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# NASH FUNCTIONS ON NONCOMPACT NASH MANIFOLDS

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## §1. INTRODUCTION

A *Nash manifold* is a semialgebraic $C^\infty$ submanifold of a Euclidean space. A *Nash function* on a Nash manifold is a $C^\infty$ function with semialgebraic graph. Let $M$ be a Nash manifold. Let $\mathcal{N}$ denote the sheaf of Nash function germs on $M$. (We write $\mathcal{N}_M$ if we need to emphasize $M$.) Let $\mathcal{O}$ (or $\mathcal{O}_M$) denote the sheaf of $C^\omega$ function germs on $M$. We call a sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ *finite* if there exists a finite open semialgebraic covering $\{U_i\}$ of $M$ such that for each $i$, $\mathcal{I}|_{U_i}$ is generated by Nash functions on $U_i$. (See [S] and [C-R-S$_2$] for elementary properties of sheaves of $\mathcal{N}$-ideals and $\mathcal{N}$-modules.) Let $\mathcal{N}(M)$ denote the ring of Nash functions on $M$ and let $\mathcal{O}(M)$ denote the ring of $C^\omega$ functions on $M$.

[C-R-S$_2$] showed that the following three elementary conjectures are equivalent, and [C-R-S$_1$] gave a positive answer to the conjectures in the case where the manifold of domain $M$ is compact.

**Separation conjecture.** *Let $M$ be a Nash manifold. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. Then $\mathfrak{p}\mathcal{O}(M)$ is a prime ideal of $\mathcal{O}(M)$.*

**Global equation conjecture.** *For the same $M$ as above, every finite sheaf $\mathcal{I}$ of $\mathcal{N}_M$-ideals is generated by global Nash functions on $M$.*

**Extension conjecture.** *For the same $M$ and $\mathcal{I}$ as above, the following natural homomorphism is surjective:*

$$H^0(M,\mathcal{N}) \longrightarrow H^0(M,\mathcal{N}/\mathcal{I}).$$

If these conjectures hold true, then the following conjecture also holds [C-R-S$_2$].

**Factorization conjecture.** *Given a Nash function $f$ on a Nash manifold $M$ and a $C^\omega$ factorization $f=f_1f_2$, there exist Nash functions $g_1$ and $g_2$ on $M$ and positive $C^\omega$ functions $\varphi_1$ and $\varphi_2$ such that $\varphi_1\varphi_2=1$, $f_1=\varphi_1g_1$ and $f_2=\varphi_2g_2$.*

In the present paper, we prove the conjectures in the noncompact case. It suffices to show the following theorem.

**Theorem.** *Let $M\subset \mathbf{R}^n$ be a noncompact Nash manifold. Let $U$ and $V$ be open semialgebraic subset of $M$ such that $M=U\cup V$. Let $\mathcal{I}$ be a sheaf of $\mathcal{N}_M$-ideals such that $\mathcal{I}|_U$ and $\mathcal{I}|_V$ are generated by global cross-sections on $U$ and $V$ respectively. Then $\mathcal{I}$ itself is generated by global cross-sections on $M$.*

The following proof of this theorem is completely different to the proof in [C-R-S$_1$] in the compact case. The proof in [C-R-S$_1$] is algebraic and based on the Néron desingularization. On the other hand, the present proof is geometric, and the key is Lemma 1 (Proposition VI.2.8 in [S]) of the next section on extension of Nash functions to a compact domain.

We refer meanings and a history of the conjectures to [C-R-S$_{1,2}$].

## §2. Proof of the theorem

A manifold stands for a manifold without boundary unless otherwise specified. A manifold with corners is, by definition, not a manifold but locally diffeomorphic to an open subset of $\mathbf{R}_+^n$, where $\mathbf{R}_+ = \{x \in \mathbf{R}: x \geq 0\}$. Let $M$ be a manifold with corners. $\operatorname{Int} M$—the interior of $M$—is the subset of $M$ where $M$ is locally diffeomorphic to $\mathbf{R}^n$. $\partial M$—the boundary of $M$—is the complement. A manifold with boundary is a manifold with corners such that the boundary is a manifold.

An *abstract Nash manifold* of dimension $m$ is a $C^\omega$ manifold with a finite system of coordinate neighborhoods $\{\psi_i: U_i \to \mathbf{R}^m\}$ such that for each pair $i$ and $j$, $\psi_i(U_i \cap U_j)$ is an open semialgebraic subset of $\mathbf{R}^m$ and the map

$$\psi_j \circ \psi_i^{-1}: \psi_i(U_i \cap U_j) \longrightarrow \psi_j(U_i \cap U_j)$$

is a Nash diffeomorphism. A $C^1$ *Nash manifold* is a $C^1$ semialgebraic submanifold of a Euclidean space. An *abstract* $C^1$ *Nash manifold* is a $C^1$ manifold with a finite system of coordinate neighborhoods of $C^1$ semialgebraic class. Note that a Nash manifold is an abstract Nash manifold, but an abstract Nash manifold is not necessarily *affine*, i.e., an abstract Nash manifold cannot be always Nash imbedded in a Euclidean space (Mazur). On the other hand, a $C^1$ Nash manifold is an abstract $C^1$ Nash manifold and, conversely, an abstract $C^1$ Nash manifold is affine (Theorem III.1.1 in [S]).

For a Nash manifold with corners $M$, we say that *the boundary of $M$ is shrunk* if we replace $M$ with $M -$ (a small closed semialgebraic neighborhood of $\overline{\partial M} - \partial M$ in $\overline{\partial M}$). We call the replaced manifold with corners a *Nash submanifold with shrunk corners of $M$*.

The index $_x$ denotes the stalk of a sheaf at $x$ or the germ of a set or a map at $x$.

**Note.** The theorem holds true if the closure $\overline{M}$ of $M$ in $\mathbf{R}^n$ is compact and contained in a Nash manifold $M'$ of the same dimension as $M$ and if $\mathcal{I}$ can be extended to a coherent sheaf $\mathcal{I}'$ of $\mathcal{N}_{M'}$-ideals on $M'$ for the following reason.

