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Graphical Methods for Determining/Estimating Optimal Repair-Limit Replacement Policies

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Abstract: In this paper, we consider two kinds of repair-limit replacement models and develop the corresponding graphical methods to estimate the optimal repair-time limits which minimize the expected costs per unit time in the steady-state. Then, both the total time on test statistics and the Lorenz statistics play important roles. Some analytical results are derived to describe the relationship between two models. Numerical examples are devoted to illustrate the asymptotic optimality of non-parametric estimators for the optimal repair-limit policies.

Keywords: repair-limit replacement policies, graphical methods, total time on test statistics, Lorenz statistics, partial ordering, non-parametric estimation, stochastic optimization models.

1. INTRODUCTION

In general, system maintenance models may be classified into two categories; preventive maintenance and corrective maintenance. The preventive maintenance is executed in advance to avoid a catastrophic failure state. On the other hand, the corrective maintenance is made to place recovery actions efficiently after failures occur. The repair-limit replacement problems determine how to design the recovery mechanism of a system using two maintenance alternatives; repair and replacement, in terms of cost minimization. First this problem was considered by Hestings [1] for army vehicles and proposed three methods of optimizing the repair-limit policies by simulation, hill-climbing and dynamic programming. Nakagawa and Osaki [2, 3], Okumoto and Osaki [4], Nguyen and Murthy [5, 6] and Kaio and Osaki [7] reformulated the Hastings' original model from the viewpoint of renewal reward processes and discussed different repair-limit replacement problems.

In the most repair-limit replacement problems [2-7], it is assumed that the distribution function of the completion time to repair a failed unit is arbitrary but known. This assumption seems to be rather strong in many practical situations. To this end, practitioners have to determine the repair-time limit under incomplete information on the repair-time distribution in most cases. Applying a graphical idea by Bergman [8] and Bergman and Klefsjö [9-11], the authors [12-15] analyzed several repair-limit replacement problems and proposed the corresponding non-parametric methods to estimate the optimal repair-time limits from the complete sample of repair-time data.

Then the total time on test (TTT) concept (see, *e.g.* Barlow and Campo [16] and Barlow [17]) is very useful to develop the estimation procedure. In other words, some kinds of repair-limit replacement problems are reduced to the well-known age replacement-type problem [8-11] and can be solved by the similar graphical technique based on the TTT concept. However, such a specific method does not always available for all maintenance problems. For instance, if the cost criterion such as the expected cost per unit time in the steady-state can not be represented by two variables; the repair-time distribution function and the associated scaled TTT transform, the method mentioned above will lose its usefulness. Hence, alternative devices instead of the TTT statistics should be applied to the graphical and non-parametric estimation method for different types of maintenance problems.

In this paper, we propose estimation methods based on the Lorenz curve as well as the TTT statistics for two repair-limit replacement problems. The Lorenz curve was introduced first by Lorenz [18] in Economics to describe income distributions. Since the Lorenz curve is essentially equivalent to the Pareto curve used in the quality control, it will be one of the most important statistics in every social science. The more general and tractable definition of the Lorenz curve was made by Gastwirth [19]. Goldie [20] proved the strong consistency of the empirical Lorenz curve and discovered several convergence properties of it. Chandra and Singpurwalla [21] and Klefsjö [22] investigated the relationship between the TTT statistics and the Lorenz statistics and derived a few aging and partial ordering properties. Recently, Aly [23] developed the testing for the Lorenz ordering.

It should be noted that the underlying repair-limit replacement problems have to be analyzed by using different devices from their cost structure, respectively. In other words, both devices are not used at the same time. The paper is organized as follows. In Section 2, we introduce two repair-limit replacement problems considered by Nakagawa and Osaki [2, 3] and develop the corresponding graphical methods to derive the optimal repair-time limits which minimize the expected costs per unit time in the steady-state. In Section 3, we obtain the comparative results to describe the relationship between two models. In Section 4, the statistical estimation problems are discussed. We show analytically and numerically that estimators of the optimal repair-time limits have strong consistency. Finally, the paper is concluded with some remarks.

2. REPAIR-LIMIT REPLACEMENT MODELS

2.1 Model 1

Consider a single unit system, where each spare is provided only by an order after a lead time L (> 0) and each failed unit is repairable. The original unit begins operating at time 0. The mean lifetime for each unit is $1/\lambda$ (> 0). When the unit has failed, the repair is started immediately. If the repair is completed up to the time limit for repair $t_0 \in [0, \infty)$, then the unit is installed at that time. It is assumed that the unit once repaired is presumed as good as new. However, if the repair time is greater than t_0, *i.e.* the repair is not completed after the time t_0, the repair is retired and the failed unit is scrapped. Then, the spare unit is ordered immediately and delivered after the lead time L. The time required for replacement is negligible for convenience. The repair time for each unit has an arbitrary distribution $G(t)$ with density $g(t)$ and finite mean $1/\mu$ (> 0), where the function $G(\cdot)$ is assumed to have an inverse function, *i.e.* $G^{-1}(\cdot)$, and to be absolutely continuous and strictly increasing. Without any loss of generality, we assume $G(0) = 0$ and $\lim_{t\to\infty} G(t) = 1$. Under these model assumptions, we define the time interval from the start of the operation to the following start as one cycle. The configuration of Model 1 is depicted in Fig. 1.

Next, we consider the cost structure. The costs considered in this paper are the following;

k_r (> 0): a cost per unit repair time

k_s (> 0): a cost per shortage period

c (> 0): an ordering cost for each spare unit.

We make the assumption;

(A-1) $k_r L < c$.

