_0$ in (2.7). Then we have

$$(2.8) \qquad T^\alpha \le (T^{\frac{\alpha}{2}} S^\beta T^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \qquad \text{for } \alpha \ge \alpha_0 \text{ and } \beta \ge \beta_0.$$

Raise each side of (2.8) to the power $\frac{\alpha-\alpha_0}{\alpha} \in [0,1]$ by Löwner-Heinz theorem, we have the first inequality of the following (2.9)

$$(2.9) \qquad \begin{aligned} T^{\alpha-\alpha_0} &\le (T^{\frac{\alpha}{2}} S^\beta T^{\frac{\alpha}{2}})^{\frac{\alpha-\alpha_0}{\alpha+\beta}} \\ &= T^{\frac{\alpha}{2}} S^{\frac{\beta}{2}} (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}})^{\frac{\alpha-\alpha_0}{\alpha+\beta}-1} S^{\frac{\beta}{2}} T^{\frac{\alpha}{2}} \qquad \text{by Lemma F.} \end{aligned}$$

refining (2.9) and taking inverses of both sides, we obtain (2.1).

Proof of Theorem H-i. If $A \ge B \ge 0$, then the following (2.10) holds

$$(2.10) \qquad A^{q+r} \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{q+r}{p+r}} \quad \text{for } p \ge q, q \in [0,1] \text{ and } r \ge 0$$

by (ii) of Theorem F since $(1+r)\frac{p+r}{q+r} \ge p+r$ and $\frac{p+r}{q+r} \ge 1$ in this case.

In the case $t=0$. (1.2) is valid by (2.10) in this case.

In the case $p=q=t \in [0,1]$. Let $C = A^{\frac{-t}{2}} B^t A^{\frac{-t}{2}}$. As $I \ge C \ge 0$ holds by Löwner-Heinz theorem, $A^r \ge A^{\frac{r}{2}} C^s A^{\frac{r}{2}}$ for $s \ge 1$, that is, (1.2) holds in this case.

In the case $p > t > 0$. Put $X = (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1}{p-t}}$. Then we have $A^{\frac{t}{2}} X^{p-t} A^{\frac{t}{2}} = B^p$ and $A \ge (A^{\frac{t}{2}} X^{p-t} A^{\frac{t}{2}})^{\frac{1}{p}}$ by the hypothesis $A \ge B \ge 0$. Put $\beta_0 = t \in (0,1]$ and $\alpha_0 = p-t > 0$. Then $A \ge (A^{\frac{\beta_0}{2}} X^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{1}{\alpha_0+\beta_0}}$, and

$$A^{\beta_0} \ge (A^{\frac{\beta_0}{2}} X^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$$

holds by Löwnew-Heinz theorem. Put $\alpha = (p-t)s$ and $\beta = r$. Then $\alpha \ge \alpha_0$ and $\beta \ge \beta_0$ hold since $s \ge 1$ and $r \ge t$ hold, so that Theorem 1 ensures the following inequality

$$(A^{\frac{\beta}{2}} X^\alpha A^{\frac{\beta}{2}})^{\frac{\alpha_0+\beta}{\alpha+\beta}} \le A^{\frac{\beta}{2}} X^{\alpha_0} A^{\frac{\beta}{2}},$$

that is, we have

$$(2.11) \qquad \begin{aligned} &\{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{p-t+r}{(p-t)s+r}} \\ \le &A^{\frac{r}{2}} A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}} A^{\frac{r}{2}}. \end{aligned}$$

Raising each side of (2.11) to the power $\frac{q-t+r}{p-t+r} \in [0,1]$ by Löwner-Heinz theorem, we have the first inequality of the following (2.12)

$$\begin{aligned} &\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \\ (2.12)\qquad &\leq (A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{q+r-t}{p+r-t}} \\ &\leq A^{q-t+r} \end{aligned}$$

and the last inequality holds by replacing r by $r-t \geq 0$ in (2.10), so the proof of Theorem H-i is complete.

3 Results on functions

Theorem H-f [9]. *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1], q \geq 0$ and $p \geq \max\{q,t\}$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$.

Theorem H-f is proved as an immediate consequence of the following Theorem 2.

Theorem 2. *Let S and T be positive invertible operators on a Hilbert space such that $S^{\beta_0} \geq (S^{\frac{\beta_0}{2}}T^{\alpha_0}S^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then for fixed $\delta \geq -\beta_0$,*

$$f(\alpha,\beta) = S^{\frac{-\beta}{2}}(S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}}S^{\frac{-\beta}{2}}$$

is a decreasing function of both α and β for $\alpha \geq \max\{\delta,\alpha_0\}$ and $\beta \geq \beta_0$.

