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THOM'S CONJECTURE ON TRIANGULATIONS OF MAPS

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§1. INTRODUCTION

Let $f_i\colon X_i \to Y_i$, $i = 1, 2$, be proper C^0 maps between closed sets in Euclidean spaces. We call f_1 and f_2 *$\mathcal{R}$-$\mathcal{L}$ equivalent* if there exist homeomorphisms $\eta\colon Y_1 \to Y_2$ and $\tau\colon X_1 \to X_2$ such that $\eta \circ f_1 = f_2 \circ \tau$. We call f_1 *triangulable* if it is $\mathcal{R}$-$\mathcal{L}$ equivalent to a PL map between closed polyhedra in Euclidean spaces.

Thom [T] conjectured that a so-called "Thom map", which Thom called une application stratifiée sans éclatement, is triangulable. In the present paper we solve the conjecture in a more general form. Partial solutions were given by Teissier [Te] and Proposition IV.1.10 in [S].

A *tube system* $\{T_j = (|T_j|, \pi_j, \rho_j)\}_{j=1,\dots,k}$ for a C^∞ stratification $\{Y_j\}_{j=1,\dots,k}$ with $Y = \cup_j Y_j \subset \mathbf{R}^n$ and $\dim Y_j < \dim Y_{j+1}$ consists of one tube T_j at each Y_j, where $\pi_j\colon |T_j| \to Y_j$ is a C^∞ open tubular neighborhood of Y_j in $\mathbf{R}^n$ and ρ_j is a non-negative C^∞ function on $|T_j|$ such that $\rho_j^{-1}(0) = Y_j$ and each point y of Y_j is a unique and non-degenerate critical point of $\rho_j|_{\pi_j^{-1}(y)}$. We call a tube system $\{T_j\}$ *strongly controlled* if for each pair j and j' with $j < j'$, the following property holds true:

$$\mathrm{ct}(T_j, T_{j'}) \qquad \pi_j \circ \pi_{j'} = \pi_j \quad \text{and} \quad \rho_j \circ \pi_{j'} = \rho_j \quad \text{on} \quad |T_j| \cap |T_{j'}|,$$

and (sc) the map $(\pi_j, \rho_j)|_{Y_{j'} \cap |T_j|}$ is a C^∞ submersion into $Y_j \times \mathbf{R}$. Note that any Whitney stratification admits a strongly controlled tube system. An example of a C^∞ stratification which admits a strongly controlled tube system but is not a Whitney stratification is {the x-axis, $\{(x, y, z) \in \mathbf{R}^3\colon y = z^2 \sin x/z,\ z \neq 0\}$}.

Let $\{X_{i,j}\}_{\substack{j=1,\dots,k\\ i=1,\dots,k_j}}$ and $\{Y_j\}_{j=1,\dots,k}$ be C^∞ stratifications of sets X and Y in $\mathbf{R}^n$, respectively, such that $\dim X_{i,j} < \dim X_{i+1,j}$ and $\dim Y_j < \dim Y_{j+1}$, and let $f\colon X \to Y$ be a C^∞ map (i.e., the restriction to X of a C^∞ map $\tilde f$ between neighborhoods of X and Y) such that each restriction $f|_{X_{i,j}}$ is a submersion into Y_j. Let $\{T_j = (|T_j|, \pi_j, \rho_j)\}$ be a strongly controlled tube system for $\{Y_j\}$, and let $\{T_{i,j} = (|T_{i,j}|, \pi_{i,j}, \rho_{i,j})\}$ be a tube system for $\{X_{i,j}\}$. We call $\{T_{i,j}\}$ *strongly controlled over* $\{T_j\}$ if the following conditions are satisfied. (sc1) For each (i, j), it holds that $f \circ \pi_{i,j} = \pi_j \circ \tilde f$ on $|T_{i,j}|$. (sc2) For each j, $\{T_{i,j}\}_{i=1,\dots,k_j}$ is a strongly controlled tube system for $\{X_{i,j}\}_{i=1,\dots,k_j}$. (sc3) For any pair (i, j) and (i', j') with $j < j'$, it holds that $\pi_{i,j} \circ \pi_{i',j'} = \pi_{i,j}$ on $|T_{i,j}| \cap |T_{i',j'}|$, and $(\pi_{i,j}, f)|_{X_{i',j'} \cap |T_{i,j}|}$ is a C^∞ submersion into the C^∞ manifold $\{(x, y) \in X_{i,j} \times (Y_{j'} \cap |T_j|)\colon f(x) = \pi_j(y)\}$. (An example of $f\colon X \to Y$ where there do not exist such tube systems $\{T_j\}$ and $\{T_{i,j}\}$ is the blow-up of S^n, $n > 1$, at a point of S^n.)

Theorem. *Let $\{X_{i,j}\}$ and $\{Y_j\}$ be C^∞ stratifications of closed sets $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^n$ respectively, and let $f\colon X \to Y$ be a C^∞ proper map such that each restriction $f|_{X_{i,j}}$ is a submersion into Y_j. Assume there exist a strongly controlled tube system $\{T_j\}$ for $\{Y_j\}$ and a tube system $\{T_{i,j}\}$ for $\{X_{i,j}\}$ strongly controlled over $\{T_j\}$. Then f is triangulable.*

The theorem is proved by a theory developed in [S] and hence can be proved also in the semialgebraic, subanalytic and $\mathfrak{X}$ categories. (See [S] for the definition of $\mathfrak{X}$.) (In the subanalytic and $\mathfrak{X}$ cases, we argue in the C^r category for a positive integer r). In the following proof we use integrations of vector fields. But we can avoid this in the above important special cases as shown in [S]. Note also that we can construct effectively a triangulation, i.e., polyhedra X' and Y' and homeomorphisms $\tau\colon X' \to X$ and $\eta\colon Y' \to Y$ such that $\eta^{-1} \circ f \circ \tau$ is PL in the cases. Hence the following assertion seems true.

Let k, l, $m \in \mathbf{N}$. The cardinal number of the $\mathcal{R}$-$\mathcal{L}$ equivalence classes of all proper semialgebraic Thom maps between closed semialgebraic sets in $\mathbf{R}^k$ whose graphs are defined by equalities or inequalities of l-polynomials of degree $\leq m$ is bounded by some recursive function in variables (k, l, m).

For the proof it suffices to find an effective method of choosing a Thom stratification $f\colon \{X_{i,j}\} \to \{Y_j\}$ of a Thom map $f\colon X \to Y$, because we can effectively construct strongly controlled tube systems $\{T_{i,j}\}$ and $\{T_j\}$ of a Thom stratification $f\colon \{X_{i,j}\} \to \{Y_j\}$ [S]. (See [G-al] for the definitions of a Thom map and a Thom stratification.) Therefore, we can prove the above assertion if we replace the phrase "Thom maps" with the one "Thom stratifications $f\colon \{X_{i,j}\} \to \{Y_j\}$" and add the condition that $\{X_{i,j}\}$ and $\{Y_j\}$ are defined by l-polynomials as graph f.

§2. C^∞ TRIANGULATIONS

In this paper, K and L always denote simplicial complexes in some Euclidean space. Let $|K|$ denote the underlying polyhedron of K. For a point x in $|K|$, let $\mathrm{st}(x, K)$ denote the subcomplex of K generated by the simplexes containing x. We denote by K^k the k-skeleton of K for a non-negative integer k. For a simplex or a manifold σ, $\mathrm{Int}\,\sigma$ and $\partial\sigma$ denote the interior and the boundary of σ respectively. If $K \subset L$, the *simplicial neighborhood* $N(K, L)$ of K in L is the smallest subcomplex of K whose underlying polyhedron is a neighborhood of $|K|$ in $|L|$. If a subset W of $|L|$ is the underlying polyhedron of a subcomplex of L, we call the subcomplex $L|_W$. For each simplex σ of K, let v_σ denote the barycenter of σ. The *barycentric subdivision* K' of K consists of all the simplexes spanned by $v_{\sigma_1}, \cdots, v_{\sigma_k}$ for $\sigma_1 \subset \cdots \subset \sigma_k \in K$.

A C^∞ *map* $h\colon K \to \mathbf{R}^n$ is a continuous map $h\colon |K| \to \mathbf{R}^n$ such that all the restrictions $h|_\sigma$, $\sigma \in K$, are of class C^∞. Let $b \in |K|$. We define $dh_b\colon |\mathrm{st}(b, K)| \to \mathbf{R}^n$ by

$$dh_b(x) = d(h|_\sigma)_b(x - b) \quad \text{for} \quad \sigma \in \mathrm{st}(b, K), \quad x \in \sigma.$$

We call h a C^∞ *imbedding* if h and dh_b for all $b \in |K|$ are homeomorphisms onto the images. Let $Z \subset \mathbf{R}^n$. A C^∞ *triangulation* of Z is a pair of K and a C^∞ imbedding $h\colon K \to \mathbf{R}^n$ such that $h(|K|) = Z$. (A *triangulation* of Z consists of K and a homeomorphism from $|K|$ to Z.) An *approximation* of h is a C^∞ map

$\hat{h}\colon \hat{K} \to \mathbf{R}^n$ such that $\hat{K}$ is a subdivision of K,

$$|h(x) - \hat{h}(x)| \le c \quad \text{for} \quad x \in |K|,$$

and

$$|dh_b(x) - d\hat{h}_b(x)| \le c|x - b| \quad \text{for} \quad b \in |K|, \quad x \in |\operatorname{st}(b, K')|$$

for a small positive number c.

Let $\alpha\colon K_1 \to K_2$ be a simplicial map between finite simplicial complexes in $\mathbf{R}^n$. By induction on $\dim K_1$ we define the *mapping cylinder* $C_\alpha(K_1, K_2)$ of α which is a simplicial complex in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and whose underlying polyhedron can be regarded as the mapping cylinder $C_\alpha(|K_1|, |K_2|)$ of the topological map $\alpha\colon |K_1| \to |K_2|$. Let K_1 and K_2 be given in $\mathbf{R}^n \times 0 \times 0 \subset \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and $0 \times \mathbf{R}^n \times 1 \subset \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ respectively, and let K'_1 and K'_2 denote the barycentric subdivision of K_1 and K_2 respectively. If $\dim K_1 = -1$, i.e., $K_1 = \emptyset$, then set $C_\alpha(K_1, K_2) = K'_2$. Let $\dim K_1 = k$ and assume we have already defined the mapping cylinder $C_\alpha(K_1^{k-1}, K_2)$. For $\sigma \in K_1 - K_1^{k-1}$, let a_σ denote the middle point of the barycenters of σ and of $\alpha(\sigma)$ in $\mathbf{R}^n \times \mathbf{R}^n \times 1/2$. We set

$$C_\alpha(K_1, K_2) = C_\alpha(K_1^{k-1}, K_2)$$
$$\cup \bigcup_{\sigma \in K_1 - K_1^{k-1}} \{a_\sigma,\ \sigma_1,\ a_\sigma * \sigma_1 \colon \sigma_1 \in K'_1|_\sigma \cup K'_2|_{\alpha(\sigma)} \cup C_{\alpha|_{\partial\sigma}}(K_1|_{\partial\sigma}, K_2|_{\alpha(\partial\sigma)})\},$$

where $a_\sigma * \sigma_1$ denotes the cone with vertex a_σ and base σ_1.

