数理解析研究所講究録
1042 巻 1998 年 78-83

Dynamical systems of Certain non-holomorphic maps

東海大理学部 内村桂輔 (Keisuke Uchimura)

1 Introduction

Complex dynamical systems given by quadratic polynomials

$$f_c(z) = z^2 + c$$

have many well-known important analytic and geometric properties. When $c = -2$, $f_{-2}(z)$ has relation to a Chebyshev polynomial. The Julia set of $f_{-2}(z)$ is the interval $[-2, 2]$. For any z in $[-2, 2]$, we set

$$z = \psi(\theta) = e^{\pi\theta i} + e^{-\pi\theta i},$$

where ψ is a function from $\mathbf{R}$ to $[-2, 2]$. Clearly

$$\psi(\theta) = \psi(\theta + 2) = \psi(2 - \theta).$$

Then we introduce an equivalence relation $\sim$ generated by $\theta \sim \theta + 2$, $\theta \sim 2 - \theta$. Hence we have an induced map

$$\psi : \mathbf{R}/\sim\ \to [-2, 2].$$

Let t be a transformation on $\mathbf{R}/\sim$ defined by $t(\theta) = 2\theta$. Then two dynamical systems $\{[-2, 2], f_{-2}\}$ and $\{\mathbf{R}/\sim,\ t\}$ are topological conjugate and the induced map ψ is a topological conjugacy. $\{\mathbf{R}/\sim,\ t\}$ is equivalent to the dynamical system given by a tent map on $[0, 1]$.

Two-dimensinal extension of these dynamical systems is considered. Let

$$z = \Psi(\sigma, \tau) = e^{2\pi\sigma i} + e^{-2\pi\tau i} + e^{2\pi(\tau-\sigma)i}.$$

Then Ψ maps $\mathbf{R}^2$ to a subset S in the complex plane $\mathbf{C}$. Clearly

$$\Psi(\sigma, \tau) = \Psi(\sigma + 1, \tau) = \Psi(\sigma, \tau + 1).$$

Then Ψ may be considered as a map from two-dimensional tours T^2 to S. Further

$$\Psi(\sigma, \tau) = \Psi(\sigma, \sigma - \tau) = \Psi(-\sigma + \tau, \tau) = \Psi(1 - \tau, 1 - \sigma).$$

Then we introduce an equivalence relation $\sim$ on T^2 defined by

$$(\sigma,\tau) \sim (\sigma, \sigma-\tau) \sim (-\sigma+\tau, \tau) \sim (1-\tau, 1-\sigma). \tag{1.1}$$

Hence we have an induced map

$$\Psi : T^2/\sim \to S.$$

An extension of $f_{-2}(z)$ is a map :

$$F_2(z) = z^2 - 2\bar{z}.$$

Then

$$F_2(\Psi(\sigma,\tau)) = \Psi(2\sigma, 2\tau).$$

Let d be a double angle map of $T^2/\sim$ onto itself. That is,

$$d(\sigma,\tau) = (2\sigma, 2\tau).$$

Then $\{S, F_2\}$ and $\{T^2/\sim, d\}$ are topological conjugate. Hence $F_2(z)$ is a two-dimension extension of $f_{-2}(z)$. Uchimura [1996] shows that $\{S, F_2\}$ is chaotic.

2 The dynamics of F_2

We study the dynamical system given by

$$F_2(z) = z^2 - 2\bar{z}. \tag{2.1}$$

We shall show that F_2 partition the complex plane $\mathbf{C}$ into two sets. One is a closed domain S and the other is its complement. The dynamics of F_2 on S is chastic and the complement of S is the basin of ∞ of F_2.

Since $F_2(z)$ is not holomorphic, we regard the function F_2 as a function of two real variables. That is,

$$F_2((x,y)) = (x^2 - y^2 - 2x, 2xy + 2y). \tag{2.2}$$

Here $z = x + iy$. The Jacobian determinant of F_2 is $4x^2 + 4y^2 - 4$. So the set of critical values is an algebraic curve of the fourth degree which is known as Steiner's hypocycloid. It takes the form

$$(x^2 + y^2 + 9)^2 + 8(-x^3 + 3xy^2) - 108 = 0. \tag{2.3}$$

Let S be a closed region bounded by Steiner's hypocycloid (2.3). See [Koornwinder, 1974].