This assumption implies that the unit ordering cost is greater than the repair cost during the interval $[0, L]$, *i.e.* until the delivery of a new unit. For an infinite planning horizon, it will be appropriate to adopt an expected cost per unit time in the steady-state. Since the mean time of one cycle is

$$T_1(t_0) = \int_0^{t_0} (1/\lambda + t)dG(t) + \int_{t_0}^{\infty} (1/\lambda + t_0 + L)dG(t)$$

$$= \quad 1/\lambda + \int_0^{t_0} \overline{G}(t)dt + L\overline{G}(t_0) \tag{1}$$

and the total expected cost for one cycle is

$$V_1(t_0) = (k_r + k_s)\int_0^{t_0} \overline{G}(t)dt + (k_s L + c)\overline{G}(t_0), \tag{2}$$

where $\overline{G}(t) = 1 - G(t)$, then the cost per unit time in the steady-state is, from the well-known renewal reward argument [24, p. 52],

$$\begin{aligned} C_1(t_0) &\equiv \lim_{t\to\infty} \frac{[\text{the total cost on } (0,t]]}{t} \\ &= V_1(t_0)/T_1(t_0) \end{aligned} \tag{3}$$

and the problem is to determine the optimal repair-time limit $t_0{}^*$ such as

$$C_1(t_0^*) = \min_{0\le t_0<\infty} C_1(t_0). \tag{4}$$

It is straightforward to seek t_0^* by differentiating $C_1(t_0)$ with respect to t_0, but we here employ the following graphical method. Define the scaled total time on test (TTT) transform of the repair-time distribution $p \equiv G(t)$ by

$$\phi_1(p) \equiv \mu \int_0^{G^{-1}(p)} \overline{G}(t)dt, \quad (0 \le p \le 1), \tag{5}$$

where

$$G^{-1}(p) = \inf\{t \ge 0 : G(t) \ge p\}. \tag{6}$$

The curve $\ell_1 = (p, \phi_1(p)) \in [0,1] \times [0,1]$ is called *the scaled TTT transform* or simply *the scaled TTT curve.* We shall propose a graphical method to solve the problem in Eq.(4) on the scaled TTT curve.

The following result is due to Koshimae, Dohi, Kaio and Osaki [13].

THEOREM 2.1: Suppose the assumption (A-1) for Model 1. The minimization problem in Eq.(4) is equivalent to obtain p^* $(0 \le p^* \le 1)$ satisfying

$$\min_{0\le p\le 1} : M_1(p, \phi_1(p)) \equiv \frac{\phi_1(p) + \xi}{p + \eta}, \tag{7}$$

where

$$\xi \equiv \frac{(k_s L + c)\mu}{(c - k_r L)\lambda} > 0 \tag{8}$$

and

$$\eta \equiv -\Big(1 + \frac{(k_r + k_s)}{(k_r L - c)\lambda}\Big). \tag{9}$$

From THEOREM 2.1, the optimal policy is $p^* = G(t_0^*)$ which minimizes the tangent slope from $(-\eta_1, -\xi_1)$ to the curve ℓ_1.

More precisely, we characterize the optimal policy from the aging property of $G(t)$.

DEFINITION 2.2: (1) The repair-time distribution $G(t)$ is IHR (DHR) if and only if the hazard rate $r(t) = g(t)/\overline{G}(t)$ is increasing (decreasing).
(2) $G(t)$ is IHR (DHR) if and only if $\phi_1(p)$ is concave (convex) in $p \in [0,1]$.

The relationship (2) between the aging and the scaled TTT transform was proved by Barlow and Campo [16]. In the plane $(x,y) = (-\infty,+\infty)\times(-\infty,+\infty)$, define the following three points

$$\mathrm{B} \equiv (x_B, y_B) = (-\eta, -\xi), \tag{10}$$

$$\mathrm{Z} \equiv (x_Z, y_Z) = \Big(\frac{(k_s L + c)r(0)}{(k_r L - c)\lambda}, -\xi\Big) \tag{11}$$

and

$$\mathrm{I} \equiv (x_I, y_I) = \Big(-\eta, 1 + \frac{(k_r + k_f)\mu}{(k_r L - c)\lambda r(\infty)}\Big). \tag{12}$$

THEOREM 2.3: (1) Suppose that the scaled TTT curve ℓ_1 is strictly convex under the assumption (A-1).

(i) If $x_B > x_Z$ and $y_B > y_I$, then there exists a unique optimal solution $p^* = G(t_0^*)$ $(0 < t_0^* < \infty)$ minimizing the expected cost per unit time in the steady-state given by Eq. (3), where p^* is given by the x-coordinate in the point of contact for the curve ℓ_1 from the point B, where

$$\max(0, -\eta) < p^* < 1. \tag{13}$$

(ii) If $x_B \le x_Z$, then the optimal repair-limit policy is $p^* = G(0) = 0$.

(iii) If $y_B \le y_I$, then the optimal repair-limit policy is $p^* = G(\infty) = 1$.

(2) Suppose that the scaled TTT curve ℓ_1 is concave under the assumption (A-1). Then, the optimal solution is $p^* = 0$ or $p^* = 1$.

PROOF: Differentiating $M_1(p, \phi_1(p))$ with respect to p and setting it equal to zero implies

$$q_1(p) \equiv \phi_1(p)'(p+\eta) - (\phi_1(p) + \xi) = 0, \tag{14}$$

where

$$\phi_1(p)' = \frac{\mu}{r(G^{-1}(p))} \tag{15}$$

and the symbol ' denotes the differentiation. Further, we have

$$q_1(p)' = \phi_1(p)''(p+\eta). \tag{16}$$

When the scaled TTT curve ℓ_1 is strictly convex, then $q_1(p)' > 0$ and the function $M_1(p, \phi_1(p))$ is strictly convex in p.

In the plane $(x,y) \in (-\infty,+\infty) \times (-\infty,+\infty)$, we define the point $\mathrm{B} = (x_B, y_B)$. Since the tangent line for the point $(p^*, \phi_1(p^*))$ on the curve ℓ_1 is

$$y = \frac{\mu}{r(G^{-1}(p^*))}(p - p^*) + \phi_1(p^*), \tag{17}$$

the condition that the point B is over the above tangent line is $q_1(p^*) = 0$. Define the intersection $\mathrm{Z} = (x_Z, y_Z)$ of the tangent line for the origin $\mathrm{O} = (0,0)$ on the curve ℓ_1 and $y = -\xi$. If the

$x-$coordinate of B is strictly greater than the $x-$coordinate of Z, $q_1(0) < 0$, otherwise, $q_1(0) \geq 0$ under the assumption (A-1). Similarly, define the intersection I $= (x_I, y_I)$ of the tangent line for the point U $= (1,1)$ on the ℓ_1 and $x = -\eta$. If the $y-$coordinate of B is strictly greater than the $y-$ coordinate of Z, $q_1(1) > 0$, otherwise, $q_1(1) \leq 0$ under (A-1). From these, we obtain the results (1).