We show some good properties of $C_\alpha(K_1, K_2)$. Clearly it is a simplicial complex in $\mathbf{R}^n \times \mathbf{R}^n \times [0,1]$, K'_1 and K'_2 are subcomplexes of $C_\alpha(K_1, K_2)$, and there is a natural simplicial map $C_\alpha(K_1, K_2) \to K'_2$, which is a retraction and carries the barycenter of a simplex σ of K_1 and the above-mentioned a_σ to the barycenter of $\alpha(\sigma)$. Given a commutative diagram of simplicial maps

$$\begin{array}{ccc} L_1 & \xrightarrow{\beta} & L_2 \\ \downarrow & & \downarrow \\ K_1 & \xrightarrow{\alpha} & K_2, \end{array}$$

there exists a natural simplicial map $C_\beta(L_1, L_2) \to C_\alpha(K_1, K_2)$. On the other hand, $C_{\mathrm{id}}(K_1, K_1)$ is naturally and simplicially isomorphic to the barycentric subdivision L of the cell complex $K_1 \times \{0, 1, [0,1]\}$. Hence we have a natural simplicial map $L \to C_\alpha(K_1, K_2)$, which equals the identity map on $|K_1| \times 0$ and α on $|K_1| \times 1$. Through this map we identify $|C_\alpha(K_1, K_2)|$ with the mapping cylinder of the topological map α.

Let M be a subset of $\mathbf{R}^n$. We call M a C^∞ *manifold possibly with corners of dimension* m if it is locally C^∞ diffeomorphic to an open subset of $\mathbf{R}_+^m$, where $\mathbf{R}_+ = [0, \infty[$. Note that such an M admits the canonical C^∞ stratification $\{Z_i\}_{i=0,\dots,m}$ such that each Z_i is the subset of $\cup_{j=0}^i Z_j$ where $\cup_{j=0}^i Z_j$ is locally C^∞ diffeomorphic to $\mathbf{R}^i$. *Faces* of M are the closures of the connected components of Z_i. For a face M' of M of dimension m', set $\operatorname{Sing} M' = M' \cap \cup_{i=0}^{m'-1} Z_i$.

For continuous maps $\psi_i\colon A_i \to B$, $i = 1, 2$, let $A_1 \times_{(\psi_1,\psi_2)} A_2$ denote the fibre product — $\{(a_1, a_2) \in A_1 \times A_2 \colon \psi_1(a_1) = \psi_2(a_2)\}$.

The key of proof of the theorem is the following lemma, which is similar to Proposition I.3.20 in [S].

Lemma 1. *Let M and M_1 be compact C^∞ manifolds possibly with corners. Let $\varphi\colon M \to M_1$ be a surjective C^∞ submersion which carries surjectively and submersively any face of M to some face of M_1. Let M' be a face of M. Let (L, g) and (K, h) be C^∞ triangulations of M_1 and a neighborhood of a union of subfaces of M' in M, respectively, such that $g^{-1} \circ \varphi \circ h$ is a PL map from $|K|$ to $|L|$. Shrink the neighborhood of the union and subdivide K. Then keeping the property that $g^{-1} \circ \varphi \circ h$ is PL, we can extend h to a C^∞ triangulation of a neighborhood of M' in M.*

Proof of Lemma 1. We can assume that the given neighborhood is a neighborhood of $\operatorname{Sing} M'$ in M. Recall the following assertion in the proof of Proposition I.3.20 in [S].

Assertion. Let $n > n_1$ be non-negative integers, let $p\colon \mathbf{R}^n_+ \to \mathbf{R}^{n_1}_+$ be the projection onto the first n_1-factors, let $\alpha\colon A \to \mathbf{R}^n_+$ be a C^∞ imbedding of a finite simplicial complex A, let (B, β) be a C^∞ triangulation of $\mathbf{R}^{n_1}_+$ such that $\beta^{-1} \circ p \circ \alpha$ is PL, and let C be a compact subset of $\mathbf{R}^n_+$. Then there exist a simplicial complex A_0 and a C^∞ imbedding $\alpha_0\colon A_0 \to \mathbf{R}^n_+$ such that some subdivision of A is a subcomplex of A_0, the restriction $\alpha_0|_{|A|}\colon A_0|_{|A|} \to \mathbf{R}^n_+$ is a strong approximation of α,

$$A_1 \subset A_0, \quad \alpha_0|_{|A_1|} = \alpha|_{|A_1|}, \quad \alpha_0(|A_0|) \supset C, \quad (\overline{|A_0| - |A|}) \cap |A_1| = \emptyset,$$

and $\beta^{-1} \circ p \circ \alpha_0$ is PL, where $A_1 = \{\sigma \in A \colon \alpha(\sigma) \cap C = \emptyset\}$.

It is easy to see that $h^{-1}(M')$ and $h^{-1}(\operatorname{Sing} M')$ are the underlying polyhedra of some subcomplexes of K. Set $U = h(|N(K|_{h^{-1}(\operatorname{Sing} M')}, K)|)$. Then U is a compact neighborhood of $\operatorname{Sing} M'$ in M, and we can assume $U \cap \overline{M - h(|K|)} = \emptyset$. (Here replace K with its barycentric subdivision if necessary.) Let $\{C_i\}_{i=1,\dots,k}$ be a covering of $\overline{M' - h(|K|)}$ by compact sets such that for each i, there exist an open neighborhood V_i of C_i in M and C^∞ imbeddings $\tau_i\colon V_i \to \mathbf{R}^m_+$ and $\theta_i\colon \varphi(V_i) \to \mathbf{R}^{m_1}_+$, where $m = \dim M$ and $m_1 = \dim M_1$, such that $V_i \cap U = \emptyset$, and the composite $\theta_i \circ \varphi \circ \tau_i^{-1}\colon \tau_i(V_i) \to \mathbf{R}^{m_1}_+$ is the restriction of the projection of $\mathbf{R}^m_+$ onto the first m_1-factors.

Let $0 < l < k$ be an integer. Assume we have already constructed a C^∞ triangulation (K_{l-1}, h_{l-1}) of a neighborhood of $U \cup \bigcup_{i=1}^{l-1} C_i$ in M such that $g^{-1} \circ \varphi \circ h_{l-1}$ is PL, some subdivision of K is a subcomplex of K_{l-1}, $h_{l-1}|_{|K|}$ is a strong approximation of h, and $h = h_{l-1}$ on $h^{-1}(U)$. Then it suffices to obtain (K_l, h_l) with the corresponding properties.

Subdividing finely L and then K_{l-1}, we can assume that (i) for $\sigma \in K_{l-1}$, if $h_{l-1}(\sigma) \cap C_l \neq \emptyset$ then $h_{l-1}(\sigma) \subset V_l$, (ii) for $\sigma_1, \sigma_2 \in L$, if $\sigma_1 \cap \sigma_2 \neq \emptyset$ and $g(\sigma_1) \cap \varphi(C_l) \neq \emptyset$ then $g(\sigma_2) \subset \varphi(V_l)$, and (iii) for $\sigma \in K_{l-1}$ and $\sigma_1 \in L$, if $h_{l-1}(\sigma) \cap C_l \neq \emptyset$ and $\varphi \circ h_{l-1}(\sigma) \cap g(\sigma_1) \neq \emptyset$ then $g(\sigma_1) \cap \varphi(C_l) \neq \emptyset$. Let D denote the complex generated by $\sigma \in L$ with $g(\sigma) \cap \varphi(C_l) \neq \emptyset$.

Apply the assertion to

$$n = m_1, \quad n_1 = 0,$$
$$(A, \alpha) = (\{\sigma \in L\colon g(\sigma) \subset \varphi(V_l)\}, \theta_l \circ (g|_{|A|})),$$
$$(B, \beta) = (\{0\}, \mathrm{id}), \quad \text{and} \quad C = \overline{[0,c]^n - \alpha(|A|)}$$

for a large number c. Then by (ii) we have a C^∞ triangulation (A_0, α_0) of a neighborhood of $[0,c]^n$ in $\mathbf{R}^n_+$ such that $A_0 \supset D$ and $\alpha_0 = \alpha$ on $|D|$. Repeat a similar argument for $c_1 = c, c_2, \dots \to \infty$. Then we obtain a C^∞ triangulation $(\tilde{B}, \tilde{\beta})$ of $\mathbf{R}^{m_1}_+$ such that $\tilde{B} \supset D$ and $\tilde{\beta} = \theta_l \circ g$ on $|D|$.

In consideration of application of the assertion, set newly

$$n = m, \quad n_1 = m_1,$$
$$(A, \alpha)$$
$$= (\text{the complex generated by } \sigma \in K_{l-1} \text{ with } h_{l-1}(\sigma) \cap C_l \neq \emptyset,\ \tau_l \circ (h_{l-1}|_{|A|})),$$
$$(B, \beta) = (\tilde{B}, \tilde{\beta}), \quad \text{and} \quad C = \tau_l(C_l).$$

By (i), α is well-defined. By (iii), $\alpha(|A|) \subset \beta(|D|)$. Hence $\beta^{-1} \circ p \circ \alpha$ $(= g^{-1} \circ \theta_l^{-1} \circ p \circ \tau_l \circ h_{l-1} = g^{-1} \circ \varphi \circ h_{l-1})$ is PL. Thus the conditions in the assertion are satisfied. Let $\alpha_0\colon A_0 \to \mathbf{R}^m_+$ be a resulting C^∞ imbedding. Set $\check{K}_{l-1} = \{\sigma \in K_{l-1}\colon h_{l-1}(\sigma) \cap C_l = \emptyset\}$. Remember that

$$(A_0, \alpha_0) = (A, \alpha) \quad \text{on} \quad |\{\sigma \in A\colon h_{l-1}(\sigma) \cap C_l = \emptyset\}|,$$

and regard

$$|A_0| \cap |\check{K}_{l-1}| = |\{\sigma \in A\colon h_{l-1}(\sigma) \cap C_l = \emptyset\}|.$$

Let E' denote the barycentric subdivision of a simplicial complex E as always. Then the family $A'_0 \cup \check{K}'_{l-1}$ is a simplicial complex. Let K_l denote the complex. We can assume that $\alpha_0(|A_0|) \subset \tau_l(V_l)$. Set

$$h_l = \begin{cases} \tau_l^{-1} \circ \alpha_0 & \text{on} \quad |A_0| \\ h_{l-1} & \text{on} \quad |\check{K}_{l-1}| - |A_0|. \end{cases}$$

Then this map is well-defined and a C^∞ imbedding by 8.8 in [M], and (K_l, h_l) fulfills the requirements. □

§3. Vector fields and removal data

Let X, Y, $\{X_{i,j}\}$, $\{Y_j\}$, $f\colon X \to Y$, $\{T_{i,j}\}$ and $\{T_j\}$ be the same as in the theorem except for the assumption that f is proper. Assume $\dim Y_j < \dim Y_{j+1}$ and $\dim X_{i,j} < X_{i+1,j}$. Let the set of indexes of $\{X_{i,j}\}$ be $\overline{H} = \{(i,j) \in \mathbf{N}^2\colon 1 \le j \le k,\ 1 \le i \le k_j\}$. Set $H = \overline{H} - \{(k_k, k)\}$. Give a lexicographic order to H and $\overline{H}$ so that $(i,j) < (i',j')$ if $j < j'$ or $j = j'$ and $i < i'$.