It is easy to find a compact Nash manifold with boundary $M''$ with $M \subset \operatorname{Int} M''$ and $M'' \subset M'$. Using the double of $M''$, we easily construct a compact Nash manifold $M^{(3)}$ and a Nash map $\rho: M^{(3)} \to M''$ such that $\rho|_{\rho^{-1}(\operatorname{Int} M'')}: \rho^{-1}(\operatorname{Int} M'') \to \operatorname{Int} M''$ is a trivial double covering. Let $\Omega$ be a union of connected components of $\rho^{-1}(\operatorname{Int} M'')$ such that $\rho|_\Omega$ is a diffeomorphism onto $\operatorname{Int} M''$. Let $\mathcal{I}^{(3)}$ denote the pull back of $\mathcal{I}'$ by $\rho$. Then $\mathcal{I}^{(3)}$ is finite and hence generated by global cross-sections. Hence $\mathcal{I}'|_{\operatorname{Int} M''}$ and then $\mathcal{I}$ are generated by global cross-sections, because we can identify $\mathcal{I}'|_{\operatorname{Int} M''}$ with $\mathcal{I}^{(3)}|_\Omega$.

Hence we will imbed $M$ in a Euclidean space so that the image has such properties. The following lemma assures it.

**Lemma 1 (Proposition VI.2.8 in [S]).** *Let $M$ be a noncompact Nash manifold, and let $f\colon M \to \mathbf{R}^m$ be a bounded Nash map. Then there exists a compact Nash manifold with corners $M'$ and a Nash diffeomorphism $\pi\colon \operatorname{Int} M' \to M$ such that $f \circ \pi$ can be extended to a Nash map $M' \to \mathbf{R}^m$.*

Using this lemma, we shall reduce the theorem to the following lemma.

**Lemma 2.** *Let $M'$ and $M''$ be (not necessarily compact) Nash submanifolds of $\mathbf{R}^n$ without boundary and with corners, respectively, such that $\overline{M'}$ is compact and contained in a Nash manifold of the same dimension, $\overline{M''}$ is a compact Nash manifold with corners and $\operatorname{Int} M'' = \operatorname{Int} \overline{M''}$ (i.e., $M'' =$ (a compact Nash manifold with corners) $-$ (a closed semialgebraic subset of the boundary)). Let $p\colon M'' \to \mathbf{R}^n$ be a Nash map such that $p|_{\operatorname{Int} M''}$ is a Nash imbedding into $M'$ and $p(\partial M'')$ is contained in $\overline{M'} - M'$. Shrink the boundary of $M''$. Then the abstract Nash manifold $M' \cup_{p|_{\operatorname{Int} M''}} M''$, defined to be the union of $M'$ and $M''$ pasted by the Nash diffeomorphism $p|_{\operatorname{Int} M''}\colon \operatorname{Int} M'' \to p(\operatorname{Int} M'')$, is affine.*

***Proof of the theorem.*** We can assume that $M$ is bounded in $\mathbf{R}^n$ because $\mathbf{R}^n$ is Nash diffeomorphic to $S^n -$ a point. Let the dimension of $M$ be $m$. By the separation theorem of Mostowski [M], we have a Nash function $\psi$ on $M$ such that $-2 \le \psi \le 2$, $\psi > 1$ on $M - V\,(= U - V)$, and $\psi < -1$ on $M - U\,(= V - U)$. Replace $M$ with graph $\psi$. Then we can assume that $M - V$ and $M - U$ have distance. Apply Lemma 1 to the inclusion map $M \to \mathbf{R}^n$. Then we assume, moreover, that $\overline{M}$ is a Nash manifold with corners. Let $\varphi$ be a positive Nash function on $M$ such that $\varphi(x) \to 0$ as $M \ni x \to$ a point of $\partial M$.

Let $f_1, \dots, f_k \in H^0(U, \mathcal{I}|_U)$ and $g_1, \dots, g_k \in H^0(V, \mathcal{I}|_V)$ be generators of $\mathcal{I}|_U$ and $\mathcal{I}|_V$ respectively. Multiplying small positive Nash functions, we can assume the generators are all bounded. Note that the restrictions of the both generators to $U \cap V$ are generators of $\mathcal{I}|_{U\cap V}$. Hence by I.6.5 in [S] there exist Nash functions $\alpha_{i,j}$ and $\beta_{i,j}$ on $U \cap V$, $i, j = 1, \dots, k$, such that for each $i$,

$$(*) \qquad f_i = \sum_{j=1}^{k} \alpha_{i,j} g_j \quad \text{and} \quad g_i = \sum_{j=1}^{k} \beta_{i,j} f_j \quad \text{on} \quad U \cap V.$$

Shrink $U$ and $V$ keeping the property that $M - V$ and $M - U$ have distance. Then by Łojasiewicz Inequality, all $\varphi^l \alpha_{i,j}$ and $\varphi^l \beta_{i,j}$ are bounded for a positive integer $l$. Apply Lemma 1 to $f_i|_{U\cap V}$, $g_i|_{U\cap V}$, $\varphi^l \alpha_{i,j}$, $\varphi^l \beta_{i,j}$, $\varphi|_{U\cap V}$ and the inclusion map $U \cap V \to \mathbf{R}^n$. Then there exists a compact Nash manifold with corners $X$ and a Nash diffeomorphism $\pi\colon \operatorname{Int} X \to U \cap V$ such that all the Nash functions on $\operatorname{Int} X$: $f_i \circ \pi$, $g_i \circ \pi$, $(\varphi^l \alpha_{i,j}) \circ \pi$, $(\varphi^l \beta_{i,j}) \circ \pi$ and $\varphi \circ \pi$ can be extended to $X$, and $\overline{\pi}^{-1}(M - U)$ and $\overline{\pi}^{-1}(M - V)$ have distance, where $\overline{\pi}$ denotes the extension of (the inclusion map) $\circ\, \pi\colon \operatorname{Int} X \to \mathbf{R}^n$ to $X \to \mathbf{R}^n$.