Secondly, consider the case where $G(t)$ is IHR. In this case, $\phi_1(p)$ becomes a concave function of p. If the $x-$coordinate of B is strictly negative and if the slope of the straight line BO is strictly smaller than that of the line BU, we have

$$(k_s L + c)/\lambda - \Big\{k_r L - c + (k_r + k_s)\Big\}/\mu < 0, \tag{18}$$

which is equivalent to the condition of $M_1(0, \phi_1(0)) < M_1(1, \phi_1(1))$. Conversely, if $x_B < 0$ and if the slope of the straight line BO is not small than that of the line BU, $M_1(0, \phi_1(0)) \geq M_1(1, \phi_1(1))$ is satisfied. On the other hand, the condition $x_B \geq 0$ implies $M_1(0, \phi_1(0)) \geq M_1(1, \phi_1(1))$. Thus the proof is completed.

EXAMPLE 2.4: Nguyen and Murthy [6] and Dohi, Matsushima, Kaio and Osaki [14] considered the repair-limit replacement models with imperfect repair. Suppose that the repair is imperfect. The mean lifetime when the repair is completed is $1/\lambda_1$ (> 0). Also, a new unit delivered after ordering fails for an infinite time horizon and then the mean lifetime is $1/\lambda_2$ (> 0). Defining the time interval from the start of repair to the following start of repair as one cycle, the mean time of one cycle is

$$T_1(t_0) = 1/\lambda_1 + \int_0^{t_0} \overline{G}(t)dt + (L + 1/\lambda_2 - 1/\lambda_1)\overline{G}(t_0). \tag{19}$$

Replacing the assumption (A-1) to $k_r L + (k_r + k_s)(1/\lambda_2 - 1/\lambda_1) < c$, we have $\min_{0 \leq p \leq 1}$: $M_i(p, \phi_1(p)) \equiv (\phi_1(p) + \xi_i)/(p + \eta_i)$, where

$$\xi_i \equiv \frac{(k_s L + c)\mu}{\{c - k_r L - (k_r + k_s)(1/\lambda_2 - 1/\lambda_1)\}\lambda_1} > 0 \tag{20}$$

and

$$\eta_i \equiv -\Big(1 + \frac{(k_r + k_s)}{\{k_r L - c + (k_r + k_s)(1/\lambda_2 - 1/\lambda_1)\}\lambda}\Big). \tag{21}$$

Therefore, the repair-limit replacement model with imperfect repair is reduced to the similar problem with a different point B$_i = (-\eta_i, -\xi_i)$.

EXAMPLE 2.5: Suppose that the repair-time distribution is the Weibull distribution with the shape parameter $\alpha = 0.8$ and the scale parameter $\beta = 2.0$. The other model parameters are $1/\lambda = 0.8000$, $L = 0.2000$, $c = 6.5000$, $k_r = 4.0000$ and $k_s = 6.0000$. The determination of the optimal repair-time limit for Model 1 is presented in Fig. 2. In this case, we have B $= (-0.4035, -0.4768)$ and the optimal point with minimum slope from B is $(p^*, \phi_1(p^*)) = (0.4280, 0.3162)$. Thus, the optimal repair-time limit is $t_0^* = G^{-1}(0.4280) = 0.9659$.

2.2 Model 2

Let us consider the similar, but somewhat different model from Model 1. The original unit begins operating at time 0. When the unit has failed, the decision maker has to select repair or replacement. Suppose that the decision maker has a subjective probability distribution function on the repair-completion time $G(t)$ with finite mean $1/\mu$ (> 0). If he or she estimates that the repair is completed up to the time limit $t_0 \in [L, \infty)$, then the repair is immediately started at the failure time. However, if he or she estimates that the repair time is greater than t_0, then the failed unit is scrapped at the failure time, the spare unit is ordered immediately and delivered after the lead time L. The configuration of Model 2 is illustrated in Fig. 3.

If we evaluate the expected cost per unit time in the steady-state objectively, it will be

$$C_2(t_0) \equiv V_2(t_0)/T_2(t_0), \tag{22}$$

where

$$V_2(t_0) = (k_s + k_r)\int_0^{t_0} tdG(t) + (k_s L + c)\overline{G}(t_0) \tag{23}$$

and

$$T_2(t_0) = 1/\lambda + \int_0^{t_0} tdG(t) + L\overline{G}(t_0). \tag{24}$$

Hence the problem is to determine the optimal repair-time limit $t_0{}^*$ such as

$$C_2(t_0^*) = \min_{0 \le t_0 < \infty} C_1(t_0). \tag{25}$$

In addition to the assumption (A-1), we need

(A-2) $(k_r + k_s) > C_2(t_0)$ for $t_0 \in [0, \infty)$.

The assumption (A-2) will be required in order to avoid an unrealistic case. To this end, if the reverse inequality holds, the repair and shortage have to always occur at the same time in the steady-state with probability one.

To develop the similar graphical technique to Model 1, we define the Lorenz transform of the repair-time distribution $G(t)$ by

$$\phi_2(p) \equiv \mu \int_0^{G^{-1}(p)} xdG(x), \quad (0 \le p \le 1). \tag{26}$$

The definition above of the Lorenz transform is essentially equivalent to

$$\phi_2(p) = \mu \int_0^{p} G^{-1}(t)dt, \quad (0 \le p \le 1) \tag{27}$$

(see [19-23]). Then the curve $\ell_2 = (p, \phi_2(p)) \in [0,1] \times [0,1]$ is called *the Lorenz curve.* From the simple algebraic manipulation, we have

THEOREM 2.6: Suppose that the assumptions (A-1) and (A-2) hold for Model 2. The minimization problem in Eq.(25) is equivalent to

$$\min_{0 \le p \le 1} : M_2(p, \phi_2(p)) \equiv \frac{\phi_2(p) + \xi}{p + \eta}. \tag{28}$$

Consequently, the optimal policy is determined by $p^* = G(t_0^*)$ which minimizes the tangent slope from B $= (-\eta, -\xi) \in (-\infty, 0) \times (-\infty, 0)$ to the curve ℓ_2.