Proof of Theorem 2.

(a) *Proof of the result that $f(\alpha,\beta)$ is a decreasing function of α for $\alpha \geq \max\{\delta,\alpha_0\}$.*

The hypothesis in Theorem 2 ensures (3.1) in the same way as the proof of Theorem 1

(3.1) $$(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \geq T^{\alpha} \qquad \text{for all } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0.$$

(3.1) yields the following (3.2) by Löwner-Heinz theorem

(3.2) $$(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{u}{\alpha+\beta}} \geq T^{u} \qquad \text{for all } \alpha \geq \alpha_0, \beta \geq \beta_0 \text{ and any } u \text{ such that } \alpha \geq u \geq 0.$$

Then we have

$$\begin{aligned} g(\alpha) =& (S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} \\ =& \{(S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\alpha+u+\beta}{\alpha+\beta}}\}^{\frac{\delta+\beta}{\alpha+u+\beta}} \\ =& \{S^{\frac{\beta}{2}}T^{\frac{\alpha}{2}}(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{u}{\alpha+\beta}}T^{\frac{\alpha}{2}}S^{\frac{\beta}{2}}\}^{\frac{\delta+\beta}{\alpha+u+\beta}} \qquad \text{by Lemma F} \\ \geq& \{S^{\frac{\beta}{2}}T^{\frac{\alpha}{2}}T^{u}T^{\frac{\alpha}{2}}S^{\frac{\beta}{2}}\}^{\frac{\delta+\beta}{\alpha+u+\beta}} \\ =& (S^{\frac{\beta}{2}}T^{\alpha+u}S^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+u+\beta}} = g(\alpha+u) \end{aligned}$$

and the last inequality holds by (3.2) and Löwner-Heinz theorem since $\frac{\delta+\beta}{\alpha+u+\beta} \in [0,1]$ holds by the hypothesis on α, β and δ. Hence $f(\alpha,\beta) = S^{\frac{-\beta}{2}} g(\alpha) S^{\frac{-\beta}{2}}$ is a decreasing function of α for $\alpha \geq \max\{\delta, \alpha_0\}$.

(b) *Proof of the result that $f(\alpha,\beta)$ is a decreasing function of β for $\beta \geq \beta_0$.*

By Lemma F,

$$\begin{aligned} f(\alpha,\beta) &= S^{\frac{-\beta}{2}}(S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}}S^{\frac{-\beta}{2}} \\ &= T^{\frac{\alpha}{2}}(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{\delta-\alpha}{\alpha+\beta}}T^{\frac{\alpha}{2}} \end{aligned}$$

and (3.1) is equivalent to the following (3.3) by Lemma F

$$S^{\beta} \geq (S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \quad \text{for all } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0. \tag{3.3}$$

(3.3) yields the following (3.4) by Löwner-Heinz theorem

$$S^{v} \geq (S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{v}{\alpha+\beta}} \quad \text{for all } \alpha \geq \alpha_0, \beta \geq \beta_0 \text{ and any } v \text{ such that } \beta \geq v \geq 0. \tag{3.4}$$

Then we have

$$\begin{aligned} h(\beta) =&(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{\delta-\alpha}{\alpha+\beta}} \\ =&\{(T^{\frac{\alpha}{2}}S^{\beta}T^{\frac{\alpha}{2}})^{\frac{\alpha+\beta+v}{\alpha+\beta}}\}^{\frac{\delta-\alpha}{\alpha+\beta+v}} \\ =&\{T^{\frac{\alpha}{2}}S^{\frac{\beta}{2}}(S^{\frac{\beta}{2}}T^{\alpha}S^{\frac{\beta}{2}})^{\frac{v}{\alpha+\beta}}S^{\frac{\beta}{2}}T^{\frac{\alpha}{2}}\}^{\frac{\delta-\alpha}{\alpha+\beta+v}} \qquad \text{by Lemma F} \\ \geq&\{T^{\frac{\alpha}{2}}S^{\frac{\beta}{2}}S^{v}S^{\frac{\beta}{2}}T^{\frac{\alpha}{2}}\}^{\frac{\delta-\alpha}{\alpha+\beta+v}} \\ =&(T^{\frac{\alpha}{2}}S^{\beta+v}T^{\frac{\alpha}{2}})^{\frac{\delta-\alpha}{\alpha+\beta+v}} = h(\beta+v) \end{aligned}$$

and the last inequality holds by (3.4) and Löwner-Heinz theorem since $\frac{\delta-\alpha}{\alpha+\beta+v} \in [-1,0]$ and taking inverses. Hence $f(\alpha,\beta) = T^{\frac{\alpha}{2}}h(\beta)T^{\frac{\alpha}{2}}$ is a decreasing function of β for $\beta \geq \beta_0$.