First we modify the inclusion map of $M$ into $\mathbf{R}^n$ so that $\mathcal{I}$ can be extended to $\overline{M} - (\overline{M - U}) - (\overline{M - V})$. We can assume that the abstract Nash manifold with corners $M \cup_\pi (X - \overline{\partial X \cap \overline{\pi}^{-1}(M)})$ is affine for the following reason. Set

$$M' = M, \quad M'' = X - \overline{\partial X \cap \overline{\pi}^{-1}(M)} \quad \text{and} \quad p = \overline{\pi}|_{M''}.$$

Then the assumptions in Lemma 2 are satisfied. Hence if we let $\tilde{M}''$ be a Nash submanifold with shrunk corners of $M''$, then $M' \cup_{p|_{\mathrm{Int}\, M''}} \tilde{M}''$ is affine. Shrink $U$ and $V$ a little so that (the shrunk $U$) $\cap$ (the shrunk $V$) and ($M-$ the original $U$) $\cap$ ($M-$ the original $V$) have distance, and $\overline{(M- \text{ the shrunk } U)}$ and ($M-$ the shrunk $V$) have distance. Then the new $M \cup_\pi (X - \overline{\partial X \cap \overline{\pi}^{-1}(M)})$ coincides with $M' \cup_{p|_{\mathrm{Int}\, M''}} \tilde{M}''$ for some $\tilde{M}''$, and hence it is affine.

Set $M_1 = M \cup_\pi (X - \overline{\partial X \cap \overline{\pi}^{-1}(M)})$, let $M_1$ be contained in $\mathbf{R}^n$, and regard $M$ as a submanifold of $M_1$. Then $M_1 = \overline{M} - (\overline{M - U}) - (\overline{M - V})$, and it is a Nash manifold with corners. Set

$$U_1 = U \cup \partial M_1 \quad \text{and} \quad V_1 = V \cup \partial M_1.$$

Then we have $M_1 = U_1 \cup V_1$, and $U_1$ and $V_1$ are open in $M_1$. Since $f_i \circ \pi$, $g_i \circ \pi$ and $\varphi \circ \pi$ can be extended to $X$, we have Nash function extensions $f_{1,i}$ of $f_i$ to $U_1$, $g_{1,i}$ of $g_i$ to $V_1$, $i = 1, \dots, k$ and $\varphi_1$ of $\varphi$ to $M_1$. Hence we have sheaf extensions $\mathcal{I}_1^U$ and $\mathcal{I}_1^V$ of $\mathcal{I}$ to $M_1$ such that $\mathcal{I}_1^U|_{U_1}$ is generated by $f_{1,1}, \dots, f_{1,k}$, and $\mathcal{I}_1^V|_{V_1}$ is generated by $g_{1,1}, \dots, g_{1,k}$. By (*) and by the fact that $(\varphi^l \alpha_{i,j}) \circ \pi$ and $(\varphi^l \beta_{i,j}) \circ \pi$ can be extended to $X$, say, $\alpha_{1,i,j}$ and $\beta_{1,i,j}$ respectively, we have

$$\varphi_1^l \mathcal{I}_1^U \subset \mathcal{I}_1^V \quad \text{and} \quad \varphi_1^l \mathcal{I}_1^V \subset \mathcal{I}_1^U.$$

Let $M_2 \subset \mathbf{R}^n$ be a Nash manifold of dimension $m$ such that $M_1 \subset M_2$ and any connected component of $M_2$ touches $M_1$. Here also we can assume $\overline{M_2}$ is a compact Nash manifold with corners. Set

$$U_2 = U_1 \cup (M_2 - M_1) \quad \text{and} \quad V_2 = V_1 \cup (M_2 - M_1).$$

Then we have $M_2 = U_2 \cup V_2$, and $U_2$ and $V_2$ are open in $M_2$. Choose $M_2$ so small that $f_{1,i}$, $g_{1,i}$, $\alpha_{1,i,j}$, $\beta_{1,i,j}$ and $\varphi_1$ can be extended to $U_2$, $V_2$, $U_2 \cap V_2$, $U_2 \cap V_2$ and $M_2$ respectively. Let $f_{2,i}$, $g_{2,i}$, $\alpha_{2,i,j}$, $\beta_{2,i,j}$ and $\varphi_2$ denote the respective extensions. Then there exist sheaves $\mathcal{I}_2^U$ and $\mathcal{I}_2^V$ of $\mathcal{N}_{M_2}$-ideals such that $\mathcal{I}_2^U|_{M_1} = \mathcal{I}_1^U$, $\mathcal{I}_2^V|_{M_1} = \mathcal{I}_1^V$, $\mathcal{I}_2^U|_{U_2}$ is generated by $f_{2,1}, \dots, f_{2,k}$, $\mathcal{I}_2^V|_{V_2}$ is generated by $g_{2,1}, \dots, g_{2,k}$ and

$$\varphi_2^l \mathcal{I}_2^U \subset \mathcal{I}_2^V \quad \text{and} \quad \varphi_2^l \mathcal{I}_2^V \subset \mathcal{I}_2^U.$$

Second, we want to extend $\mathcal{I}_2^U$ to $\overline{M_2 - V_2}$. Apply Lemma 1 to $f_{2,1}, \dots, f_{2,k}$, $\varphi_2|_{U_2}$ and the inclusion map $U_2 \to \mathbf{R}^n$. Then we have a compact Nash manifold with corners $Y$ and a Nash diffeomorphism $\tau$: $\mathrm{Int}\, Y \to U_2$ such that each $f_{2,i} \circ \tau$ and $\varphi_2 \circ \tau$ can be extended to $Y$, and $\overline{\tau}^{-1}(M_2 - U_2)$ and $\overline{\tau}^{-1}(M_2 - V_2)$ have distance, where $\overline{\tau}$ is defined by $\tau$ as $\overline{\pi}$. Let $M_3$ denote the following abstract Nash manifold with corners:

$$M_2 \cup_{\tau|_{\mathrm{Int}\, Y}} (\mathrm{Int}\, Y \cup$$
$$(\text{a small open semialgebraic neighborhood of } \partial Y \cap \overline{\tau^{-1}(U_2 - V_2)} \text{ in } \partial Y)).$$