PROOF: The expected cost per unit time in the steady-state in Eq.(22) becomes $C_2(t_0) = (k_s + k_r) - K(t_0)$, where

$$K(t_0) \equiv \frac{(k_s + k_r)\mu/\lambda + \mu(k_r L - c)\overline{G}(t_0)}{\mu/\lambda + \mu L\overline{G}(t_0) + \int_0^{t_0} tdG(t)}. \tag{29}$$

Since $K(t_0) > 0$ for $t_0 \in [0, \infty)$ from (A-2), it is found that $(k_s + k_r)/\lambda > c - k_r L$ from (A-1) and then $\eta > 0$. Hence the underlying problem is $\min_{0 \le t_0 < \infty} 1/K(t_0)$ and yields Eq.(28).

THEOREM 2.7: (1) Suppose that the assumptions (A-1) and (A-2) hold. Then there exists a unique optimal solution $p^* = G(t_0^*)$ $(0 < t_0^* < \infty)$ minimizing the expected cost per unit time in the steady-state given by Eq. (28), where p^* is given by the x-coordinate in the point of contact for the curve ℓ_2 from the point B.

PROOF: Differentiating $M_2(p, \phi_2(p))$ with respect to p and setting it equal to zero implies

$$q_2(p) \equiv \phi_2(p)'(p+\eta) - (\phi_2(p) + \xi), \tag{30}$$

where $\phi_2(p)' = \mu G^{-1}(p)$. Further, we have

$$q_2(p)' = \phi_2(p)''(p+\eta) > 0 \tag{31}$$

and the function $M_2(p, \phi_2(p))$ is strictly convex in p, since $\phi_2(p)'' = \mu/g(G^{-1}(p)) > 0$. From $q_2(0) = -\xi < 0$ and $q_2(1) \to \infty$, the result is proved.

EXAMPLE 2.8: Suppose that the repair-time distribution is the Weibull distribution with the shape parameter $\alpha = 0.8$ and the scale parameter $\beta = 2.0$. The other model parameters are $1/\lambda = 0.8000$, $L = 0.2000$, $c = 6.5000$, $k_r = 4.0000$ and $k_s = 6.0000$. The determination of the optimal repair-time limit for Model 2 is presented in Fig. 4. In this case, we have $\mathrm{B} = (-0.4035, -0.4768)$ and the optimal point with minimum slope from B is $(p^*, \phi_2(p^*)) = (0.5530, 0.1407)$. Thus, the optimal repair-time limit is $t_0^* = G^{-1}(0.5530) = 1.5255$.

3. COMPARATIVE RESULTS

We compare two repair-limit replacement models. From Eqs. (3) and (22), it can be seen that $T_1(t_0) = T_2(t_0) + t_0\overline{G}(t_0)$ and $V_1(t_0) = V_2(t_0) + (k_r + k_s)t_0\overline{G}(t_0)$ for a fixed t_0. When $t_0 = 0$ and $t_0 \to \infty$, it is obvious that $C_1(t_0) = C_2(t_0)$. Thus, we pay our attention to the case of $t_0 \in (0, \infty)$. Now, define $C_1(t_1^*) = \min_{0<t_0<\infty} C_1(t_0)$, $C_2(t_2^*) = \min_{0\le t_0<\infty} C_2(t_0)$, $M_1(p_1^*, \phi_1(p_1^*)) = \min_{0<p<1} M_1(p, \phi_1(p))$ and $M_2(p_2^*, \phi_2(p_2^*)) = \min_{0<p<1} M_2(p, \phi_2(p))$.

THEOREM 3.1: (i) Under the assumption (A-2), $C_1(t_0) > C_2(t_0)$ holds for a fixed $t_0 \in [0, \infty)$. (ii) Under the assumptions (A-1) and (A-2), if $\phi_j(p)$ $(j = 1, 2)$ is monotonically increasing and strictly convex in $p \in (0, 1)$, then $p_1^* < p_2^*$ and $t_1^* < t_2^*$.

PROOF: (i) $C_1(t_0) > C_2(t_0)$ if and only if $(k_r + k_s)t_0\overline{G}(t_0)T_2(t_0) > t_0\overline{G}(t_0)V_2(t_0)$. Since $t_0\overline{G}(t_0) > 0$ for a fixed $t_0 \in (0, \infty)$ and (A-2), the result is derived. (ii) Chandra and Singpurwalla [21] proved the following relation;

$$\phi_1(p) = \phi_2(p) + \mu(1-p)G^{-1}(p). \tag{32}$$

Hence, we have $\phi_1(p) > \phi_2(p)$ for $p \in (0, 1)$. Since $\phi_j(p)$ $(j = 1, 2)$ is monotonically increasing and strictly convex in $p \in (0, 1)$, it is straightforward to see $p_1^* < p_2^*$ and $t_1^* < t_2^*$. The proof is completed.

Next, we develop the comparative results on respective repair-limit replacement models. Suppose that there are two repairmen with different repair abilities. We classify two repairmen into Repairman 1 and Repairman 2, respectively. Their repair times X_1 and X_2 are non-negative random variables with distribution functions $G_j(t)$ $(j = 1, 2)$ and the same finite mean $1/\mu$, respectively. We require the following definition on the stochastic ordering [25].

DEFINITION 3.2: (1) X_1 is usually stochastic-ordered with respect to X_2 (denoted as $X_1 \le_{\mathrm{st}} X_2$) if $\overline{G_1}(t) \le \overline{G_2}(t)$.
(2) X_1 is star-shaped stochastic-ordered with respect to X_2 (denoted as $X_1 \le_* X_2$) if $G_2^{-1}(G_1(t))/t$ is increasing in $t \in (0, G_1^{-1}(1))$.

THEOREM 3.3: In Model 1, define the optimal repair-time limits for two repairmen with the same mean repair time $1/\mu$ as $t_{11}^* = G_1^{-1}(p_{11}^*)$ and $t_{12}^* = G_2^{-1}(p_{12}^*)$, respectively, where p_{11}^* and p_{12}^* are the solutions for Eq. (7) with $G_j(t)$ $(j = 1, 2)$. If $\phi_1(p)$ is monotonically increasing and strictly convex in $p \in [0, 1]$ and if the repair time for Repairman 1 is smaller than that for Repairman 2 in the usual stochastic ordering, then $t_{11}^* \leq t_{12}^*$.