A *vector field* v^Y on $\{Y_j\}$ consists of one C^∞ vector field v_j on each Y_j. We call v^Y *controlled* if for each pair j and j',

$$\mathrm{cv}(T_j, T_{j'}) \qquad \left.\begin{aligned} d\pi_j v_{j'y} &= v_{j\pi_j(y)} \\ d\rho_j v_{j'y} &= 0 \end{aligned}\right\} \quad \text{for} \quad y \in Y_{j'} \cap U_j,$$

where U_j is some neighborhood of Y_j in $|T_j|$. If only the former equality is assumed, we call v^Y *weakly controlled.* We call a vector field $v^X = \{v_{i,j}\}$ on $\{X_{i,j}\}$ *controlled over* v^Y if the former equality of $\mathrm{cv}(T_{i,j}, T_{i',j'})$ for each pair (i,j) and (i',j'), the latter for each pair (i,j) and (i',j), and the following equality for each (i,j) hold:

$$df v_{i,jx} = v_{jf(x)} \quad \text{for} \quad x \in X_{i,j}.$$

Let $v^Y = \{v_j\}$ be a vector field on $\{Y_j\}$. For each j, let $\omega_j\colon \Omega_j \to Y_j$, $\Omega_j \subset Y_j \times \mathbf{R}$, be the maximal C^∞ flow defined by v_j. Set $\Omega = \cup \Omega_j$ and define a map $\omega\colon \Omega \to Y$ by $\omega|_{\Omega_j} = \omega_j$ for each j. We call ω the *flow* of v^Y. We call v^Y *locally integrable* if Ω is open in $Y \times \mathbf{R}$ and the flow is continuous.

Assume X and Y are compact. Let $0 < \varepsilon_{k-1} \ll \cdots \ll \varepsilon_1 \ll \infty$ be numbers. Then for $j \le l$, (1) the following set is a C^∞ submanifold possibly with corners of Y_l:

$$Y_{j,l} = Y_l \cap |T_j| - \rho_1^{-1}(]0, \varepsilon_1/2[) - \cdots - \rho_{j-1}^{-1}(]0, \varepsilon_{j-1}/2[),$$

(2) if $j < l$, the restriction of (π_j, ρ_j) to $Y_{j,l} \cap \rho_j^{-1}(]0, 2\varepsilon_j])$ is a C^∞ submersion into $Y_{j,j} \times]0, 2\varepsilon_j]$, and (3) the sets $Y_{j,j}$ and $\cup_{j' \ge j} Y_{j,j'} \cap \rho_j^{-1}([0, 2\varepsilon_j])$ are compact. We call $\varepsilon = \{\varepsilon_j\}_{j=1,\dots,k-1}$ with such properties a *removal data* of $\{T_j\}_{j=1,\dots,k}$.

A *removal data* $\varepsilon = \{\varepsilon_{i,j}\}_{(i,j)\in H}$ of $\{T_j, T_{i,j}\}_{(i,j)\in \overline{H}}$ is such that the following eight conditions are satisfied. Let $(i_1, j_1) \le (i_2, j_2) \in \overline{H}$. (1) Each $\varepsilon_{i,j}$ is a small positive number. Set $\varepsilon_{k_j,j} = \varepsilon_j$. (2) $\{\varepsilon_j\}_{j=1,\dots,k-1}$ is a removal data of $\{T_j\}_{j=1,\dots,k}$. (3) The following set is a C^∞ manifold possibly with corners:

$$\begin{aligned} X_{i_1,j_1,i_2,j_2} &= X_{i_2,j_2} \cap |T_{i_1,j_1}| \cap (\rho_{j_1} \circ f)^{-1}([0, 2\varepsilon_{j_1}]) \\ &- \bigcup_{j<j_1} (\rho_j \circ f)^{-1}(]0, \varepsilon_j/2[) - \bigcup_{i<i_1} \rho_{i,j_1}^{-1}(]0, \varepsilon_{i,j_1}/2[). \end{aligned}$$

(Here we ignore $(\rho_{j_1} \circ f)^{-1}([0, 2\varepsilon_{j_1}])$ if $j_1 = k$.) (4) If $j_1 = j_2$ and if $i_1 < i_2$, the restriction of $(\pi_{i_1,j_1}, \rho_{i_1,j_1})$ to $X_{i_1,j_1,i_2,j_2} \cap \rho_{i_1,j_1}^{-1}(]0, 2\varepsilon_{i_1,j_1}])$ is a C^∞ submersion into $X_{i_1,j_1,i_1,j_1} \times]0, 2\varepsilon_{i_1,j_1}]$. (5) If $j_1 < j_2$, the restriction of $(\pi_{k_{j_1},j_1}, \rho_{j_1} \circ f)$ to $X_{k_{j_1},j_1,i_2,j_2}$ is a C^∞ submersion into $X_{k_{j_1},j_1,k_{j_1},j_1} \times]0, 2\varepsilon_{j_1}]$. (6) If $j_1 < j_2$ and if $i_1 < k_{j_1}$, the restriction of $(\pi_{i_1,j_1}, f, \rho_{i_1,j_1})$ to $X_{i_1,j_1,i_2,j_2} \cap \rho_{i_1,j_1}^{-1}([\varepsilon_{i_1,j_1}/2, 2\varepsilon_{i_1,j_1}])$ is a C^∞ submersion into $(X_{i_1,j_1,i_1,j_1} \times_{(f,\pi_{j_1})} (Y_{j_1,j_2} \cap \rho_{j_1}^{-1}(]0, 2\varepsilon_{j_1}]))) \times [\varepsilon_{i_1,j_1}/2, 2\varepsilon_{i_1,j_1}]$. (7) The set $\cup_{(i,j)\ge(k_{j_1},j_1)} X_{k_{j_1},j_1,i,j}$ is compact. (8) If $i_1 < k_{j_1}$, the set $\cup_{(i,j)\ge(i_1,j_1)} X_{i_1,j_1,i,j} \cap \rho_{i_1,j_1}^{-1}([0, 2\varepsilon_{i_1,j_1}])$ is compact.

It is easy to see existence of a removal data of $\{T_j, T_{i,j}\}_{(i,j)\in\overline{H}}$. Indeed, it suffices to choose $\{\varepsilon_{i,j}\}$ so that $0 < \varepsilon_{1,1} \ll \infty$ and $\varepsilon_{i,j} \gg \varepsilon_{i',j'}$ if $(i,j) < (i',j')$. (Only condition (6) is nontrivial. For each $(i_3, j_1) > (i_1, j_1)$, the restriction of (π_{i_3,j_1}, f)

to $X_{i_2,j_2} \cap |T_{i_3,j_1}|$ and $(\pi_{i_1,j_1}, \rho_{i_1,j_1})$ to $X_{i_3,j_1} \cap |T_{i_1,j_1}|$ are C^∞ submersion into $X_{i_3,j_1} \times_{(f,\pi_{j_1})} (Y_{j_2} \cap |T_{j_1}|)$ and $X_{i_1,j_1} \times \mathbf{R}$, respectively, by conditions (sc2) and (sc3). Hence (6) holds.)

In the case where f is proper and the connected components of Y_j are bounded in $\mathbf{R}^n$, we need to and can easily generalize the above definition of a removal data. For each j, let $\{Y_j^l\}_{l\in\Gamma_j}$ denote the family of the connected components of Y_j. Replace the above $\{\varepsilon_{i,j}\}$, $X_{i_1,j_1,i_2,j_2}, \ldots$ with $\{\varepsilon_{i,j,l}\}_{(i,j)\in H, l\in\Gamma_j}$,

$$X_{i_2,j_2} \cap f^{-1}(Y_{j_2}^{l_2}) \cap \pi_{i_1,j_1}^{-1}(f^{-1}(Y_{j_1}^{l_1})) \cap (\rho_{j_1}\circ f)^{-1}([0, 2\varepsilon_{j_1,l_1}])$$
$$- \bigcup_{j<j_1, l\in\Gamma_j} (\rho_j \circ f)^{-1}(]0, \varepsilon_{j,l}/2[) - \bigcup_{i<i_1} \rho_{i,j_1}^{-1}(]0, \varepsilon_{i,j_1,l_1}/2[)$$
$$\text{for} \quad l_1 \in \Gamma_{j_1} \text{ and } l_2 \in \Gamma_{j_2},$$

... . Then the generalization is clear. We omit the details.

If we undo the assumption that the connected components of Y_j are bounded, the generalization becomes complicated. See [S] for it. We need not consider this case in the present paper by the following lemma.