Then by the same reason as above, $M_3$ is affine, and we can assume $M_2 \subset M_3 \subset \mathbf{R}^n$. Set

$$U_3 = U_2 \cup (M_3 - M_2) \quad \text{and} \quad V_3 = V_2.$$

Then we have

$$\overline{M_2 - V_2} \subset U_3, \quad M_3 = U_3 \cup V_3 \quad \text{and} \quad U_3 \cap V_3 = U_2 \cap V_2,$$

and $U_3$ and $V_3$ are open in $M_3$. Since $f_{2,i} \circ \tau$ and $\varphi_2 \circ \tau$ are extended to $Y$, $f_{2,i}$ and $\varphi_2$ can be extended to Nash functions $f_{3,i}$ on $U_3$ and $\varphi_3$ on $M_3$. Let $\mathcal{I}_3^U$ denote the sheaf of $\mathcal{N}_{U_3}$-ideals on $U_3$ (not on $M_3$) generated by $f_{3,1}, \dots, f_{3,k}$.

Third, as the above extension of $M_1$ to $M_2$ and then to $M_3$, we obtain a Nash manifold $M_4$ of dimension $m$, open semialgebraic subsets $U_4$ and $V_4$ of $M_4$, Nash functions $f_{4,i}$ on $U_4$, $g_{4,i}$ on $V_4$, $i = 1, \dots, k$ and $\varphi_4$ on $M_4$, and sheaves $\mathcal{I}_4^U$ of $\mathcal{N}_{U_4}$-ideals and $\mathcal{I}_4^V$ of $\mathcal{N}_{V_4}$-ideals such that

$$\begin{gathered} \overline{M} \subset M_4, \quad M_4 = U_4 \cup V_4, \\ U_4 \cap M = U, \quad V_4 \cap M = V, \\ f_{4,i}|_U = f_i, \quad g_{4,i}|_V = g_i, \quad \varphi_4|_M = \varphi, \\ \varphi_4^l \mathcal{I}_4^U \subset \mathcal{I}_4^V, \quad \varphi_4^l \mathcal{I}_4^V \subset \mathcal{I}_4^U \quad \text{on} \quad U_4 \cap V_4, \end{gathered} \tag{$**$}$$

$\mathcal{I}_4^U$ is generated by $f_{4,1}, \dots, f_{4,k}$, and $\mathcal{I}_4^V$ is generated by $g_{4,1}, \dots, g_{4,k}$.

Finally, we define a sheaf $\mathcal{I}_4$ of $\mathcal{N}_{M_4}$-ideals so that for each $x \in M_4$,

$$\mathcal{I}_{4x} = \begin{cases} \{h \in \mathcal{N}_x \colon \varphi_{4x}^{l'} h \in \mathcal{I}_{4x}^U \text{ for some } l'\} & \text{if } x \in U_4 \\ \{h \in \mathcal{N}_x \colon \varphi_{4x}^{l'} h \in \mathcal{I}_{4x}^V \text{ for some } l'\} & \text{if } x \in V_4. \end{cases}$$

By $(**)$, $\mathcal{I}_4$ is a well-defined coherent sheaf, and by the fact that $\varphi$ is positive, it is an extension of $\mathcal{I}$. Hence the theorem follows from the note. □

***Proof of Lemma 2.*** Let $\dim M' = m$. Regard $M' \cup_{p|_{\operatorname{Int} M''}} M''$ as an abstract $C^1$ Nash manifold with corners which is of class $C^\omega$ around its boundary. By Theorem III.1.1 in [S], there exists its $C^1$ Nash imbedding into a Euclidean space, say, $\mathbf{R}^{n'}$. By the proof of Theorem III.1.1, the imbedding map can be of class $C^\omega$ around the boundary. Hence the image can be of class $C^\omega$ around the boundary. By Theorem III.1.3, ibid., and its proof, the image is modified to be a Nash manifold with corners through a $C^1$ Nash diffeomorphism of class $C^\omega$ around the boundary. Consequently, we have a Nash manifold with corners $M_1 \subset \mathbf{R}^{n'}$ and a $C^1$ Nash diffeomorphism $\rho\colon M_1 \to M' \cup_{p|_{\operatorname{Int} M''}} M''$ of class $C^\omega$ around $\partial M_1$. Here by the same arguments as before, we can assume $\overline{M_1}$ is compact and contained in a Nash manifold $M_2$ of dimension $m$. It suffices to approximate $\rho$ by a Nash map in the $C^1$ topology, because a strong $C^1$ Nash approximation of a $C^1$ Nash diffeomorphism in the $C^1$ topology is a diffeomorphism by Lemma II.1.7, ibid. (See Chapter II, ibid., for the topology.) Define a $C^1$ Nash map $\xi\colon M_1 \to \mathbf{R}^n$ by

$$\xi = \begin{cases} \rho & \text{on} \quad \rho^{-1}(M') \\ p \circ \rho & \text{on} \quad \rho^{-1}(M''). \end{cases}$$

Then $\xi(M_1) \subset \overline{M'}$, $\xi$ is of class $C^\omega$ around $\partial M_1$, and $\xi|_{\operatorname{Int} M_1}$ is a $C^1$ diffeomorphism onto $M'$.

Shrink $\partial M_1$. Then there exists a strong Nash approximation $\xi'$ of $\xi$ in the $C^1$ topology such that $\xi' = \xi$ on $\partial M_1$ and $\xi'(\operatorname{Int} M_1) = M'$ for the following reason.