PROOF: From the well-known result by Barlow [17], we find that $\phi_1(G_1(t)) \geq \phi_1(G_2(t))$ if and only if $X_1 \leq_* X_2$. Immediately, we have $p_{11}^* \leq p_{12}^*$, $G_1(t_{11}^*) \leq G_2(t_{12}^*)$ and $G_2^{-1}(G_1(t_{11}^*)) \leq t_{12}^*$, since $\phi_1(p)$ is monotonically increasing and strictly convex in $p \in [0, 1]$. Since the usual stochastic ordering $\leq_{\rm st}$ includes the star-shaped ordering $\leq_*$, $G_2^{-1}(G_1(t_{11}^*)) \leq t_{12}^*$ and $G_1(t) \geq G_2(t)$ yield $G_1(t_{11}^*) \leq G_2(t_{12}^*)$ and $t_{11}^* \leq t_{12}^*$.

THEOREM 3.4: In Model 2, define the optimal repair-time limits for two repairmen with the same mean repair time $1/\mu$ as $t_{21}^* = G_1^{-1}(p_{21}^*)$ and $t_{22}^* = G_2^{-1}(p_{22}^*)$, respectively, where p_{21}^* and p_{22}^* are the solutions for Eq. (28) with $G_j(t)$ $(j = 1, 2)$. If the repair time for Repairman 1 is smaller than that for Repairman 2 in the usual stochastic ordering, then $t_{21}^* \leq t_{22}^*$.

PROOF: Chandra and Singpurwalla [21] proved $\phi_2(G_1(t)) \geq \phi_2(G_2(t))$ if $X_1 \leq_* X_2$. The rest part of the proof is similar to THEOREM 3.3.

4. NON-PARAMETRIC ESTIMATION METHODS

Based on the graphical ideas in Section 2, we propose statistical methods to estimate the optimal repair-limit policies for two models. Suppose that the optimal repair-time limit has to be estimated from an ordered complete sample $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ of repair times from an absolutely continuous repair-time distribution G, which is unknown. The estimator of $G(t) = p$ is the empirical distribution given by

$$G_n(x) = \begin{cases} i/n & \text{for } x_i \leq x < x_{i+1}, \quad (i = 0, 1, 2, \cdots, n-1) \\ 1 & \text{for } x_n \leq x. \end{cases} \tag{33}$$

Then the scaled TTT statistics based on this sample are

$$\phi_{1i} \equiv S_i / S_n, \tag{34}$$

where

$$S_i \equiv \sum_{j=1}^{i} (n - j + 1)(x_j - x_{j-1}), \quad (i = 1, 2, \cdots, n;\ S_0 = 0). \tag{35}$$

By plotting the point $(i/n, \phi_{1i})$, $(i = 0, 1, 2, \cdots, n)$, and connecting them by line segments, we obtain the so-called *scaled TTT plot*, $\ell_{1n} \in [0, 1] \times [0, 1]$. On the other hand, the empirical Lorenz curve is

$$\phi_{2i} \equiv \sum_{i=1}^{[np]} x_i / \sum_{i=1}^{n} x_i, \tag{36}$$

where $[a]$ is the greatest integer in a. Similarly, plotting the point $(i/n, \phi_{2i})$, $(i = 0, 1, 2, \cdots, n)$, and connecting them by line segments, we obtain the *empirical Lorenz curve* $\ell_{2n} \in [0, 1] \times [0, 1]$.

As empirical counterparts of THEOREM 2.1 and THEOREM 2.6, we propose non-parametric estimators of the repair-time limits for respective models.

THEOREM 4.1: The optimal repair-time limit for Model j $(= 1, 2)$ can be estimated by $\hat{t_{jn}} = x_{i^*}$, where

$$\left\{ i^* \mid \min_{0 \leq i \leq n} \frac{\phi_{ji} + \xi}{i/n + \eta} \right\}. \tag{37}$$

The proof is omitted for brevity. We consider the following two examples for better understanding of the result above.

EXAMPLE 4.2: The repair-time data were made by the random number following the Weibull distribution with shape parameter $\alpha = 0.2$ and scale parameter $\beta = 2.0$. The other model parameters are $1/\lambda = 0.8000$, $L = 0.2000$, $c = 6.5000$, $k_r = 4.0000$ and $k_s = 6.0000$. The scaled TTT plot based on the 200 sample data for Model 1 is shown in Fig. 5. Since B $= (-0.4035, -0.4771)$, the optimal point with minimum slope from B becomes $(i^*/n, \phi_{1i^*}) = (119/200, \phi_{1,119}) = (0.5980, 0.4873)$. Thus, the estimator of the optimal repair-time limit $\hat{t}_{1,200} = x_{119} = 1.7006$.

EXAMPLE 4.3: Under the same model parameters with EXAMPLE 4.2 except for $\alpha = 0.8$, the empirical Lorenz curve based on the 200 sample data for Model 2 is shown in Fig. 6. Since B $= (-0.4035, -0.4771)$, the optimal point with minimum slope from B becomes $(i^*/n, \phi_{2i^*}) = (110/200, \phi_{2,110}) = (0.5829, 0.1742)$. Thus, the estimator of the optimal repair-time limit $\hat{t}_{2,200} = x_{110} = 1.6492$.

Of our interest is the investigation of asymptotic properties of the estimators $\hat{t}_{jn}$, $(j = 1, 2)$ in THEOREM 4.1. The following theorem guarantees the asymptotic optimality of the estimators above.

THEOREM 4.4: (i) The expected cost $C_j(\hat{t_{jn}})$, $(j = 1, 2)$ of using the repair-time limit $\hat{t}_{jn}$ tends with probability one to $C_j(t_0^*)$ as n tends to infinity if t_0^* is positive and finite.
(ii) The minimum expected cost per unit time in the steady-state $C_j(t_0^*)$, $(j = 1, 2)$ may be estimated by

$$\hat{C}_j(t_{jn}^*) = \frac{(k_r + k_s)\phi_j(p_{jn}^*)/\mu_n + (k_s L + c)(1 - p_{jn}^*)}{1/\lambda + \phi_j(p_{jn}^*)/\mu_n + L(1 - p_{jn})}, \tag{38}$$

where $1/\mu_n$ is empirical mean of the repair time. Then the estimator is strongly consistent.
(iii) If a unique optimal repair-time limit exists then $\hat{t}_{jn}$ $(j = 1, 2)$ is strongly consistent.