Lemma 2. *In the theorem, we can assume that each connected component of Y_j is bounded in $\mathbf{R}^n$.*

Proof of Lemma 2. In this proof we shall frequently shrink $|T_{i,j}|$ and $|T_j|$ without telling. Considering the unions of strata of same dimensions, we assume $\dim Y_j = j$, $j = 0, \ldots, k$, only now. It is easy to construct a C^∞ proper function ξ on $\mathbf{R}^n$ such that for each $y \in Y_j$, ξ is constant on $\pi_j^{-1}(y)$, $\mathbf{Z} + [-1/3, 1/3] = \cup_{z\in\mathbf{Z}}[z-1/3, z+1/3]$ is common C^∞ regular values of all ξ and $\xi|_{Y_j}$, $j \neq 0$, and $\xi(Y_0) \cap (\mathbf{Z} + [-1/3, 1/3]) = \emptyset$. Set

$$Y_j' = Y_j - \xi^{-1}(\mathbf{Z}) \quad \text{and} \quad Y_j'' = Y_{j+1} \cap \xi^{-1}(\mathbf{Z}).$$

Clearly $\{Y_j', Y_j''\}$ is a C^∞ stratification of Y such that the connected components of the strata are bounded in $\mathbf{R}^n$ and each Y_j is the union of Y_j' and Y_{j-1}''. We want to construct a strongly controlled tube system $\{T_j' = (|T_j'|, \pi_j', \rho_j'),\ T_j'' = (|T_j''|, \pi_j'', \rho_j'')\}$ for $\{Y_j', Y_j''\}$.

Set

$$|T_j'| = |T_j| - \xi^{-1}(\mathbf{Z}), \quad \rho_j' = \rho_j \quad \text{on} \quad |T_j'| \quad \text{and}$$
$$|T_j''| = |T_{j+1}| \cap \xi^{-1}(\mathbf{Z} +]-1/3, 1/3[).$$

Let ξ' be a C^∞ function on $\mathbf{R}$ such that

$$\xi'(x) = (x-z)^2 \quad \text{on} \quad [z-1/3, z+1/3] \quad \text{for each } z \in \mathbf{Z}.$$

Set

$$\rho_j'' = \rho_{j+1} + \xi' \circ \xi \quad \text{on} \quad |T_j''|.$$

For the moment, set $\pi_j' = \pi_j$, which we need to modify.

We want to define π''_j first on $Y_{j+1} \cap |T''_j|$ so that for $j < j'$,

$$\left.\begin{aligned} \pi''_j \circ \pi_{j+1} &= \pi_{j+1} \circ \pi''_{j'} \\ \rho''_j \circ \pi''_{j'} &= \rho''_j \end{aligned}\right\} \quad \text{on} \quad Y_{j'+1} \cap |T''_j|.$$

Shrink $|T''_j|$ sufficiently. Assume that there exist a vector field $\{v_{j+1}\}$ on $\{Y_{j+1} \cap |T''_j|\}$ such that $v_{j+1}\xi = 1$, and for $j < j'$,

$$\text{cv}'(j+1, j'+1) \qquad \left.\begin{aligned} d\pi_{j+1} v_{j'+1y} &= v_{j+1\pi_{j+1}(y)} \\ d\rho''_j v_{j'+1y} &= 0 \end{aligned}\right\} \quad \text{for} \quad y \in Y_{j'+1} \cap |T''_j|.$$

Define π''_j on $Y_{j+1} \cap |T''_j|$ so that $\{\pi''^{-1}_j(y)\}_{y \in Y''_j}$ is the integral curves of v_{j+1}. Then π''_j satisfies the required properties. Extend π''_j to $|T''_j|$ by setting $\pi''_j = \pi''_j \circ \pi_{j+1}$. Then it is easy to see that $\{T''_j\}$ is a strongly controlled tube system for $\{Y''_j\}$, and for $j < j'$, the former equality of $\text{ct}(T''_j, T'_{j'})$ and (sc) for $(\pi''_j, \rho''_j)|_{Y_{j'} \cap |T''_j|}$ hold.

We now construct v_j. Since $\xi|_{Y_1}$ is C^∞ regular at $Y_1 \cap \xi^{-1}(\mathbf{Z})$, there clearly exists v_1. Assume that we have already constructed v_j for all $j < k$. It suffices to construct v_k. Moreover, consider the following downward induction. Let $l < k$ be a nonnegative integer. Assume we have defined v_k on $Y_k \cap |T''_{k-1}| \cap (\cup_{l<j<k-1} |T''_j|)$ so that $\text{cv}'(j+1, k)$ hold on $Y_k \cap |T''_{k-1}| \cap |T''_j|$ for all j with $l < j < k-1$. Then it suffices to extend v_k to $Y_k \cap |T''_{k-1}| \cap |T''_l|$ so that $\text{cv}'(l+1, k)$ holds on $Y_k \cap |T''_{k-1}| \cap |T''_l|$, because we easily extend v_k defined on $Y_k \cap |T''_{k-1}| \cap (\cup_{j<k-1} |T''_j|)$ to $Y_k \cap |T''_{k-1}|$ by using a C^∞ partition of unity.

Note that $\text{cv}'(l+1, k)$ for v_k holds on $Y_k \cap |T''_{k-1}| \cap |T''_l| \cap (\cup_{l<j<k-1} |T''_j|)$. Indeed, the former equality follows from $\text{ct}(T_{l+1}, T_{j+1})$, $\text{cv}'(j+1, k)$ and $\text{cv}'(l+1, j+1)$, and we have

$$\begin{aligned} &d\rho''_l v_{ky} = d\rho_{l+1} v_{ky} + d(\xi' \circ \xi) v_{ky} \\ =&d\rho_{l+1} \circ d\pi_{j+1} v_{ky} + d(\xi' \circ \xi) \circ d\pi_{j+1} v_{ky} \\ =&d\rho''_l \circ d\pi_{j+1} v_{ky} = d\rho''_l v_{j+1\pi_{j+1}(y)} = 0 \\ &\qquad\qquad \text{for} \quad y \in Y_k \cap |T''_{k-1}| \cap |T''_l| \cap |T''_j|,\ l < j < k-1. \end{aligned}$$

Forget T''_j, $l < j < k-1$, and consider only T''_l. For sufficiently small $|T''_{k-1}|$, the map $(\pi_{l+1}, \rho''_l)|_{Y_k \cap |T''_{k-1}| \cap |T''_l|}$ is a C^∞ submersion into $Y_{l+1} \times \mathbf{R}$. Hence we have a C^∞ vector field v_{kl} on $Y_k \cap |T''_{k-1}| \cap |T''_l|$ such that $v_{kl}\xi = 1$ and $\text{cv}'(l+1, k)$ holds. Consequently, pasting v_k and v_{kl} by a partition of unity, we can extend v_k to $Y_k \cap |T''_{k-1}| \cap |T''_l|$. To be precise, let θ be a C^∞ function on Y_k such that $0 \le \theta \le 1$, $\theta = 1$ outside $Y_k \cap$(a sufficiently small neighborhood of $\cup_{l<j<k-1} Y''_j$ in $\mathbf{R}^n$) and $\theta = 0$ on $Y_k \cap$(a smaller one). Shrink $|T''_j|$, $l \le j < k-1$. Define v_k to be $\theta v_{kl} + (1-\theta) v_k$ on $Y_k \cap |T''_{k-1}| \cap |T''_l| \cap (\cup_{l<j<k-1} |T''_j|)$, v_{kl} on $Y_k \cap |T''_{k-1}| \cap |T''_l| - (\cup_{l<j<k-1} |T''_j|)$ and v_k on $Y_k \cap |T''_{k-1}| \cap (\cup_{l<j<k-1} |T''_j|) - |T''_l|$. Then v_k satisfies the required conditions. Thus we obtain $\{T''_j\}$.

It is easy to see that for $j < j'$, $\mathrm{ct}(T'_j, T'_{j'})$, the former equality of $\mathrm{ct}(T''_j, T'_{j'})$, (sc) and the conditions of a tube system hold. If $j+1 < j'$, then the latter of $\mathrm{ct}(T''_j, T'_{j'})$ also holds because

$$\begin{aligned}\rho''_j \circ \pi'_{j'} &= \rho''_j \circ \pi_{j'} = \rho_{j+1} \circ \pi_{j'} + \xi' \circ \xi \circ \pi_{j'} \\ &= \rho_{j+1} + \xi' \circ \xi = \rho''_j \quad \text{on} \quad |T''_j| \cap |T'_{j'}|.\end{aligned}$$

But the latter of $\mathrm{ct}(T''_j, T'_{j+1})$ is not correct. (We need not consider $\mathrm{ct}(T'_j, T''_{j'})$ because we can choose $|T'_j|$ and $|T''_{j'}|$ so that they do not intersect.) We modify π'_{j+1} so that this holds as follows.

Shrinking $|T_{j+1}|$ we assume $\rho_{j+1} \le 1$. Let $V_1 \subset V_2$ be small open neighborhoods of $Y''_j \times \mathbf{Z} \times 0$ in $Y''_j \times \mathbf{R} \times [0,1]$ such that $\overline{V_1} \subset V_2$, and the image of $\overline{V_2}$ under the projection $Y''_j \times \mathbf{R} \times [0,1] \to Y''_j \times \mathbf{R}$ is contained in $(\pi''_j, \xi)(Y_{j+1} \cap |T''_j|)$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a C^∞ diffeomorphism of $Y''_j \times (\mathbf{R} - \mathbf{Z}) \times [0,1]$ such that

$$\alpha = \mathrm{id} \quad \text{on} \quad Y''_j \times (\mathbf{R} - \mathbf{Z}) \times 0,$$
$$\alpha_1(y,s,t) = y, \quad \alpha_3(y,s,t) = t, \quad \text{and}$$
$$\alpha_2(y,s,t) = \begin{cases} \pm((s-z)^2 + t)^{1/2} + z \\ \qquad \text{on} \quad V_1 \cap (Y''_j \times ([z-1/3, z+1/3] - z) \times [0,1]), \quad z \in \mathbf{Z} \\ s \qquad \text{outside} \quad V_2, \end{cases}$$

whose existence is easily shown if V_1 is sufficiently small.

Modify π'_{j+1} to be

$$((\pi''_j, \xi)|_{Y_{j+1} \cap |T''_j|})^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi''_j, \xi, \rho_{j+1}) \quad \text{on} \quad |T'_{j+1}| \cap |T''_j|,$$

and do not change π'_{j+1} on $|T'_{j+1}| - |T''_j|$. Then it is clear that $\{T'_j\}$ is a tube system and (sc) is satisfied. Note that π'_{j+1} does not change outside $(\pi''_j, \xi, \rho_{j+1})^{-1}(V_2)$. Hence $\mathrm{ct}(T'_{j'+1}, T'_{j+1})$ can hold for any $j' < j$ because we can choose small V_2 and shrink $|T'_{j'+1}|$ so that $(\pi''_j, \xi, \rho_{j+1})^{-1}(V_2)$ and $|T'_{j'+1}|$ do not intersect.