Consequently we have finished a proof of Theorem 2 by (a) and (b).

Proof of Theorem H-f. We consider the case $p > t > 0$. Put $X = (A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{\frac{1}{p-t}}$. Then we have $A^{\frac{t}{2}}X^{p-t}A^{\frac{t}{2}} = B^{p}$ and $A \geq (A^{\frac{t}{2}}X^{p-t}A^{\frac{t}{2}})^{\frac{1}{p}}$ by the hypothesis $A \geq B \geq 0$. Put $\beta_0 = t \in (0,1]$ and $\alpha_0 = p-t > 0$. Then $A \geq (A^{\frac{\beta_0}{2}}X^{\alpha_0}A^{\frac{\beta_0}{2}})^{\frac{1}{\alpha_0+\beta_0}}$, so that

$$A^{\beta_0} \geq (A^{\frac{\beta_0}{2}}X^{\alpha_0}A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$$

holds by Löwnew-Heinz theorem. Put $\alpha = (p-t)s, \beta = r$ and $\delta = q-t$. The hypothesis $t \in (0,1], q \geq 0$ and $p \geq \max\{q,t\}$ in Theorem H-f satisfy the conditions required on α, β and δ in Theorem 2, that is, $\delta \geq -\beta_0$, $\alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \beta_0$. Applying Theorem 2,

$$\begin{aligned} f(\alpha,\beta) =&A^{\frac{-\beta}{2}}(A^{\frac{\beta}{2}}X^{\alpha}A^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}}A^{\frac{-\beta}{2}} \\ =&A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \\ =&G_{p,q,t}(A,B,r,s) \end{aligned}$$

is decreasing for $r \geq t$ and $s \geq 1$, so the proof in the case $p > t > 0$ is complete.

In the case $t = 0$, Theorem H-f easily follows by [7, Theorem 3].

In the case $p = t \geq q \geq 0$. Let $C = A^{\frac{-t}{2}} B^t A^{\frac{-t}{2}}$. Then $I \geq C \geq 0$ by Löwner-Heinz theorem, so that $A^r \geq A^{\frac{r}{2}} C^s A^{\frac{r}{2}}$ holds since $I \geq C \geq 0$ and $s \geq 1$, and again by Löwner-Heinz theorem

$$A^u \geq (A^{\frac{r}{2}} C^s A^{\frac{r}{2}})^{\frac{u}{r}} \qquad \text{for } r \geq u \geq 0. \tag{3.5}$$

Then we obtain

$$\begin{aligned}
G_{t,q,t}(A, B, r, s) &= A^{\frac{-r}{2}} (A^{\frac{r}{2}} C^s A^{\frac{r}{2}})^{\frac{q-t+r}{r}} A^{\frac{-r}{2}} \\
&= C^{\frac{s}{2}} (C^{\frac{s}{2}} A^r C^{\frac{s}{2}})^{\frac{q-t}{r}} C^{\frac{s}{2}} \qquad \text{by Lemma F} \\
&= C^{\frac{s}{2}} \{(C^{\frac{s}{2}} A^r C^{\frac{s}{2}})^{\frac{r+u}{r}}\}^{\frac{q-t}{r+u}} C^{\frac{s}{2}} \\
&= C^{\frac{s}{2}} \{C^{\frac{s}{2}} A^{\frac{r}{2}} (A^{\frac{r}{2}} C^s A^{\frac{r}{2}})^{\frac{u}{r}} A^{\frac{r}{2}} C^{\frac{s}{2}}\}^{\frac{q-t}{r+u}} C^{\frac{s}{2}} \qquad \text{by Lemma F} \\
&\geq C^{\frac{s}{2}} \{C^{\frac{s}{2}} A^{\frac{r}{2}} A^u A^{\frac{r}{2}} C^{\frac{s}{2}}\}^{\frac{q-t}{r+u}} C^{\frac{s}{2}} \\
&= C^{\frac{s}{2}} (C^{\frac{s}{2}} A^{r+u} C^{\frac{s}{2}})^{\frac{q-t}{r+u}} C^{\frac{s}{2}} \\
&= A^{\frac{-(r+u)}{2}} (A^{\frac{r+u}{2}} C^s A^{\frac{r+u}{2}})^{\frac{q-t+r+u}{r+u}} A^{\frac{-(r+u)}{2}} = G_{t,q,t}(A, B, r+u, s)
\end{aligned} \tag{3.6}$$