PROOF: See for $j = 1$ (Model 1) Bergman [8]. For $j = 2$ (Model 2), it is straightforward from convergence theorems for empirical Lorenz curve proved by Goldie [20].

We examine numerically the strong consistency of estimators proposed for two repair-limit replacement models. Since the real optimal repair-time limit can be calculated under the known repair-time distribution, we investigate the convergence property of estimators to the real value.

EXAMPLE 4.5: For Model 1, suppose that the repair-time distribution is the Weibull distribution with the shape parameter $\alpha = 0.2$ and the scale parameter $\beta = 2.0$. The other model parameters are $1/\lambda = 0.8000$, $L = 0.2000$, $c = 6.5000$, $k_r = 4.0000$ and $k_s = 6.0000$. Then the optimal repair-time limit and the minimum expected cost become $t_0^* = 0.1299$ and $C_1(t_0^*) = 0.2050$, respectively. On the other hand, the asymptotic behaviour of estimators for the optimal repair-time limit and their associated minimum expected costs are depicted in Figs. 7 and 8. From these figures, we observe that the estimators converge to the corresponding real values around which the number of data is 50.

EXAMPLE 4.6: For Model 2, suppose that the repair-time distribution is the Weibull distribution with the shape parameter $\alpha = 0.8$ and the scale parameter $\beta = 2.0$. The other model parameters are similar to EXAMPLE 3.5. Then the optimal repair-time limit and the minimum expected cost become $t_0^* = 1.5255$ and $C_2(t_0^*) = 0.7217$, respectively. On the other hand, the asymptotic behaviour of estimators for the optimal repair-time limit and their associated minimum expected costs are depicted in Figs. 9 and 10. From these figures, we observe that the estimators converge to the corresponding real values around which the number of data is 80.

5. CONCLUDING REMARKS

In this paper, we have proposed graphical methods to estimate the optimal repair-time limits for two kinds of repair-limit replacement models, based on the total time on test statistics and the Lorenz statistics. It has been shown that both estimators provided are non-parametric and have strong consistency. These statistical properties will be useful for practitioners to determine the maintenance plan if they can obtain a sufficiently large number of repair-time data. If we have to determine the optimal repair-limit policy from small sample, then any statistical technique such as Jack-Knife method should be applied to make the number of given sample increase or existing parametric methods should be used to specify the repair-time distribution with some theoretical distribution functions.

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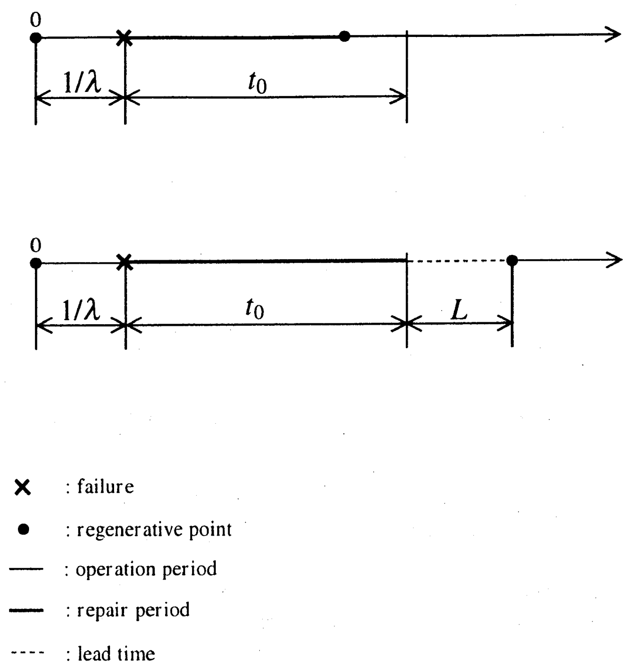

Figure 1: Configuration of Model 1.

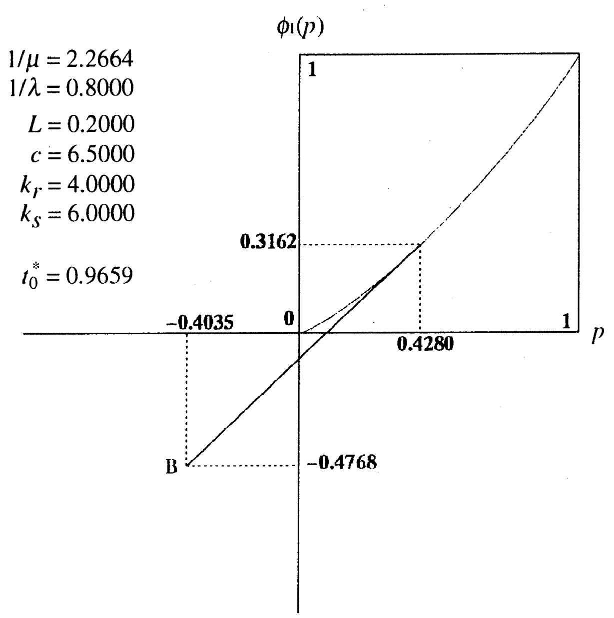

Figure 2: Determination of the optimal repair-time limit based on the scaled TTT transform (Model 1).

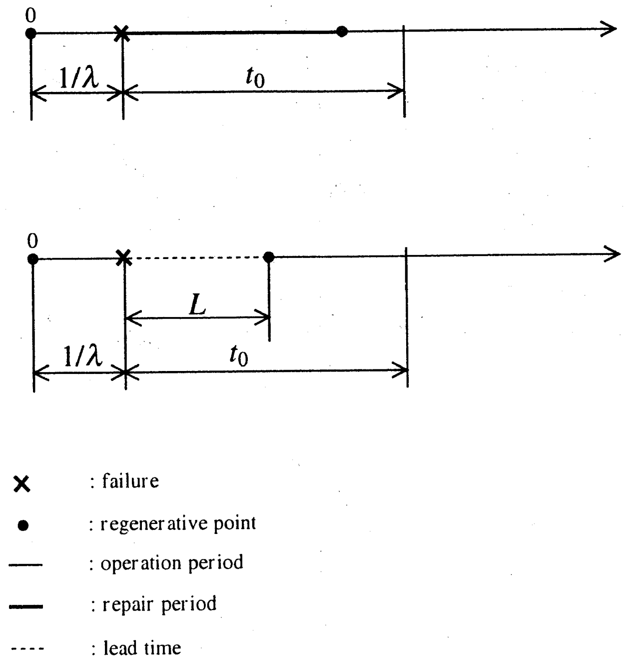

Figure 3: Configuration of Model 2.