Moreover, we have

$$\begin{aligned}\pi''_j \circ \pi'_{j+1} &= \pi''_j \circ ((\pi''_j, \xi)|_{Y_{j+1} \cap |T''_j|})^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi''_j, \xi, \rho_{j+1}) \\ &= \alpha_1 \circ (\pi''_j, \xi, \rho_{j+1}) = \pi''_j \\ &\text{on} \quad (\pi''_j, \xi, \rho_{j+1})^{-1}(V_1) \cap \xi^{-1}([z-1/3, z+1/3] - z), \quad z \in \mathbf{Z},\end{aligned}$$

and

$$\begin{aligned}\rho''_j \circ \pi'_{j+1} &= \rho_j \circ ((\pi''_j, \xi)|_{Y_{j+1} \cap |T''_j|})^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi''_j, \xi, \rho_{j+1}) \\ &\quad + \xi' \circ \xi \circ ((\pi''_j, \xi)|_{Y_{j+1} \cap |T''_j|})^{-1} \circ (\alpha_1, \alpha_2) \circ (\pi''_j, \xi, \rho_{j+1}) \\ &= 0 + \xi' \circ \alpha_2 \circ (\pi''_j, \xi, \rho_{j+1}) = (\xi - z)^2 + \rho_{j+1} = \rho''_j \quad \text{on the same domain.}\end{aligned}$$

Therefore, if we shrink $|T''_j|$, $\mathrm{ct}(T''_j, T'_{j+1})$ holds.

If $j' < j$, $\mathrm{ct}(T''_{j'}, T'_{j+1})$ continues to hold. Indeed, this is clear on $|T''_{j'}| \cap |T'_{j+1}| - (\pi''_j, \xi, \rho_{j+1})^{-1}(V_2)$. Shrink $|T'_{j+1}|$ and $|T''_j|$ so that $|T'_{j+1}| \cap |T''_j| \subset (\pi''_j, \xi, \rho_{j+1})^{-1}(V_1)$. Then, on $|T''_{j'}| \cap |T'_{j+1}| \cap (\pi''_j, \xi, \rho_{j+1})^{-1}(V_2)$, we have

$$\begin{aligned}(\pi''_{j'}, \rho''_{j'}) \circ \pi'_{j+1} &= ((\pi''_{j'}, \rho''_{j'}) \circ \pi''_j) \circ \pi'_{j+1} \\ &= (\pi''_{j'}, \rho''_{j'}) \circ (\pi''_j \circ \pi'_{j+1}) = (\pi''_{j'}, \rho''_{j'}) \circ \pi''_j = (\pi''_{j'}, \rho''_{j'}).\end{aligned}$$

Thus a strongly controlled tube system $\{T'_j, T''_j\}$ is constructed.

From now on we remove the assumption $\dim Y_j = j$, and we change the definition of Y''_j for

$$Y''_j = Y_j \cap \xi^{-1}(\mathbf{Z}).$$

In the same way as above, set

$$X'_{i,j} = X_{i,j} - (\xi \circ f)^{-1}(\mathbf{Z}) \quad \text{and} \quad X''_{i,j} = X_{i,j} \cap (\xi \circ f)^{-1}(\mathbf{Z}).$$

We want to define a tube system $\{T'_{i,j} = (|T'_{i,j}|, \pi'_{i,j}, \rho'_{i,j}), T''_{i,j} = (|T''_{i,j}|, \pi''_{i,j}, \rho''_{i,j})\}$ for $\{X'_{i,j}, X''_{i,j}\}$ strongly controlled over $\{T'_j, T''_j\}$. Let $\tilde{f}$ denote the extension of f in condition (sc1) of strong controlledness.

Set

$$|T'_{i,j}| = |T_{i,j}| - (\xi \circ \tilde{f})^{-1}(\mathbf{Z}), \quad |T''_{i,j}| = |T_{i,j}| \cap (\xi \circ \tilde{f})^{-1}(\mathbf{Z} +]-1/3, 1/3[),$$

$$\left.\begin{aligned} \pi'_{i,j} &= \pi_{i,j} \\ \rho'_{i,j} &= \rho_{i,j} \end{aligned}\right\} \quad \text{on} \quad |T'_{i,j}|, \quad \text{and}$$

$$\rho''_{i,j} = \rho_{i,j} + \xi' \circ \xi \circ \tilde{f} \quad \text{on} \quad |T''_{i,j}|.$$

The definition of $\pi''_{i,j}$ is similar to that of π''_j as follows. Shrink $|T''_{i,j}|$ sufficiently. Then there exist C^∞ imbeddings

$$\theta_{i,j} \colon X_{i,j} \cap |T''_{i,j}| \longrightarrow X''_{i,j} \times \mathbf{R}$$

of the form $(\theta^*_{i,j}, \xi \circ f)$ such that

$$\begin{aligned} \theta^*_{i,j} &= \mathrm{id} \quad \text{on} \quad X''_{i,j}, \\ f \circ \theta^*_{i,j} &= \pi''_j \circ f \quad \text{on} \quad X_{i,j} \cap |T''_{i,j}|, \\ \pi_{i,j} \circ \theta^*_{i',j'} &= \theta^*_{i,j} \circ \pi_{i,j} \quad \text{on} \quad X_{i',j'} \cap |T''_{i',j'}| \cap |T''_{i,j}|, \quad \text{and} \\ \rho''_{i,j} \circ \theta^*_{i',j'} &= \rho''_{i,j} \quad \text{on the same domain if } j = j'. \end{aligned}$$

Set $\pi''_{i,j} = \theta^*_{i,j}$ on $X_{i,j} \cap |T''_{i,j}|$, and extend it to $|T''_{i,j}|$ by setting $\pi''_{i,j} = \pi''_{i,j} \circ \pi_{i,j}$.

The tube system $\{T'_{i,j}, T''_{i,j}\}$ satisfies the required conditions except that

$$f \circ \pi'_{i,j} = \pi'_j \circ \tilde{f} \quad \text{on} \quad |T'_{i,j}|.$$

But we can modify $\pi'_{i,j}$ so that this equality holds in the same way that we did π'_j. We omit the details. Thus we prove the lemma. □

Lemma 3 (I.3.2 in [G-al] and its proof). *Let X, Y, $\{X_{i,j}\}$, $\{Y_j\}$, $f\colon X \to Y$, $\{T_{i,j}\}$ and $\{T_j\}$ be the same as in the theorem except for the assumption that f is proper. Assume* $\dim Y_1 < \dim Y_j$ *for* $j \neq 1$.

Given a C^∞ vector field v_1 on Y_1, there exists a controlled vector field on $\{Y_j\}$ which is an extension of v_1.

Given a weakly controlled vector field $v^Y = \{v_j\}$ on $\{Y_j\}$ and a vector field $\{v_{i,1}\}_i$ on $\{X_{i,1}\}_i$ controlled over $\{v_1\}$, there exists a vector field on $\{X_{i,j}\}_{i,j}$ which is an extension of $\{v_{i,1}\}_i$ and controlled over v^Y.

[G-al] treats only Thom maps. But the proof works in our situation. See [S].

Lemma 4 (I.4.6 in [G-al]). *In the same situation as in Lemma 3, a controlled vector field on $\{Y_j\}$ and a vector field on $\{X_{i,j}\}$ controlled over a locally integrable vector field on $\{Y_j\}$ are locally integrable.*

§4. PROOF OF THE THEOREM

Proof of the theorem. Assume $\dim Y_j < \dim Y_{j+1}$ and $\dim X_{i,j} < X_{i+1,j}$. Let the sets of indexes H and $\overline{H}$ and an order in H and $\overline{H}$ be given as in §3. By Lemma 2 we can assume that each connected component of Y_j is bounded in $\mathbf{R}^n$. But, only for simplicity of notations, we assume, moreover, that Y is compact. The following arguments work in the noncompact case. (See a generalization of the definition of a removal data in §3.) Let a removal data $\varepsilon = \{\varepsilon_{i,j}\}_{(i,j)\in H}$ of $\{T_i, T_{i,j}\}_{(i,j)\in\overline{H}}$ be fixed. Set $\varepsilon_{k_j,j} = \varepsilon_j$.

Set

$$Y_j^\varepsilon = Y_j - \bigcup_{l<j} \rho_l^{-1}(]0,\varepsilon_l[), \ \ j = 1,\ldots,k,$$

which are compact C^∞ manifolds possibly with corners. We want C^∞ triangulations (L_j, g_j) of Y_j^ε such that for $j < j'$, the restriction of $g_j^{-1} \circ \pi_j \circ g_{j'}$ to a neighborhood of $g_{j'}^{-1}(\rho_j^{-1}(\varepsilon_j))$ in $|L_{j'}|$ is a PL map to $|L_j|$. We call the property PL(j,j'). (Proposition I.3.20 in [S] shows the existence. But we repeat the proof because we shall use the idea.)

We construct the triangulations by induction. If we apply Lemma 1 to the constant map $Y_1^\varepsilon \to 0$, existence of (L_1, g_1) follows. Let $1 \le l_1 < l_2 \le k$ be integers. Assume we have constructed (L_j, g_j) for all j with $j < l_2$ and a C^∞ triangulation (L_{l_2}, g_{l_2}) of a neighborhood of $Y_{l_2}^\varepsilon \cap (\cup_{l_1<j<l_2}\rho_j^{-1}(\varepsilon_j))$ in $Y_{l_2}^\varepsilon$ with PL(j,l_2) for all j with $l_1 < j < l_2$. Then shrinking the neighborhood we need to extend (L_{l_2}, g_{l_2}) to a C^∞ triangulation of a neighborhood of $Y_{l_2}^\varepsilon \cap (\cup_{l_1\le j<l_2}\rho_j^{-1}(\varepsilon_j))$ with PL(l_1,l_2). Let $l_1 < j < l_2$. By PL(l_1,j), PL(j,l_2) and ct(T_{l_1},T_j), the restriction of $g_{l_1}^{-1}\circ\pi_{l_1}\circ g_{l_2}$ to a neighborhood of $g_{l_2}^{-1}(\rho_{l_1}^{-1}(\varepsilon_{l_1})\cap\rho_j^{-1}(\varepsilon_j))$ in $|L_{l_2}|$ is a PL map to $|L_{l_1}|$. Note that $Y_{l_2}^\varepsilon\cap\rho_{l_1}^{-1}(\varepsilon_{l_1})$ is a disjoint union of faces of $Y_{l_2}^\varepsilon$, and $Y_{l_2}^\varepsilon\cap\rho_{l_1}^{-1}(\varepsilon_{l_1})\cap(\cup_{l_1<j<l_2}\rho_j^{-1}(\varepsilon_j))$ is a union of subfaces of $Y_{l_2}^\varepsilon\cap\rho_{l_1}^{-1}(\varepsilon_{l_1})$. Hence by Lemma 1 we can extend (L_{l_2}, g_{l_2}) as required. Thus we have a C^∞ triangulation (L_{l_2}, g_{l_2}) of a neighborhood of $\partial Y_{l_2}^\varepsilon$ in $Y_{l_2}^\varepsilon$ with PL(j,l_2) for all $j < l_2$. A further extension to whole $Y_{l_2}^\varepsilon$ follows from Lemma 1 applied to the map $Y_{l_2}^\varepsilon \to 0$. Therefore, there exist (L_j, g_j), $j = 1,\ldots,k$.