Let $\tilde{M}' \subset \mathbf{R}^n$ be a Nash manifold that contains $\overline{M'}$ and is of dimension $m$. Shrink $\partial M_1$. Then by Lemma 3 below, there exists a Nash function $\varphi$ on $M_1$ with zero set $= \partial M_1$. Let $U$ be a small open semialgebraic neighborhood of $\partial M_1$ in $M_2$ where $\varphi|_{U \cap M_1}$ and $\xi|_{U \cap M_1}$ can be extended as a Nash function and a Nash map to $\tilde{M}'$ respectively. Set $M_3 = M_1 \cup U$, and let $\tilde{\varphi}\colon M_3 \to \mathbf{R}$ and $\tilde{\xi}\colon M_3 \to \tilde{M}'$ denote the respective extensions. Apply Theorem II.5.2 in [S] to $\tilde{\varphi}$, $\tilde{\xi}$, $M_3$ and $\tilde{M}'$. Then there exists a Nash approximation $\tilde{\xi}'\colon M_3 \to \tilde{M}'$ of $\tilde{\xi}$ in the $C^1$ topology such that $\tilde{\xi}' = \tilde{\xi}$ on $\tilde{\varphi}^{-1}(0)$ and $\tilde{\xi}'(M_3) = \tilde{\xi}(M_3)$. If we set $\xi' = \tilde{\xi}'|_{M_1}$ then $\xi'$ is a Nash approximation of $\xi$ in the $C^1$ topology and satisfies the required conditions.

Moreover, $\xi'|_{\operatorname{Int} M_1}$ can be a Nash diffeomorphism onto $M'$ for the following reason.

First we prove that $\xi'|_{\operatorname{Int} M_1}$ can be an immersion. For each $i = 1, \dots, n$, let $v_i$ denote the Nash vector field on $M_1$ such that for each $x \in M_1$,

$$\left(\frac{\partial}{\partial x_i}\right)_x = v_{ix} + (\text{a vector normal to the tangent space of } M_1 \text{ at } x).$$

For a $C^1$ map $\chi = (\chi_1, \dots, \chi_n)\colon M_1 \to \mathbf{R}^n$, let $\alpha(\chi)$ denote the sum of the squares of the minors of degree $m$ of the $n \times n$ matrix whose $(i,j)$-element is $v_i \chi_j$. Then $\chi|_{\operatorname{Int} M_1}$ is an immersion if and only if $\alpha(\chi)$ is positive on $\operatorname{Int} M_1$. It follows from Łojasiewicz Inequality and the property $\alpha(\xi) > 0$ on $\operatorname{Int} M_1$ that $\alpha(\xi') > 0$ on $\operatorname{Int} M_1$ if we choose $\xi'$ so that $\xi' - \xi$ is the product of $\varphi^l$ and a $C^1$ Nash map close to the zero map in the $C^1$ topology for a large integer $l$ and for the above $\varphi$. Hence $\xi'|_{\operatorname{Int} M_1}$ can be an immersion.

Second, we see that $\xi'|_{\operatorname{Int} M_1}$ can be injective. For a map $\chi\colon M_1 \to \mathbf{R}^n$, let $\beta(\chi)\colon M_1 \times M_1 \to \mathbf{R}^n$ be defined by

$$\beta(\chi)(x_1, x_2) = \chi(x_1) - \chi(x_2) \quad \text{for} \quad (x_1, x_2) \in M_1 \times M_1.$$

Let $\Delta$ denote the diagonal of $M_1 \times M_1$. Then $\chi|_{\operatorname{Int} M_1}$ is injective if and only if

$$(*) \qquad \beta(\chi)^{-1}(0) = \Delta \quad \text{in} \quad \operatorname{Int} M_1 \times \operatorname{Int} M_1,$$

the zero set of $\beta(\xi)$ contains $\Delta$ and is contained in $\partial M_1 \times \partial M_1 \cup \Delta$, and the rank of the Jacobian matrix of $\beta(\xi)$ at each point of $\operatorname{Int} \Delta$ equals $m$. Note that $\dim \Delta = m$. Let $l$ be a large integer and let $\gamma\colon M_1 \times M_1 \to \mathbf{R}^n$ be a $C^1$ Nash map which vanishes on $\Delta$ and is close to the zero map in the $C^1$ topology. Then by Łojasiewicz Inequality, it is easy to see that the zero set of the map

$$M_1 \times M_1 \ni (x_1, x_2) \longrightarrow \beta(\xi)(x_1, x_2) + (\varphi^{2l}(x_1) + \varphi^{2l}(x_2))\gamma(x_1, x_2) \in \mathbf{R}^n$$

coincides with the zero set of $\beta(\xi)$. Choose $\xi'$ so that $\xi' - \xi$ is the product of $\varphi^{l'}$ and a $C^1$ Nash map close to the zero map in the $C^1$ topology for a much larger integer $l'$. Then $\beta(\xi')$ is of the above form. Hence $\xi'$ has the property $(*)$. Thus $\xi'|_{\operatorname{Int} M_1}$ can be injective.

By the above two facts, $\xi'|_{\mathrm{Int}\,M_1}$ can be a diffeomorphism onto $M'$ because $(\xi - \xi')(x)$ converges to $0 \in \mathbf{R}^n$ as a point $x$ in $\mathrm{Int}\,M_1$ converges to a point of $\overline{M}_1 - \mathrm{Int}\,M_1$.

Define a Nash map $\rho'\colon \mathrm{Int}\,M_1 \to M' \cup_{p|_{\mathrm{Int}\,M''}} M''$ to be $\xi'$. Choose $\xi'$ so that the map $\xi' - \xi\colon M_1 \to \mathbf{R}^n$ is the product of $\varphi^l$ and a $C^1$ Nash map $M_1 \to \mathbf{R}^n$ for a sufficiently large integer $l$ (Theorem II.5.2, ibid.). Then by Łojasiewicz Inequality we can extend $\rho'$ to a semialgebraic homeomorphism $\rho'\colon M_1 \to M' \cup_{p|_{\mathrm{Int}\,M''}} M''$ which equals $\rho$ on $\partial M_1$. Clearly $\rho'|_{\mathrm{Int}\,M_1}$ is a Nash diffeomorphism onto $M'$. Hence Lemma 2 follows if we can choose $\xi'$ so that $\rho'$ is a Nash diffeomorphism around $\partial M_1$. For that it suffices to prove the following assertion.

*Let $\pi$ be the semialgebraic homeomorphism of $M_1$ such that $\rho \circ \pi = \rho'$. Then we can choose $\xi'$ so that $\pi$ is a Nash diffeomorphism around $\partial M_1$.*

It follows from $\rho \circ \pi = \rho'$ that $\xi \circ \pi = \xi'$. Since $\pi$ is unique and since $\pi = \mathrm{id}$ on $\partial M_1$, the problem is local at $\partial M_1$. Hence we can reduce the above assertion to the next one.