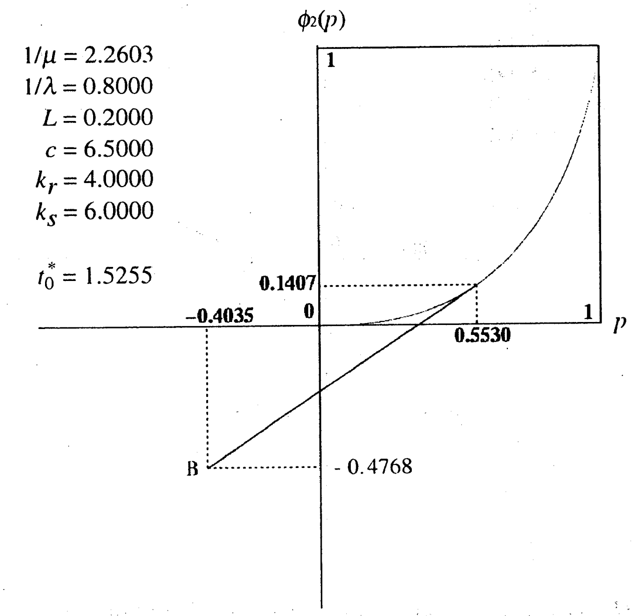

Figure 4: Determination of the optimal repair-time limit based on the Lorenz transform (Model 2).

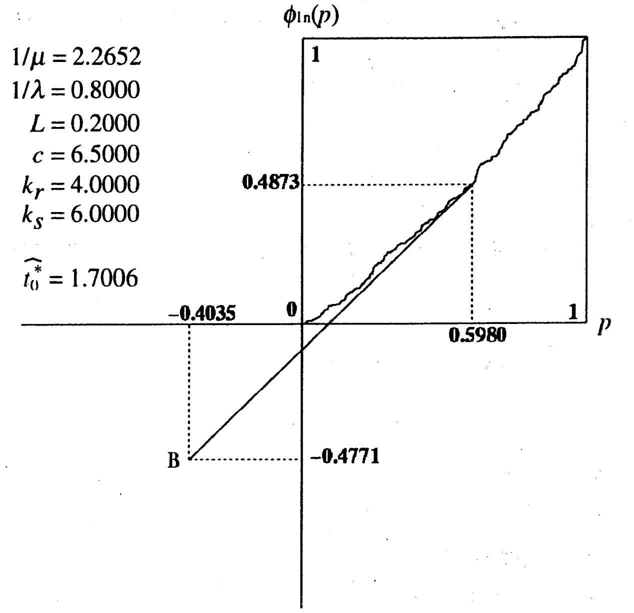

Figure 5: Estimation of the optimal repair-time limit based on the scaled TTT plot (Model 1).

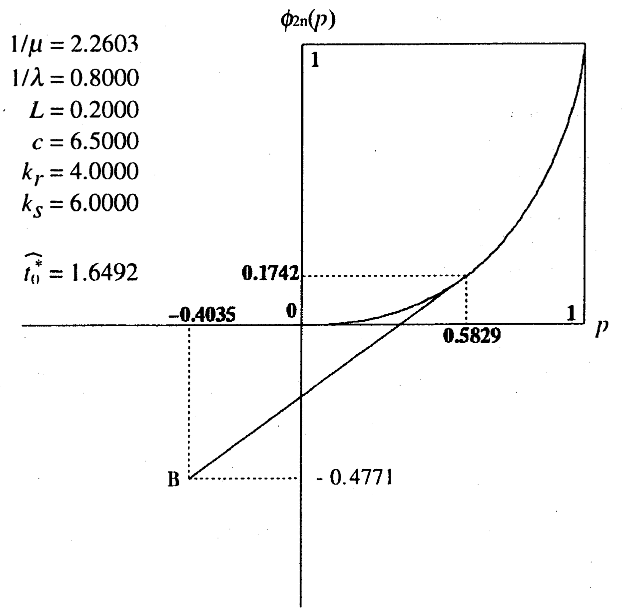

Figure 6: Estimation of the optimal repair-time limit based on the empirical Lorenz curve (Model 2).

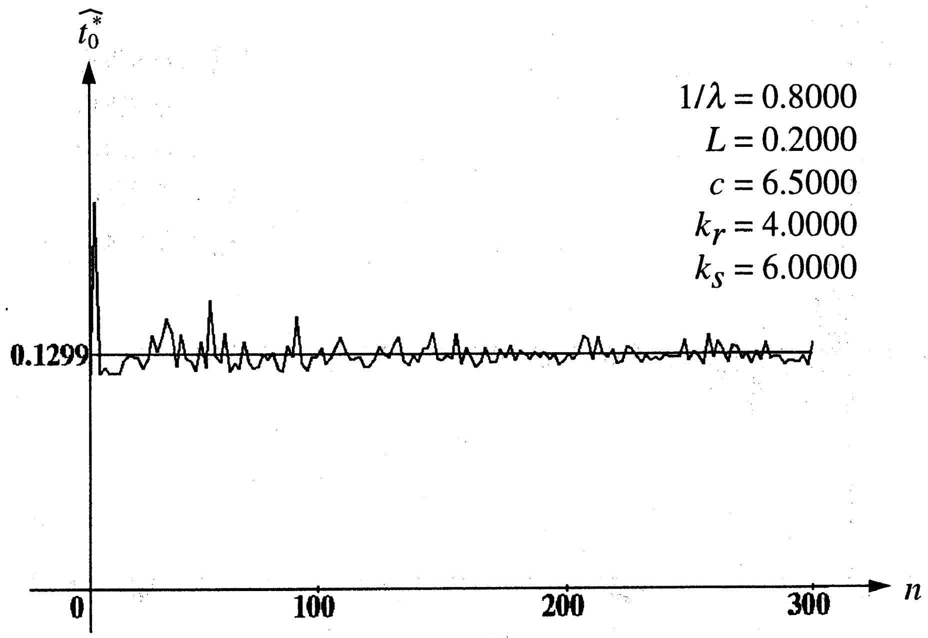

Figure 7: Asymptotic property of the estimated repair-time limit (Model 1)