Note that for $1 \le j < j' \le k$, $g_{j'}^{-1}(\rho_j^{-1}(\varepsilon_j))$ is the underlying polyhedron of a subcomplex of $L_{j'}$. For a simplicial complex K, let K' and $\hat{K}$ denote the barycentric and some subdivisions of K respectively.

Set

$$Y_j^+ = Y - \bigcup_{l<j} \rho_l^{-1}([0,\varepsilon_l[), \ \ j = 1,\ldots,k.$$

Note that

$$Y_1^+ = Y, \quad Y_k^+ = Y_k^\varepsilon \quad \text{and} \quad Y_j^+ = Y_{j+1}^+ \cup (Y_j^+ \cap \rho_j^{-1}([0,\varepsilon_j])), \ \ j = 1,\ldots,k-1.$$

We want to construct (not necessarily C^∞) triangulations (L_j^+, g_j^+) of Y_j^+ such that for $1 \le j < j' \le k$, $g_{j'}^{-1}(\rho_j^{-1}(\varepsilon_j))$ is the underlying polyhedron of some subcomplex

$L^+_{j'}(j)$ of $L^+_{j'}$, the map $\alpha^+_{j'}(j)\colon |L^+_{j'}(j)| \to |L_j|$ is PL,

$$L^+_j = (\widehat{L^+_{j+1}})' \cup C_{\alpha^+_{j+1}(j)}(\widehat{L^+_{j+1}}(j), \hat{L}_j),$$
$$\widehat{L^+_{j+1}}(j)' = (\widehat{L^+_{j+1}})' \cap C_{\alpha^+_{j+1}(j)}(\widehat{L^+_{j+1}}(j), \hat{L}_j),$$
$$g^+_j|_{|L^+_{j+1}|} = g^+_{j+1} \quad \text{and} \quad g^+_j|_{|L_j|} = g_j,$$

where
$$\alpha^+_{j'}(j) = g_j^{-1} \circ \pi_j \circ (g^+_{j'}|_{|L^+_{j'}(j)|}).$$

(This is shown in the proof of Corollary I.3.21 in [S]. We shall need the same procedure.)

We define (L^+_j, g^+_j) by downward induction on j. Clearly we set $L^+_k = L'_k$ and $g^+_k = g_k$. Let $1 \le j < k$ be an integer, and assume (L^+_{j+1}, g^+_{j+1}). Set

$$g^+_j = \begin{cases} g^+_{j+1} & \text{on} \quad |L^+_{j+1}| \\ g_j & \text{on} \quad |L_j|. \end{cases}$$

We need to subdivide L^+_{j+1} and L_j so that $\alpha^+_{j+1}(j)\colon \widehat{L^+_{j+1}}(j) \to \hat{L}_j$ is a simplicial map and then to extend g^+_j to $C_{\alpha^+_{j+1}(j)}(|L^+_{j+1}(j)|, |L_j|)$. The former requirement is clearly fulfilled since $\alpha^+_{j+1}(j)$ is PL. For the latter it suffices to find a homeomorphism $\theta_j\colon Y^+_j \cap \rho_j^{-1}(]0, \varepsilon_j]) \to (Y^+_j \cap \rho_j^{-1}(\varepsilon_j)) \times]0, \varepsilon_j]$ of the form (θ^*_j, ρ_j) such that $\pi_j \circ \theta^*_j = \pi_j$ and $\theta^*_j = \mathrm{id}$ on $Y^+_j \cap \rho_j^{-1}(\varepsilon_j)$. Indeed, by such θ_j we can identify $Y^+_j \cap \rho_j^{-1}([0, \varepsilon_j])$ with $C_{\pi_j|_{Y^+_j \cap \rho_j^{-1}(\varepsilon_j)}}(Y^+_j \cap \rho_j^{-1}(\varepsilon_j), Y^\varepsilon_j)$, and we can naturally extend g^+_j to $C_{\alpha^+_{j+1}(j)}(|L^+_{j+1}(j)|, |L_j|)$. It is clear by $\mathrm{ct}(T_{j'}, T_j)$, $\mathrm{PL}(j', j)$ for $j' < j$ and by the properties of a mapping cylinder that $(L^+_j, \text{the extension})$ satisfies all the requirements.

Existence of θ_j immediately follows if we apply Thom's Second Isotopy Lemma to the sequence of maps $Y^+_j \cap \rho_j^{-1}(]0, \varepsilon_j]) \overset{(\pi_j, \rho_j)}{\longrightarrow} \pi_j(Y^+_j \cap \rho_j^{-1}(\varepsilon_j)) \times]0, \varepsilon_j] \overset{\mathrm{proj}}{\longrightarrow}]0, \varepsilon_j]$. (Note that $\pi_j(Y^+_j \cap \rho_j^{-1}(\varepsilon_j))$ does not necessarily coincide with Y^ε_j. We will show a more precise construction of θ_j later because we need another additional property.) Thus we have the required (L^+_j, g^+_j).

Set

$$X^\varepsilon_{i,j} = X_{i,j} - \bigcup_{j'<j} (\rho_{j'} \circ f)^{-1}(]0, \varepsilon_{j'}[) - \bigcup_{i'<i} \rho^{-1}_{i',j}(]0, \varepsilon_{i',j}[) \quad \text{for} \quad (i,j) \in \overline{H},$$

which also are compact C^∞ manifolds possibly with corners. We will construct C^∞ triangulations $(K_{i,j}, h_{i,j})$ of $X^\varepsilon_{i,j}$ with the following three properties. (1) For $(i,j) \in \overline{H}$, the map $g_j^{-1} \circ f \circ h_{i,j}\colon |K_{i,j}| \to |L_j|$ is PL. Let $(i_1, j_1) < (i_2, j_2) \in \overline{H}$. (2) If $j_1 < j_2$, the restriction of $h^{-1}_{i_1,j_1} \circ \pi_{i_1,j_1} \circ h_{i_2,j_2}$ to a neighborhood of $h^{-1}_{i_2,j_2}((\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}) \cap \rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}]) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}))$ in $h^{-1}_{i_2,j_2}(\rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}]) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}))$ is a PL map to $|K_{i_1,j_1}|$. (Here we ignore $\rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}])$ if $i_1 = k_{j_1}$.) (3) If $j_1 = j_2$, the restriction of $h^{-1}_{i_1,j_1} \circ \pi_{i_1,j_1} \circ h_{i_2,j_2}$ to a neighborhood of $h^{-1}_{i_2,j_2}(\rho^{-1}_{i_1,j_1}(\varepsilon_{i_1,j_1}))$ in $|K_{i_2,j_2}|$ is a PL map to $|K_{i_1,j_1}|$.

As in the case of Y^ε_j, we construct them by induction. Existence of $(K_{1,1}, h_{1,1})$ with (1) is clear by Lemma 1. Let $(i_1, j_1) < (i_2, j_2) \in \overline{H}$. Assume we have $(K_{i,j}, h_{i,j})$ for all $(i, j) < (i_2, j_2)$ and a C^∞ triangulation $(K_{i_2,j_2}, h_{i_2,j_2})$ of the following set with property (1) for (i_2, j_2), (2) for any pair $(i', j') < (i_2, j_2)$ with $(i', j') > (i_1, j_1)$ and (3) for any pair $(i', j_2) < (i_2, j_2)$ with $(i', j_2) > (i_1, j_1)$:

$$\bigcup_{(i,j)>(i_1,j_1),\, j<j_2} (\text{a neighborhood of } X^\varepsilon_{i_2,j_2} \cap (\rho_j \circ f)^{-1}(\varepsilon_j) \cap \pi^{-1}_{i,j}(X^\varepsilon_{i,j}) \quad \text{in } X^\varepsilon_{i_2,j_2} \cap \pi^{-1}_{i,j}(X^\varepsilon_{i,j}))$$

$$\cup \bigcup_{(i_1,j_1)<(i',j_2)<(i_2,j_2)} (\text{a neighborhood of } X^\varepsilon_{i_2,j_2} \cap \rho^{-1}_{i',j_2}(\varepsilon_{i',j_2}) \text{ in } X^\varepsilon_{i_2,j_2}).$$

We call such $(K_{i_2,j_2}, h_{i_2,j_2})$ a C^∞ triangulation of $R(i_2, j_2, i'_1, j'_1)$, where (i'_1, j'_1) denotes the minimum of the elements of $\overline{H}$ greater than (i_1, j_1). We extend $(K_{i_2,j_2}, h_{i_2,j_2})$ to a C^∞ triangulation of $R(i_2, j_2, i_1, j_1)$. Let $\varepsilon'_{j_1} > \varepsilon_{j_1}$ be a number sufficiently close to ε_{j_1}.

There are four possible cases: (i) $j_1 = j_2$, (ii) $j_1 < j_2$ and $i_1 = k_{j_1}$, (iii) $j_1 < j_2$, $i_1 < k_{j_1}$ and $i_2 = 1$ or (iv) $j_1 < j_2$, $i_1 < k_{j_1}$ and $i_2 > 1$. In case (i), the arguments on the extension are the same as in the case of Y^ε_j, because we do not need consider (2) and because (1) follows from (1) for (i_1, j_1) and (3).

Assume (ii). We easily see the following three facts. First the fibre product $|K_{i_1,j_1}| \times_{(f\circ h_{i_1,j_1}, \pi_{j_1}\circ g_{j_2})} g^{-1}_{j_2}(\rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[))$ is a polyhedron. (We treat not $g^{-1}_{j_2}(\rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}]))$ but $g^{-1}_{j_2}(\rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[))$, because $g^{-1}_{j_2}(\rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}]))$ is not always a polyhedron. But $g^{-1}_{j_2}(\rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[))$ is non-compact and hence does not admit a finite simplicial decomposition.) Second, the restriction of the map (h_{i_1,j_1}, g_{j_2}) to some simplicial complex whose underlying polyhedron is this polyhedron is a C^∞ triangulation of the fibre product $X^\varepsilon_{i_1,j_1} \times_{(f,\pi_{j_1})} (Y^\varepsilon_{j_2} \cap \rho^{-1}_{j_1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[))$, which is a C^∞ manifold possibly with corners. Third, the restriction of (π_{i_1,j_1}, f) to $X^\varepsilon_{i_2,j_2} \cap (\rho_{j_1} \circ f)^{-1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1})$ is a C^∞ submersion onto a union of some connected components of the preceding manifold possibly with corners and, moreover, satisfies the conditions in Lemma 1. (Lemma 1 treats only compact sets, and the present sets are not compact. But the problem is only around the compact set $X^\varepsilon_{i_2,j_2} \cap (\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1})$. Hence Lemma 1 is applicable.) Therefore, an extension of $(K_{i_2,j_2}, h_{i_2,j_2})$ to a C^∞ triangulation of $R(i_2, j_2, i_1, j_1)$ is possible.