*We can choose $\xi'$ so that for each $x \in \partial M_1$ there exists a Nash diffeomorphism germ $\tau$ of $M_{1x}$ such that $\xi_x \circ \tau = \xi'_x$.*

We can assume $M_1 \subset \mathbf{R}^m$ and $\tilde{M}' \subset \mathbf{R}^m$ since the problem is local. Let $J$ denote the Jacobian of $\xi$. Then we precisely state the above assertion as follows, which is due to [T].

*There exists such $\tau$ if for each $x \in \partial M_1$, $\xi'_x - \xi_x$ is the product of $J_x^2\varphi_x$ and a Nash map germ.*

Such $\xi'$ exists by the above construction of $\xi'$ if we have a Nash function $J'$ on $M_1$ such that $J'^{-1}(0) \subset \partial M_1$, and for each $x \in \partial M_1$, $J'_x$ is the product of $J_x$ and a Nash function germ. Let $\mathcal{J}$ denote the finite sheaf of $\mathcal{N}_{M_1}$-ideals defined to be $J\mathcal{N}_{M_1}$ around $\partial M_1$ and $\mathcal{N}_{M_1}$ outside of $\partial M_1$. Then by Lemma 3, $\mathcal{J}$ has finite generators if we shrink $\partial M_1$. The sum of the squares of the generators fulfills the requirements for $J'$.

It remains to show the last assertion. We assume $M_1 = \tilde{M}' = \mathbf{R}^m$ for simplicity of notation. Let $g\colon \mathbf{R}^m \to \mathbf{R}^m$ be the Nash map germ such that $\xi'_0 - \xi_0 = J_0^2\varphi_0 g$. By the Taylor expansion formula we have

$$\xi_0(x+y) = \xi_0(x) + y\cdot\frac{\partial \xi_0}{\partial x} + \sum_{i,j=1}^{m} y_i y_j f_{i,j}(x,y), \quad x,\ y = (y_1, \dots, y_m) \in \mathbf{R}^m,$$

for some Nash map germs $f_{i,j}\colon \mathbf{R}^{2m} \to \mathbf{R}^m$, where $\frac{\partial}{\partial x}$ denotes the Jacobian matrix. Substitute $y$ with $J_0(x)y$. Then

$$\xi_0(x + J_0(x)y) - \xi_0(x) = J_0(x)y\cdot\frac{\partial \xi_0}{\partial x}(x) + J_0^2(x)\sum_{i,j=1}^{m} y_i y_j f'_{i,j}(x,y)$$

for some Nash map germs $f'_{i,j}$. Hence we need only find a Nash map germ $y = y(x)\colon \mathbf{R}^m \to \mathbf{R}^m$ such that $y(0) = 0$ and

$$J_0(x)y(x)\cdot\frac{\partial \xi_0}{\partial x}(x) + J_0^2(x)\sum_{i,j=1}^{m} y_i(x)y_j(x)f'_{i,j}(x,y(x)) = J_0^2(x)\varphi_0(x)g(x).$$

Multiply this equality by the cofactor matrix of $\frac{\partial \xi_0}{\partial x}(x)$. Then it is equivalent to

$$y(x) + \sum_{i,j=1}^{m} y_i(x) y_j(x) f''_{i,j}(x, y(x)) = \varphi_0(x) g'(x),$$

where $f''_{i,j}$ and $g'$ are some Nash map germs. By the implicit function theorem, the last equality is solved. □

**Lemma 3.** *Let $M \subset \mathbf{R}^n$ be a Nash manifold with corners. Let $\mathcal{I}$ be a finite sheaf of $\mathcal{N}_M$-ideals on $M$ such that $\mathcal{I}_x = \mathcal{N}_x$ for $x \in \operatorname{Int} M$. Shrink $\partial M$. Then Global equation conjecture and Extension conjecture for this $\mathcal{I}$ hold true.*

***Proof.*** We can assume $\overline{M} - M$ is a point. Let $\varphi$ be the function on $M$ which measures distance from $\overline{M} - M$, and let $\varepsilon$ be a small positive number. Then $\varphi$ is of class Nash on $\varphi^{-1}(]0, \varepsilon])$ and $C^1$ regular on $(\operatorname{Int} M) \cap \varphi^{-1}(]0, \varepsilon])$ and on (each face of $\partial M) \cap \varphi^{-1}(]0, \varepsilon])$. Hence $M_1 = \varphi^{-1}([\varepsilon, \infty[)$ is a compact Nash manifold with corners. Set

$$M_2 = M - \{x \in \partial M : \varphi(x) \le \varepsilon\} \quad \text{and} \quad M_3 = \varphi^{-1}(]\varepsilon, \infty[),$$

which are Nash manifolds with corners. By the semialgebraic version of Thom's First Isotopy Lemma [C-S$_2$], we have a semialgebraic map $\tau : \varphi^{-1}(]0, \varepsilon]) \to \varphi^{-1}(\varepsilon)$ such that $\tau = \mathrm{id}$ on $\varphi^{-1}(\varepsilon)$ and $(\tau, \varphi)|_{M_2 \cap \varphi^{-1}(]0,\varepsilon])}$ is a Nash diffeomorphism onto $(M_2 \cap \varphi^{-1}(\varepsilon)) \times ]0, \varepsilon]$. Using $\tau$ we easily construct a $C^1$ Nash diffeomorphism $\pi : M_3 \to M_2$ which is the identity on a small semialgebraic neighborhood of $\partial M_3$ in $M_3$.

From the note it follows that there exists a Nash function on $M_3$ with zero set $= \partial M_3$, and $\mathcal{I}|_{M_3}$ is generated by global cross-sections. We show that $\mathcal{I}|_{M_2}$ also is generated by global cross-sections. For that it suffices to find a Nash approximation $\pi' : M_3 \to M_2$ of $\pi$ in the $C^1$ topology such that $\pi' = \mathrm{id}$ on $\partial M_3$ and the pull back of $\mathcal{I}|_{M_2}$ by $\pi'$ equals $\mathcal{I}|_{M_3}$.