We shall show that the function $F_2(z)$ restricted to the set S is a finite-to-one factor of the algebraic endomorphism of the torus given by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. We set

$$z = \exp(2\pi\sigma i) + \exp(-2\pi\tau i) + \exp(2\pi(\tau - \sigma)i). \tag{2.4}$$

Then the point (σ,τ) is regarded as a point in the torus $T \simeq \mathbf{R}^2/\mathbf{Z}^2$. From Then, we have a torus endomorphism l on T given by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. From (2.4), we know that the point (σ,τ) is equivalent to the points $(-\sigma+\tau,\tau)$, $(\sigma,\sigma-\tau)$ and $(1-\tau,1-\sigma)$. That is to say, we have to consider an equivalence relation given by

$$(\sigma,\tau)\sim(-\sigma+\tau,\tau),\ (\sigma,\tau)\sim(\sigma,\sigma-\tau),\ (\sigma,\tau)\sim(1-\tau,1-\sigma). \qquad (2.5)$$

Let $R = T/\sim$.

The set R is regarded as a closed region bounded by a regular triangle OAB with $O=(0,0)$, $A=(\frac{1}{2},\frac{-1}{2\sqrt{3}})$, $B=(\frac{1}{2},\frac{1}{2\sqrt{3}})$, in the (s,t)-plane, where

$$s=(\sigma+\tau)/2,\ t=\sqrt{3}(\sigma-\tau)/2.$$

Now we consider the symbolic dynamics. For any point $x \in R$, we define the itinerary $h(x)$ by the rule

$$h(x) = (s_0 s_1 s_2 ...)$$

where $s_j = k$ if and only if $d^j(x) \in \Delta(k)$, $(k=0,1,2,3)$. Here $\Delta(0)=\Delta DEF$, $\Delta(1)=\Delta OEF$, $\Delta(2)=\Delta ADF$, $\Delta(3)=\Delta BED$, where D,E and F are the midpoints of AB, OB and OA, respectively. Set

$$\Sigma_4 = \{(s_0 s_1 s_2 ...)\ |\ s_j = 0,1,2 \text{ or } 3\}.$$

We introduce an equivalence relation $\sim$ on Σ_4 which is defined by

$$\begin{aligned}(w_1 0 w_2) &\sim (w_1 2 w_2), \text{ for } w_1 \in \{0,1,2,3\}^*,\ (w_2) \in h(OA),\\ (w_1 0 w_2) &\sim (w_1 3 w_2), \text{ for } w_1 \in \{0,1,2,3\}^*,\ (w_2) \in h(OB),\\ (w_1 0 w_2) &\sim (w_1 1 w_2), \text{ for } w_1 \in \{0,1,2,3\}^*,\ (w_2) \in h(AB).\end{aligned}$$

Let Σ' denote the quotient $\Sigma_4/\sim$. Note that σ is naturally defined on Σ'. That is,

$$\sigma[(s_0 s_1 ... s_n ...)] = [(s_1 ... s_n ...)].$$

Clearly σ is well-defined.

Theorem 2.1. The mapping $\tilde{h}\ :\ R \to \Sigma'$ is a bijection and it makes the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{d} & R \\ \tilde{h}\downarrow & & \downarrow\tilde{h} \\ \Sigma' & \xrightarrow[\sigma]{} & \Sigma' \end{array}$$

Let Γ be a Sierpinsky gasket whose outermost triangle is OAB. Then it can be easily seen that

$$d(\Gamma) = \Gamma.$$

Let

$$\Sigma" = \{[(s_0 s_1 ... s_n ...)] \,|s_i \in \{1,2,3\}, \text{ for all } i\}.$$

The shift map σ is naturally defined on $\Sigma"$. Hence we have the following corrollary.