Assume (iii) or (iv). In these cases, the preceding arguments do not work. Indeed, the given $(K_{i_2,j_2}, h_{i_2,j_2})$ defines only a C^∞ triangulation of a neighborhood of $X^\varepsilon_{i_2,j_2} \cap (\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}) \cap \rho^{-1}_{i_1,j_1}(\varepsilon_{i_1,j_1})$ in $X^\varepsilon_{i_2,j_2} \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}) \cap \rho^{-1}_{i_1,j_1}(\varepsilon_{i_1,j_1})$, but for application of Lemma 1 in the preceding way, what is necessary is a C^∞ triangulation of a neighborhood of the same set in $X^\varepsilon_{i_2,j_2} \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}) \cap \rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}])$. Hence we need such an extension of the C^∞ triangulation.

To be precise, set

$$M = X^\varepsilon_{i_2,j_2} \cap (\rho_{j_1} \circ f)^{-1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}) \cap \rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}]),$$

which is a C^∞ manifold possibly with corners. Then we have

$$\partial M = A \cup B \cup C \cup D,$$

where $A = M \cap (\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}), \quad B = M \cap (\bigcup_{i<i_2} \rho_{i,j_2}^{-1}(\varepsilon_{i,j_2})),$

$$C = M \cap \rho_{i_1,j_1}^{-1}(\varepsilon_{i_1,j_1}) \quad \text{and} \quad D = M \cap (\bigcup_{i<i_1} \rho_{i,j_1}^{-1}(\varepsilon_{i,j_1})),$$

$h_{i_2,j_2}^{-1}(M)$ is the intersection of the open neighborhood $h_{i_2,j_2}^{-1}((\rho_{j_1} \circ f)^{-1}([\varepsilon_{j_1}, \varepsilon'_{j_1}[))$ of $h_{i_2,j_2}^{-1}(A)$ in $|K_{i_2,j_2}|$ and the closed polyhedron $h_{i_2,j_2}^{-1}(\pi_{i_1,j_1}^{-1}(X_{i_1,j_1}^{\varepsilon}) \cap \rho_{i_1,j_1}^{-1}(]0, \varepsilon_{i_1,j_1}]))$, and $M \cap \operatorname{Im} h_{i_2,j_2}$ is the union of C and a closed neighborhood U of B in M. Hence $(K_{i_2,j_2}, h_{i_2,j_2})$ induces a C^∞ triangulation, say, (K, h) for simplicity of notation, of $U \cup C$, which equals $(K_{i_2,j_2}, h_{i_2,j_2})$ around $h_{i_2,j_2}^{-1}(A)$. Shrinking U, we need to extend (K, h) to a C^∞ triangulation of $U \cup$ (a neighborhood of $A \cap C$ in M).

Assume (iii). Then $B = \emptyset$. Hence the extension follows from the following note, which is clear by condition (6) of a removal data of $\{T_{i,j}\}$.

Note: There exists a C^∞ diffeomorphism $\theta\colon M \cap \rho_{i_1,j_1}^{-1}([\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}]) \to C \times [\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}]$ of the form $(\theta^*, \rho_{i_1,j_1})$ with $\pi_{i_1,j_1} \circ \theta^* = \pi_{i_1,j_1}$ and $f \circ \theta^* = f$.

Case (iv) remains. The situation is more complicated. The note is not sufficient. Indeed, (K, h) would change if we used only the note, since $B \neq \emptyset$. Given a subset E of M such that $h^{-1}(E)$ is the underlying polyhedron of some subcomplex of K, let K_E denote the subcomplex by abuse of notation. We can assume that the closure of the interior U° of U as a subset of M coincides with U, and $|N(K_B, K)|$ does not intersect with the boundary of $|K_U|$ as a subset of $|K|$. Let $a > 1$ be a number close to 1. Let β be the simplicial function on K defined by $\beta = a$ at the vertices $|K_A^0 \cap K_C^0| - h^{-1}(U^\circ)$ and $\beta = 1$ at any other vertex. Clearly $\beta = 1$ on $|N(K_B, K)|$, and the polyhedron $\cup_{u \in |K_C|} u \times [1, \beta(u)]$ has a natural cell complex structure. Paste the barycentric subdivision of this cell complex with K' by the identification of $|K_C| \times 1$ with $|K_C|$ in $|K|$. Let $\tilde{K}$ denote this simplicial complex.

We want to define a C^∞ imbedding $\tilde{h}\colon \tilde{K} \to M$ so that $(\tilde{K}, \tilde{h})$ is the required C^∞ triangulation. By θ in the note in case (iii), we can regard (M, C) as $(C \times]\varepsilon_{i_1,j_1}/2, \varepsilon_{i_1,j_1}], C \times \varepsilon_{i_1,j_1})$, because the problem is only local around$C$. We call the latter pair $(C \times]0, 1], C \times 1)$ for simplicity of notation. Let h be of the form (h_1, h_2), where $h_1\colon |K| \to C$ and $h_2\colon |K| \to]0, 1]$. Set

$$\tilde{h} = \begin{cases} (h_1, (2-\beta)h_2) & \text{on } |K| \\ (h_1(u), t + 1 - \beta(u)) & \text{for } u \in |K_C| \text{ and } t \in [1, \beta(u)]. \end{cases}$$

Note that $\tilde{h} = h$ on $|N(K_B, K)|$. Let a be sufficiently close to 1. Then $\tilde{h}|_{K'}$ is a strong approximation of h. Hence by 8.8 in [M], $\tilde{h}|_{K'}$ is a C^∞ imbedding. On the other hand, by the above definition of $\tilde{h}$, $\tilde{h}$ outside K' also is a C^∞ imbedding. Moreover, it is clear that $\tilde{h}$ is a C^∞ triangulation of a neighborhood of $B \cup (A \cap C)$ in M.

In both cases of (iii) and (iv), we can extend $(K_{i_2,j_2}, h_{i_2,j_2})$ to a C^∞ triangulation of $R(i_2, j_2, i_1, j_1)$ in the same way as in case of (ii). That completes the induction step. Thus by induction we have a C^∞ triangulation $(K_{i_2,j_2}, h_{i_2,j_2})$ of a neighborhood of $\partial X^\varepsilon_{i_2,j_2}$ in $X^\varepsilon_{i_2,j_2}$. Its further extension to a C^∞ triangulation of X_{i_2,j_2} with (1) follows if we apply Lemma 1 to the map $f|_{X^\varepsilon_{i_2,j_2}}: X^\varepsilon_{i_2,j_2} \to Y^\varepsilon_{j_2}$.

As in the case of Y_j, note the following property. Let $(i_1, j_1) < (i_2, j_2) \in \overline{H}$. The following set is the underlying polyhedron of some subcomplex of K_{i_2,j_2}:

$$h^{-1}_{i_2,j_2}(\rho^{-1}_{i_1,j_1}(\varepsilon_{i_1,j_1})) \quad \text{if} \quad j_1 = j_2,$$
$$h^{-1}_{i_2,j_2}((\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1})) \quad \text{if} \quad i_1 = k_{j_1} \text{ and } j_1 < j_2, \quad \text{and}$$
$$h^{-1}_{i_2,j_2}((\rho_{j_1} \circ f)^{-1}(\varepsilon_{j_1}) \cap \pi^{-1}_{i_1,j_1}(X^\varepsilon_{i_1,j_1}) \cap \rho^{-1}_{i_1,j_1}(]0, \varepsilon_{i_1,j_1}])) \quad \text{otherwise.}$$

For each $(i,j) \in \overline{H}$, set

$$N_{i,j} = \begin{cases} X \cap \pi^{-1}_{i,j}(X^\varepsilon_{i,j}) \cap \rho^{-1}_{i,j}([0, \varepsilon_{i,j}]) & \text{if} \quad j = k \\ X \cap (\rho_j \circ f)^{-1}([0, \varepsilon_j]) \cap \pi^{-1}_{i,j}(X^\varepsilon_{i,j}) & \text{if} \quad i = k_j,\ j < k \\ X \cap (\rho_j \circ f)^{-1}([0, \varepsilon_j]) \cap \pi^{-1}_{i,j}(X^\varepsilon_{i,j}) \cap \rho^{-1}_{i,j}([0, \varepsilon_{i,j}]) & \text{otherwise,} \end{cases}$$
$$N'_{i,j} = N_{i,j} \cap \bigcup_{(i',j') > (i,j)} N_{i',j'} \quad \text{and} \quad X^+_{i,j} = \bigcup_{(i',j') \geq (i,j)} N_{i',j'}.$$

Since $X^+_{1,1} = X$, the theorem follows if we can construct triangulations $(K^+_{i,j}, h^+_{i,j})$ of $X^+_{i,j}$ such that the following three conditions are satisfied. For $(i,j) \in \overline{H}$, $g^{+-1}_j \circ f \circ h^+_{i,j}: |K^+_{i,j}| \to |L^+_j|$ is PL. For $(i_1, j_1) < (i_2, j_2) \in \overline{H}$, $h^{+-1}_{i_2,j_2}(N_{i_1,j_1})$ is the underlying polyhedron of some subcomplex $K^+_{i_2,j_2}(i_1, j_1)$ of $K^+_{i_2,j_2}$, and the map $\alpha^+_{i_2,j_2}(i_1, j_1): |K^+_{i_2,j_2}(i_1, j_1)| \to |K_{i_1,j_1}|$ is PL, where

$$\alpha^+_{i_2,j_2}(i_1, j_1) = h^{-1}_{i_1,j_1} \circ \pi_{i_1,j_1} \circ (h^+_{i_2,j_2}|_{|K^+_{i_2,j_2}(i_1,j_1)|}).$$

For $(i,j) \in H$, let (i', j') denote the minimum of the elements of $\overline{H}$ greater than (i,j). Then

$$K^+_{i,j} = (\widehat{K^+_{i',j'}})' \cup C_{\alpha^+_{i',j'}(i,j)}(\widehat{K^+_{i',j'}}(i,j), \hat{K}_{i,j}),$$
$$\widehat{K^+_{i',j'}}(i,j)' = (\widehat{K^+_{i',j'}})' \cap C_{\alpha^+_{i',j'}(i,j)}(\widehat{K^+_{i',j'}}(i,j), \hat{K}_{i,j}),$$
$$h^+_{i,j}|_{|K^+_{i',j'}|} = h^+_{i',j'} \quad \text{and} \quad h^+_{i,j}|_{|K_{i,j}|} = h_{i,j}.$$

Here $'$ and $\hat{}$ denote the barycentric and some subdivisions respectively.