Let $\psi$ be a global cross-section of $\mathcal{I}|_{M_3}$ with zero set $= \partial M_3$. By Theorem II.5.2 in [S] there exists a Nash approximation $\pi'$ of $\pi$ such that the map $\pi' - \pi : M_3 \to \mathbf{R}^n$ is the product of $\psi$ and a $C^1$ Nash map $\alpha : M_3 \to \mathbf{R}^n$ of class $C^\omega$ around $\partial M_3$. We need only prove that for each $a \in \partial M_3$ and for each $f \in \mathcal{N}_a$, $f$ is contained in $\mathcal{I}_a$ if and only if $f \circ \pi'_a$ is in $\mathcal{I}_a$. (Note that $\pi'(a) = a$.) As the problem is local, we can assume $M \subset \mathbf{R}^m$ and $a = 0$, where $m = \dim M$. In general, for a Nash function germ $g$ at 0 in $\mathbf{R}^m$ there exists a Nash function germ $h$ at 0 in $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}$ such that

$$g(x + zy) = g(x) + zh(x, y, z) \quad \text{for } (x, y, z) \text{ around } 0 \text{ in } \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}.$$

Hence we have

$$f \circ \pi'_0(x) = f(x + \psi_0(x)\alpha_0(x)) = f(x) + \psi_0(x) f_1(x, \alpha_0(x), \psi_0(x))$$

for some Nash function germ $f_1$ at 0 in $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}$. Therefore, $f \in \mathcal{I}_0$ if and only if $f \circ \pi'_0 \in \mathcal{I}_0$. □

**Remark.** *Global equation and Extension conjectures hold true for any real closed field $R$ which contains* $\mathbf{R}$.

We prove this in the same way as in the proof of the implication (i)$\Rightarrow$(ii) of Theorem 2.4 in [C-S$_1$]. For semialgebraic subsets $X$ and $Y$ of $\mathbf{R}^n$ and for a semialgebraic map $f\colon X \to Y$, let $X_R$, $Y_R$ and $f_R\colon X_R \to Y_R$ denote the extensions to $R$ of $X$, $Y$ and $f$ respectively.

***Proof of Global equation conjecture.*** It suffices to prove the theorem for a (not necessarily noncompact) Nash manifold $M$ in $R^n$. Let $\dim M = m$. By Theorem 2.4 in [C-S$_1$], we can assume there exists a Nash manifold $M^{\mathbf{R}} \subset \mathbf{R}^n$ such that $M$ is diffeomorphic to $M^{\mathbf{R}}_R$. Hence let $M = M^{\mathbf{R}}_R$. Moreover, by its proof we can assume $U = U^{\mathbf{R}}_R$ and $V = V^{\mathbf{R}}_R$ for some open semialgebraic sets $U^{\mathbf{R}}$ and $V^{\mathbf{R}}$ of $M^{\mathbf{R}}$. Let $f_1, \dots, f_k \in H^0(U, \mathcal{I}|_U)$ and $g_1, \dots, g_k \in H^0(V, \mathcal{I}|_V)$ be generators of $\mathcal{I}|_U$ and $\mathcal{I}|_V$ respectively. Let $\gamma_{i,j}\colon U \cap V \to R$ and $\delta_{i,j}\colon U \cap V \to R$, $i, j = 1, \dots, k$, be Nash functions such that for each $i$,

$$(*) \qquad f_i = \sum_{j=1}^{k} \gamma_{i,j} g_j \quad \text{and} \quad g_i = \sum_{j=1}^{k} \delta_{i,j} f_j \quad \text{on} \quad U \cap V.$$

Let $f\colon M \to R$ be a Nash function. Then we have a presentation

$$\operatorname{graph} f = \bigcup_{\text{finite}} \{x \in R^{n+1}\colon \varphi(x,a) = 0,\ \varphi_1(x,a) > 0, \dots, \varphi_l(x,a) > 0\},$$

where $\varphi$ and $\varphi_i$ are polynomials with coefficients in $\mathbf{Z}$ and $a$ is a $p$-uple of elements of $R$. For $b \in \mathbf{R}^p$, set

$$X_b = \bigcup_{\text{finite}} \{x \in \mathbf{R}^{n+1}\colon \varphi(x,b) = 0,\ \varphi_1(x,b) > 0, \dots, \varphi_l(x,b) > 0\}.$$

Then, as noted in the proof of Theorem 2.4 in [C-S$_1$], the set of $b$ such that $X_b$ is a Nash manifold of dimension $m$ is semialgebraic in $\mathbf{R}^p$. Moreover, by the same reason as in the proof, the set $B \subset \mathbf{R}^p$ of $b$ such that $X_b$ is the graph of a Nash function on $M^{\mathbf{R}}$ is semialgebraic. Note that $X_b \subset M^{\mathbf{R}} \times \mathbf{R}$. Set $X = \cup_{b \in B} X_b \times b$.

By Theorem 2.4 there exists a finite semialgebraic stratification $B = \cup B^i$ of $B$ into Nash manifolds such that for each $i$, $X^i = X \cap \mathbf{R}^{n+1} \times B^i$ is a Nash manifold and that there is a Nash diffeomorphism $\xi^i\colon M^{\mathbf{R}} \times B^i \to X^i$ compatible with the projection onto $B^i$. For $(x,b) \in M^{\mathbf{R}} \times B^i$, $\xi^i(x,b)$ is of the form $(\xi^i_1(x,b), \xi^i_2(x,b), b) \in M^{\mathbf{R}} \times \mathbf{R} \times B^i$. Then it is easy to see that the map $M^{\mathbf{R}} \times B^i \ni (x,b) \to (\xi^i_1(x,b), b) \in M^{\mathbf{R}} \times B^i$ is a diffeomorphism. Hence we can assume $\xi^i_1$ is the identity map of $M^{\mathbf{R}}$ and we have a Nash function $h^i\colon M^{\mathbf{R}} \times B^i \to \mathbf{R}$ such that for each $b \in B^i$, the graph of the function $h^i(\cdot, b)\colon M^{\mathbf{R}} \to \mathbf{R}$ coincides with $X_b$.