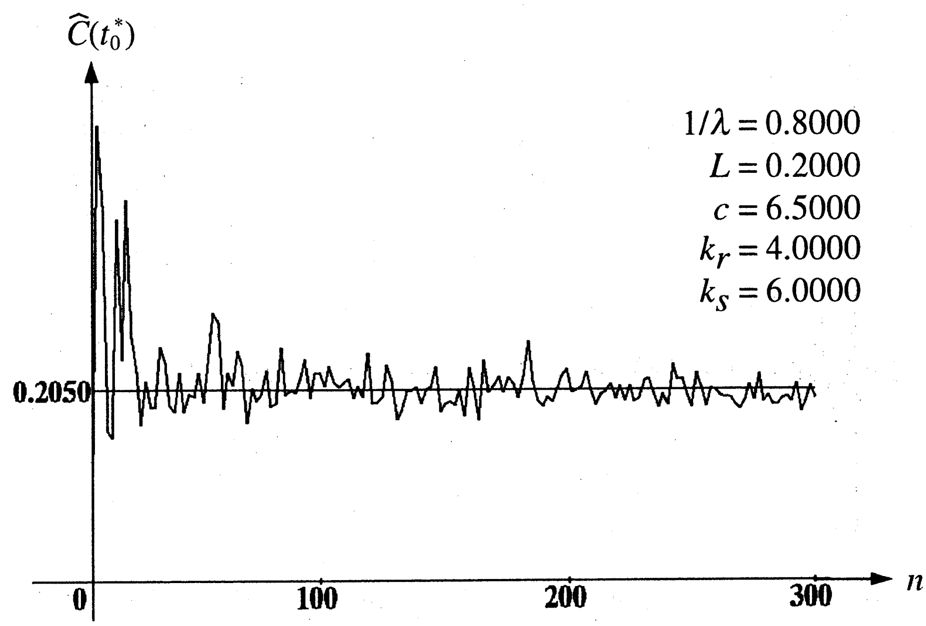

Figure 8: Asymptotic property of the estimated minimum expected (Model 1)

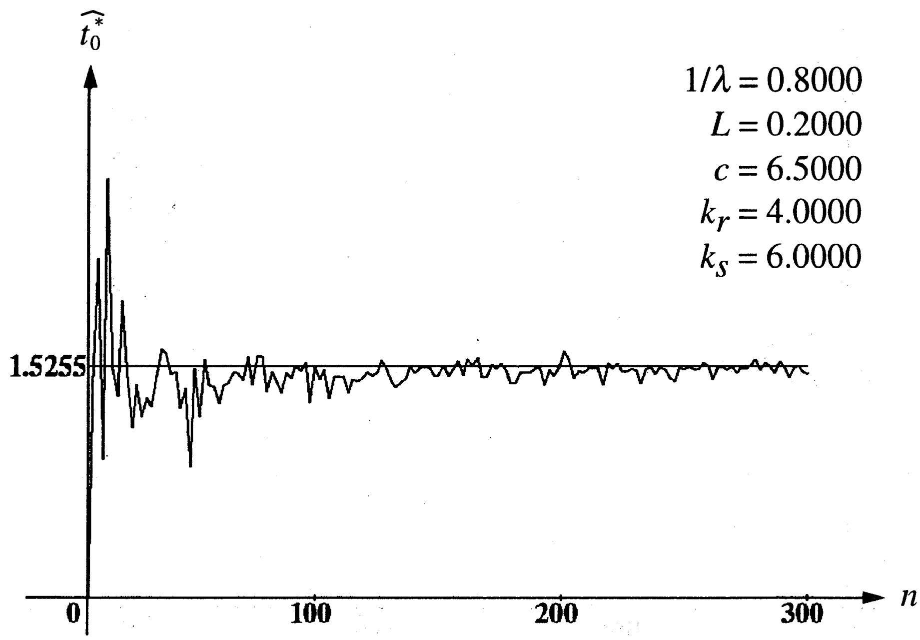

Figure 9: Asymptotic property of the estimated repair-time limit (Model 2)

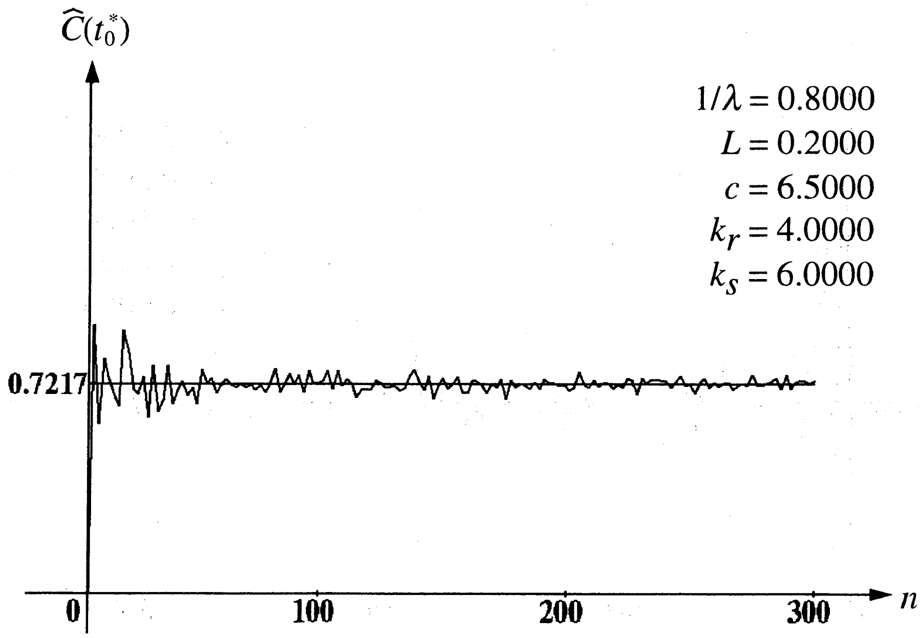

Figure 10: Asymptotic property of the estimated minimum expected (Model 2)