We construct $(K^+_{i,j}, h^+_{i,j})$ by downward induction as (L^+_j, g^+_j). Then by the same reason, it suffices to find a homeomorphism $\theta_{i,j}: N_{i,j} - X^\varepsilon_{i,j} \to N'_{i,j} \times]0,1]$ of the form $(\theta^*_{i,j}, \theta^{**}_{i,j})$ for each $(i,j) \in H$ such that

(a) $$\theta^*_{i,j} = \text{id} \quad \text{on} \quad N'_{i,j}, \quad \pi_{i,j} = \pi_{i,j} \circ \theta^*_{i,j},$$

(b) $$\rho_j \circ f = \theta^{**}_{i,j} \cdot \rho_j \circ f \circ \theta^*_{i,j} \quad \text{if} \quad j < k, \quad \text{and}$$

(c) $$\theta^*_j \circ f = \theta^*_j \circ f \circ \theta^*_{i,j} \quad \text{on} \quad N_{i,j} - (\rho_j \circ f)^{-1}(0) \quad \text{if} \quad j < k.$$

If $j = k$, $\theta_{i,j}$ is constructed as θ_j. So assume $j < k$. To distinguish elements of $\overline{H}$, we call (i, j) (i_0, j_0) and use the notation (i, j) for a general element. Since the problem is local around N_{i_0,j_0}, we assume

$$|T_{i,j}| \subset |T_{i_0,j_0}| \quad \text{and} \quad |T_j| \subset |T_{j_0}| \quad \text{for all } (i,j) > (i_0, j_0).$$

Set

$$X_{?(i,j)} = \bigcup_{(i',j')?(i,j)} X_{i',j'} \quad \text{and} \quad Y_{?j} = \bigcup_{j'?j} Y_{j'} \quad \text{for} \quad (i,j) \in \overline{H} \quad \text{and} \quad ? \in \{\geq, >\},$$

and let $\otimes Z$ or $\otimes(Z)$ in $\mathbf{R}^n \times \mathbf{R}^n$ denote the fibre product $X_{i_0,j_0} \times_{(f,\pi_{j_0})} Z$ for a subset Z of $Y_{\geq j_0}$. Define naturally a C^∞ map $\otimes f\colon X_{\geq(i_0,j_0)} \to \otimes Y_{\geq j_0}$. Then we can easily construct a strongly controlled tube system $\{\otimes T_j = (|\otimes T_j|, \otimes\pi_j, \otimes\rho_j)\}_{j \geq j_0}$ for $\{\otimes Y_j\}_{j\geq j_0}$ such that for each $j \geq j_0$,

$$\otimes|T_j| \subset |\otimes T_j|,$$
$$\left.\begin{aligned} \otimes\pi_j(x,y) &= (x, \pi_j(y)) \\ \otimes\rho_j(x,y) &= \rho_j(y) \end{aligned}\right\} \quad \text{for} \quad (x,y) \in \otimes|T_j|,$$

and $\{T_{i,j}\}_{(i,j)\geq(i_0,j_0)}$ is strongly controlled over $\{\otimes T_j\}_{j\geq j_0}$. Let $p_X\colon \otimes Y_{\geq j_0} \to X_{i_0,j_0}$ and $p_Y\colon \otimes Y_{\geq j_0} \to Y_{\geq j_0}$ denote the projections.

Let us specify the construction of θ_j^* as in the proof of I.5.8 (Thom's Second Isotopy Lemma) in [G-al]. There exists a controlled vector field $\{v_j\}_{j>j_0}$ on $\{Y_j \cap \rho_{j_0}^{-1}(]0, 2\varepsilon_{j_0}[)\}_{j>j_0}$ such that

$$(*) \qquad d\pi_{j_0} v_j = 0 \quad \text{and} \quad v_j\rho_{j_0} = 1, \;\; j > j_0.$$

(The existence follows if we apply Lemma 3 to the map $(\pi_{j_0}, \rho_{j_0})\colon Y \cap \rho_{j_0}^{-1}(]0, 2\varepsilon_{j_0}[) \to Y_{j_0} \times]0, 2\varepsilon_{j_0}[$.) Then by Lemma 4, $\{v_j\}$ is locally integrable. Hence if we define $\theta_{j_0} = (\theta_{j_0}^*, \rho_{j_0})$ so that for each $y \in Y_{j_0}^+ \cap \rho_{j_0}^{-1}(\varepsilon_{j_0})$,

$$\theta_{j_0}^{*-1}(y) = \rho_{j_0}^{-1}(]0, \varepsilon_{j_0}]) \cap (\text{the integral curve of } \{v_j\} \text{ passing through } y),$$

which is possible by condition (3) of a removal data of $\{T_j\}$, then θ_{j_0} fulfills the requirements.

Multiplying v_j by ρ_{j_0}, we replace the latter equality of $(*)$ with $v_j\rho_{j_0} = \rho_{j_0}$. Let $(*)'$ denote the new equalities. Define a C^∞ vector field v_{j_0} on Y_{j_0} to be 0. Then $v^Y = \{v_j\}_{j\geq j_0}$ is a locally integrable and weakly controlled vector field on $\{Y_j\}_{j\geq j_0}$. (Local integrability around Y_{j_0} follows from $(*)'$.)

We want to lift v^Y to a vector field v^X on $\{X_{i,j}\}_{(i,j)\geq(i_0,j_0)}$ which induces $\theta_{i,j}^*$ as v^Y does θ_j^*. First we lift v^Y to $\{\otimes Y_j\}$. Since $d\pi_{j_0}v_j = 0$, there exists uniquely a vector field $v^{\otimes Y} = \{\otimes v_j\}_{j\geq j_0}$ on $\{\otimes Y_j\}_{j\geq j_0}$ such that

$$dp_X \otimes v_{j\,x,y} = 0 \quad \text{and} \quad dp_Y \otimes v_{j\,x,y} = v_{j\,y} \quad \text{for} \quad (x,y) \in \otimes Y_j, \;\; j \geq j_0.$$

Clearly $v^{\otimes Y}$ is locally integrable and weakly controlled, and it induces the homeomorphism

$$\otimes(Y_{j_0}^+ \cap \rho_{j_0}^{-1}(]0,\varepsilon_{j_0}])) \ni (x,y) \longrightarrow (x,\theta_{j_0}(y)) \in \otimes(Y_{j_0}^+ \cap \rho_{j_0}^{-1}(\varepsilon_{j_0})) \times]0,\varepsilon_{j_0}].$$

Second, by the same reason as above we obtain a controlled vector field $\{v_{i,j_0}\}_{i>i_0}$ on $\{X_{i,j_0}\}_{i>i_0}$ such that

$$(**) \qquad d\pi_{i_0,j_0} v_{i,j_0} = 0 \quad \text{and} \quad v_{i,j_0}\rho_{i_0,j_0} = \rho_{i_0,j_0}, \ i > i_0.$$

Set $v_{i_0,j_0} = 0$ on X_{i_0,j_0}. Then $\{v_{i,j_0}\}_{i\geq i_0}$ is a locally integrable vector field on $\{X_{i,j_0}\}_{i\geq i_0}$.

Third, by Lemma 3 there exists a vector field $v^X = \{v_{i,j}\}_{(i,j)\geq(i_0,j_0)}$ on $\{X_{i,j}\}_{(i,j)\geq(i_0,j_0)}$ which is an extension of $\{v_{i,j_0}\}_{i\geq i_0}$ and such that $\{v_{i,j}\}_{(i,j)>(i_0,j_0)}$ is controlled over $v^{\otimes Y}$. Lemma 4 claims that $\{v_{i,j}\}_{(i,j)>(i_0,j_0)}$ is locally integrable. Moreover, it follows from $(*)'$, $(**)$ and controlledness of $\{v_{i,j}\}_{(i,j)>(i_0,j_0)}$ over $v^{\otimes Y}$ that v^X is locally integrable around X_{i_0,j_0}.

In the same way as we defined $\theta^*_{j_0}$, we do $\theta^*_{i_0,j_0}$ so that for each $x \in N'_{i_0,j_0}$,

$$\theta^{*-1}_{i_0,j_0}(x) = N_{i_0,j_0} \cap (\text{the integral curve of } v^X \text{ passing through } x),$$

which is possible by conditions (7) and (8) of a removal data of $\{T_{i,j}\}$, if v^X points outside of N_{i_0,j_0} at each point of N'_{i_0,j_0}. The last condition is satisfied at $N'_{i_0,j_0} \cap (\rho_{j_0}\circ f)^{-1}\{0,\varepsilon_{j_0}\}$, and hence, by weak controlledness of v^X, at a neighborhood of $N'_{i_0,j_0} \cap (\rho_{j_0}\circ f)^{-1}(0)$ in N'_{i_0,j_0}. Therefore, it suffices to choose sufficiently small ε_{j_0}. This means that when we fix $\{\varepsilon_{i,j}\}$ at the beginning of the proof, we construct also $\theta_{i,j}$.

By (b), $\theta^{**}_{i_0,j_0}$ is automatically defined on $N_{i_0,j_0} - (\rho_{j_0}\circ f)^{-1}(0)$. It is extensible to $N_{i_0,j_0} \cap (\rho_{j_0}\circ f)^{-1}(0) - X^{\varepsilon}_{i_0,j_0}$ for the following reason. Let $\omega\colon \Omega \to X_{\geq(i_0,j_0)}$, $\Omega \subset X_{\geq(i_0,j_0)} \times \mathbf{R}$, denote the flow of v^X. Then by $(*)$ we have

$$\omega(x,\log t) = \theta^{-1}_{i_0,j_0}(x,t) \quad \text{for} \quad (x,t) \in (N'_{i_0,j_0} - (\rho_{j_0}\circ f)^{-1}(0)) \times]0,1].$$

Conditions (a), (b) and (c) are satisfied. Indeed, the former equality of (a) is trivial. The latter follows from controlledness of $\{v_{i,j}\}_{(i,j)>(i_0,j_0)}$ over $v^{\otimes Y}$. (c) is clear by the definition of $\theta^*_{j_0}$ and $\theta^*_{i_0,j_0}$ and the same controlledness. □

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