Note that there exists $i$ such that $a \in B^i_R$, i.e., $f = h^i_R(\cdot, a)$.

Consequently, there exist Nash manifolds $A$ and $C$ over $\mathbf{R}$, Nash maps $F = (F_1, \dots, F_k)\colon U^{\mathbf{R}} \times A \to \mathbf{R}^k$, $G = (G_1, \dots, G_k)\colon V^{\mathbf{R}} \times A \to \mathbf{R}^k$, Nash functions

$\Gamma_{i,j}\colon (U^{\mathbf{R}} \cap V^{\mathbf{R}}) \times C \to \mathbf{R}$ and $\Delta_{i,j}\colon (U^{\mathbf{R}} \cap V^{\mathbf{R}}) \times C \to \mathbf{R}$, $i, j = 1, \dots, k$, and points $a \in A_R$ and $c \in C_R$ such that

$$F_R(\cdot, a) = (f_1, \dots, f_k), \quad G_R(\cdot, a) = (g_1, \dots, g_k),$$
$$\Gamma_{i,jR}(\cdot, c) = \gamma_{i,j} \quad \text{and} \quad \Delta_{i,jR}(\cdot, c) = \delta_{i,j}.$$

Replace $A$, $C$, $a$ and $c$ with $A \times C$, $A \times C$, $(a, c)$ and $(a, c)$ respectively. Then we can assume $A = C$ and $a = c$. Moreover, we can choose $A$, $F$, $G$, $\Gamma_{i,j}$ and $\Delta_{i,j}$ so that for each $i$,

$$(**) \qquad F_i = \sum_{j=1}^{k} \Gamma_{i,j} G_j \quad \text{and} \quad G_i = \sum_{j=1}^{k} \Delta_{i,j} F_j \quad \text{on} \quad (U^{\mathbf{R}} \cap V^{\mathbf{R}}) \times A$$

by the same reason as above, because it is possible to express by a formula of the first order theory of real closed field the fact that the equality $(**)$ holds.

By $(**)$ there exists a sheaf of $\mathcal{N}_{M^{\mathbf{R}} \times A}$-ideals $\mathcal{J}$ on $M^{\mathbf{R}} \times A$ such that $\mathcal{J}|_{U^{\mathbf{R}} \times A}$ and $\mathcal{J}|_{V^{\mathbf{R}} \times A}$ are generated by $F_1, \dots, F_k$ and $G_1, \dots, G_k$ respectively. By the theorem we have a finite number of generators $H_i$ of $\mathcal{J}$. Then it is easy to see that $H_{iR}(\cdot, a)$ generate $\mathcal{I}$. □

***Proof of Extension conjecture.*** It is sufficient to prove the following assertion.

*Let $M \subset R^n$ be a Nash manifold. Let $U$ and $V$ be open semialgebraic subsets of $M$ such that $M = U \cup V$. Let $\mathcal{I}$ be a sheaf of $\mathcal{N}_M$-ideals generated by a finite number of global Nash functions. Let $f\colon U \to R$ and $g\colon V \to R$ be Nash functions such that $f - g$ is a cross-section of $\mathcal{I}|_{U \cap V}$. Then there exists a Nash function $h\colon M \to R$ such that $h|_U - f$ and $h|_V - g$ are cross-sections of $\mathcal{I}|_U$ and $\mathcal{I}|_V$ respectively.*

Let $\varphi_i$, $i = 1, \dots, l$, be generators of $\mathcal{I}$. We have

$$f - g = \sum_{i=1}^{l} \gamma_i \varphi_i \quad \text{on} \quad U \cap V$$

for some Nash functions $\gamma_i\colon U \cap V \to R$. Then, as in the preceding proof of Global equation conjecture, we can assume $M = M^{\mathbf{R}}_R$, $U = U^{\mathbf{R}}_R$ and $V = V^{\mathbf{R}}_R$ for some Nash manifold $M^{\mathbf{R}}$ over $\mathbf{R}$ and open semialgebraic subsets $U^{\mathbf{R}}$ and $V^{\mathbf{R}}$ of $M^{\mathbf{R}}$ and we obtain a Nash manifold $A$ over $\mathbf{R}$, a point $a$ of $A_R$ and Nash functions $F\colon U^{\mathbf{R}} \times A \to \mathbf{R}$, $G\colon V^{\mathbf{R}} \times A \to \mathbf{R}$, $\Phi_i\colon M^{\mathbf{R}} \times A \to \mathbf{R}$, $i = 1, \dots, l$, and $\Gamma_i\colon (U^{\mathbf{R}} \cap V^{\mathbf{R}}) \times A \to \mathbf{R}$, $i = 1, \dots, l$, such that

$$F - G = \sum_{i=1}^{l} \Gamma_i \Phi_i \quad \text{on} \quad (U^{\mathbf{R}} \cap V^{\mathbf{R}}) \times A,$$
$$F_R(\cdot, a) = f, \quad G_R(\cdot, a) = g, \quad \Phi_{iR}(\cdot, a) = \varphi_i \quad \text{and} \quad \Gamma_{iR}(\cdot, a) = \gamma_i.$$

Let $\mathcal{J}$ be the sheaf of $\mathcal{N}_{M^{\mathbf{R}} \times A}$-ideals on $M^{\mathbf{R}} \times A$ generated by $\Phi_i$. Then, since Extension conjecture holds true for $\mathbf{R}$, there exists a Nash function $H\colon M^{\mathbf{R}} \times A \to \mathbf{R}$ such that $H|_{U^{\mathbf{R}} \times A} - F$ and $H|_{V^{\mathbf{R}} \times A} - G$ are cross-sections of $\mathcal{J}|_{U^{\mathbf{R}} \times A}$ and $\mathcal{I}|_{V^{\mathbf{R}} \times A}$ respectively. Clearly $h = H_R(\cdot, a)$ fulfills the requirements. □

**Problem.** *Open problems are Global extension and Extension conjectures for a general real closed field.*

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