Corrollary 2.1. The mapping $\tilde{h} : \Gamma \to \Sigma"$ is a bijection and it makes the following diagram commutative :

$$\begin{array}{ccc} \Gamma & \xrightarrow{d} & \Gamma \\ \tilde{h} \downarrow & & \downarrow \tilde{h} \\ \Sigma" & \xrightarrow[\sigma]{} & \Sigma" \end{array}$$

Theorem 2.2. The dynamical system $\{S, F_2\}$ is chaotic. That is,
(1) it is transitive;
(2) it is sensitive to intial conditions;
(3) the set of periodic orbits of F_2 is dense in S.

For any complex number $z \notin S$, we have the following theorem.

Theorem 2.3 For any complex number $z \notin S$,

$$(F_2)^n(z) \to \infty \quad \text{as} \quad n \to \infty.$$

3 Attracting basins of $F_{\frac{4}{3}}$

In this section we consider perturbations of the mapping $F_2(z)$. We study the dynamical systems given by

$$F_c(z) = z^2 - c\bar{z}.$$

Such a map is one of the simplest q.c. mappings except for affine mappings. The mapping also satisfies a simple property

$$\frac{\partial^2}{\partial z \partial \bar{z}} F_c^2 = -4cz,$$

for Laplace operator.

First we note that there are four fixed points and 12 periodic points of prime period 2 which are listed in Uchimura[1977].

We study the dynamics of $F_c(z)$ when $c = \frac{4}{3}$. The value $c = \frac{4}{3}$ is a special value in the sense that all attractive fixed points and attractive periodic points of period 2 lie on a critical set. We will study attractive basins of $F_{\frac{4}{3}}$.

When $c = \frac{4}{3}$, two attracting fixed points $P^2_{(1)}$ and $P^3_{(1)}$ are written as

$$P^2_{(1)} = -\frac{1}{6} + i\sqrt{\frac{5}{12}}$$

$$F^3_{(1)} = \overline{P^2_{(1)}}.$$

There are four attracting periodic points of period 2 which are written as

$$P^2_{(2)} = \omega P^2_{(1)}, P^3_{(2)} = \omega^2 P^2_{(1)}$$

$$P^4_{(2)} = \omega P^3_{(1)}, P^5_{(2)} = \omega^2 P^3_{(1)}.$$

Critical points of $F_{\frac{4}{3}}(z)$ are the zeros of Jacobian determinent of $F_{\frac{4}{3}}$. Then critical set of $F_{\frac{4}{3}}(z)$ is equal to a circle $\{z; 2 \mid z \mid = \frac{4}{3}\}$. Note that the attracting periodic points $P^2_{(1)}$, $P^3_{(1)}$, $P^2_{(2)}$, $P^3_{(2)}$, $P^4_{(2)}$, $P^5_{(2)}$ lie on the circle. Set
$A = (\frac{4}{3}, 0)$, $B = (-\frac{2}{3}, \frac{2\sqrt{3}}{3})$ and $C = (-\frac{2}{3}, -\frac{2\sqrt{3}}{3})$.

Theorem 3.1. The interiors of the triangular regions $\triangle OBD$ and $\triangle OCF$ are the immediate basins of fixed points $P^2_{(1)}$ and $P^3_{(1)}$. The interiors of $\triangle OBE$ and $\triangle OAD$ are immediate basins of periodic points $P^4_{(2)}$ and $P^5_{(2)}$. The interiors of $\triangle OCE$ and $\triangle OAF$ are immediate basin of periodic points $P^2_{(2)}$ and $P^3_{(2)}$.

By the symmetry we know that to prove Theorem 3.1 it suffices to prove the following proposition.

Proposition 3.1. Let z be any interior point of $\triangle OBD$. Then

$$(F_{\frac{4}{3}})^n(z) \to P^2_{(1)} \quad (n \to \infty).$$

This proposition is proved in Uchimura[1997].

References

Koornwinder, T. H. [1974] "Orthogonal polynimials in two variables which are eigenfunctions of two algebraically independent partial differential operators III IV", *Indag. Math.* **36**, 357-381.

Uchimura, K. [1996] " The dynamical systems associated with Chebyshev polynomials in two variables", *Int. J. Bifurcation and Chaos*, **6**, 12B, 2611-2618.

Uchimura, K. [1997] "Attracting basins of certain non-holomorphic map", *Res. Rep. Dept. Math. Tokai Univ.*, **13**, 1-13.