and the last inequality holds by (3.5) and Löwner-Heinz theorem since $\frac{q-t}{r+u} \in [-1, 0]$ and taking inverses. Consequently $G_{t,q,t}(A, B, r, s)$ is a decreasing function of both $r \geq t$ and $s \geq 1$ because $G_{t,q,t}(A, B, r, s)$ is decreasing of $s \geq 1$ by (3.6) since $I \geq C \geq 0$.

Whence the proof of Theorem H-f is complete.

4 Best possibility and counterexamples

We discuss best possibility of (1.1) in Theorem G and also we cite counterexamples related to Theorem G.

Counterexample 1. There exists a counterexample to (1.1) of Theorem G if we replace $A \geq B$ in Theorem G by $\log A \geq \log B$. Let $p = 2, t = 1, r = 2$ and $s = 2$. Then p, t, r and s satisfy the condition in Theorem G. Take A and B as

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}^2, \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}^2.$$

Then it turns out that $\log A \geq \log B$ holds since $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \geq \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ and $\log t$ is operator monotone, but $A \not\geq B$ holds and

$$A^{1-t+r} - \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} = \begin{pmatrix} 50.1594\cdots & 61.8403\cdots \\ 61.8403\cdots & 74.8485\cdots \end{pmatrix},$$

so that the eigenvalues of $A^{1-t+r} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ are $-0.5563\cdots$ and $125.5643\cdots$, therefore $A^{1-t+r} \not\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$.

Hence we can't replace $A \geq B$ in Theorem G by $\log A \geq \log B$, which is weaker than $A \geq B \geq 0$.

Counterexample 2. There exists a counterexample to (1.1) of Theorem G if r and t don't satisfy the condition $r \geq t$. Let $p = 2, s = 2, t = 1 \in [0,1]$ and $r = \frac{1}{2}$. Then $r \not\geq t$. Take A and B as

$$A = \begin{pmatrix} 28 & 44 \\ 44 & 73 \end{pmatrix}, \qquad B = \begin{pmatrix} 20 & 36 \\ 36 & 65 \end{pmatrix}.$$

Then $A \geq B \geq 0$ and

$$A^{1-t+r} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} = \begin{pmatrix} 1.9229\cdots & 0.6555\cdots \\ 0.6555\cdots & -0.0547\cdots \end{pmatrix},$$

so that the eigenvalues of $A^{1-t+r} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ are $-0.2523\cdots$ and $2.1205\cdots$, therefore $A^{1-t+r} \not\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$.

Counterexample 3. There exists a counterexample to (1.1) of Theorem G if t don't satisfy the condition $t \in [0,1]$. Let $t = 1.2 \notin [0,1], p = 2, r = 2, s = 2$. Then $r \geq t$. Take A and B as

$$A = \begin{pmatrix} 125 & 90 \\ 90 & 69 \end{pmatrix}, \qquad B = \begin{pmatrix} 125 & 90 \\ 90 & 65 \end{pmatrix}.$$

Then $A \geq B \geq 0$ and

$$A^{1-t+r} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} = \begin{pmatrix} 33.3128\cdots & 43.4624\cdots \\ 43.4624\cdots & 55.3433\cdots \end{pmatrix}$$

so that the eigenvalues of $A^{1-t+r} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ are $-0.5084\cdots$ and $89.1646\cdots$, therefore $A^{1-t+r} \not\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$.

Remark. We remark the following result. By using his skillful and excellent technique as almost same as one in [13], K. Tanahashi [14] asserts that $\frac{1-t+r}{(p-t)s+r}$ of the right hand side of (1.1) of Theorem G is best possible in the sense of the following: $A^{(1-t+r)\alpha} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{(1-t+r)\alpha}{(p-t)s+r}}$ does not hold for any $\alpha > 1$ in Theorem G.

At the end of this section, we cite the following conjecture related to Theorem H and Theorem G.

Conjecture. *There exists a counterexample to Theorem G in general for any $r < t$.*

If $t = 0$ and $r < 0$ in Theorem G, we have already obtained a counterexample.

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