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ASYMPTOTICS AND SCATTERING PROBLEM FOR THE GENERALIZED KORTEWEG-DE VRIES EQUATION

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§1 **Introduction.** We consider the asymptotic behavior in time of solutions to the Cauchy problem for the generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} u_t + (|u|^{\rho-1}u)_x + \frac{1}{3}u_{xxx} = 0, & t, x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{1.1}$$

Here u_0 is a real valued function and $\rho > 3$. We denote the Sobolev space $H^{1,1} = \{\phi \in L^2; \|\phi\|_{1,1} = \left\|\left(1+x^2\right)^{1/2}\left(1-\partial_x^2\right)^{1/2}\phi\right\|_{L^2} < \infty\}$, and the free Airy evolution group

$$U(t)\phi = \mathcal{F}^{-1}e^{it\xi^3/3}\hat{\phi}(\xi).$$

Here and below $\mathcal{F}\phi$ or $\hat{\phi}$ is the Fourier transform of the function ϕ defined by $\mathcal{F}\phi(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\phi(x)dx$. The inverse Fourier transformation $\mathcal{F}^{-1}$ is given by the formula $\mathcal{F}^{-1}\phi(x) = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\phi(\xi)d\xi$.

Our purpose in this note is to explain the following result which was proved in paper [12].

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Theorem 1.1. *We assume that the initial data* u_0 *are real,* $u_0 \in H^{1,1}$ *and* $\|u_0\|_{1,1} = \epsilon$ *is sufficiently small. Then there exists a unique global solution* $u \in C(\mathbf{R}; H^{1,1})$, *of the Cauchy problem (1.1) with* $\rho > 3$ *such that*

$$\|u(t)\|_{L^\beta} \leq \frac{C\epsilon}{(1+t)^{\frac{1}{3}-\frac{1}{3\beta}}}, \quad \|uu_x(t)\|_{L^\infty} \leq \frac{C\epsilon^2}{t^{\frac{2}{3}}(1+t)^{\frac{1}{3}}},$$

for all $t > 0$ *and for every* $\beta \in (4, \infty]$. *Furthermore we show that there exists a unique final state* $u_+ \in L^2$ *such that*

$$\|u(t) - U(t)u_+\|_{L^2} \leq C\epsilon t^{-\frac{\rho-3}{3}} \quad \text{for} \quad t \geq 1. \tag{1.2}$$

The Cauchy problem (1.1) was intensively studied by many authors and a large amount of literature is devoted to investigate it. The existence and uniqueness of solutions to (1.1) in different Sobolev spaces were proved in [9, 10, 14, 15, 16, 19, 20, 23, 27]. The smoothing properties of solutions were studied in [3, 5, 6, 15, 16] and the blow-up effect for the slowly decaying solutions of the Cauchy problem (1.1) was found in [2]. For the special cases of the KdV equation itself and the modified KdV equation ($\rho = 3$ in (1.1)) the Cauchy problem was solved by the Inverse Scattering Transform (IST) method and the large time asymptotic behavior of solutions was found (see [1, 7]). The IST method depends essentially on the nonlinear character of the equation, although in the case of MKdV equation $(\rho = 3)$ solutions decay with the same speed as in the corresponding linear case, i.e. $\sup_{x\in\mathbf{R}} |u(t,x)| \leq C(1+t)^{-1/3}$ as $t \to \infty$. Now let us give a brief survey of the previous results on the large time asymptotic behavior of solutions to (1.1) which were obtained by functional analytic methods. To state these results we introduce some function spaces. $L^p = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = \left(\int |\phi(x)|^p dx\right)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $H^{m,s}$ is defined by $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \left\|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\right\| < \infty\}$, $m, s \in \mathbf{R}$.

In paper [25] Strauss proved

Proposition 1.1. *Let* $\rho > 5$, *the initial data* $u_0 \in L^1 \cap H^{1,0}$ *and* $\epsilon = \|u_0\|_{L^1} + \|u_0\|_{H^{1,0}}$ *be sufficiently small. Then the solution* $u(t)$ *of (1.1) satisfies the time decay estimate* $\|u(t)\|_\infty \leq C\epsilon(1+t)^{-\frac{1}{3}}$ *and there exists a final state* $u_+ \in L^2$ *such that* $\lim_{t\to\infty} \|u(t) - U(t)u_+\| = 0$.

In his method W. Strauss used the following large time decay estimate $\|U(t)u_0\|_\infty \leq Ct^{-\frac{1}{3}}\|u_0\|_1$ of the L^∞ norm of solutions to the Airy equation.

Later this result on the asymptotically free evolution of solutions to (1.1) was extended to the values $\rho > (5+\sqrt{21})/2 \approx 4.79$ in [17, 18, 24, 26]. They obtained

Proposition 1.2. *Assume that* $\rho > (5+\sqrt{21})/2 \approx 4.79$, *the initial data* $u_0 \in L^{2\rho/(2\rho-1)} \cap H^{1,0}$ *and* $\epsilon = \|u_0\|_{L^{2\rho/(2\rho-1)}} + \|u_0\|_{H^{1,0}}$ *is sufficiently small. Then the solution* $u(t)$ *of (1.1) satisfies the time decay estimate* $\|u(t)\|_{2\rho} \leq C\epsilon(1+t)^{-\frac{1}{3}(1-\frac{1}{\rho})}$ *and there exists a final state* $u_+ \in L^2$ *such that* $\lim_{t\to\infty} \|u(t) - U(t)u_+\| = 0$.

Their method is based on the following L^p decay estimate $\|U(t)u_0\|_{2\rho} \leq Ct^{-\frac{1}{3}(1-\frac{1}{\rho})}\|u_0\|_{2\rho/(2\rho-1)}$ for the solutions to the Airy equation.

In paper [22] Ponce and Vega improved the above result for the values of $\rho > (9+\sqrt{73})/4 \approx 4.39$.

Proposition 1.3. *Let* $\rho > (9+\sqrt{73})/4 \approx 4.39$, *the initial data* $u_0 \in L^1 \cap H^{1,0}$ *and* $\epsilon = \|u_0\|_{L^1} + \|u_0\|_{H^{1,0}}$ *be sufficiently small. Then the same result as in Proposition 1.2 holds. Furthermore the solution* $u(t)$ *satisfies the time decay estimate*

$$\left\|\left(-\partial_x^2\right)^{1/4}u(t)\right\|_\infty \leq C\epsilon(1+t)^{-\frac{1}{2}}.$$

For the proof of Proposition 1.3 Ponce and Vega used the L^p decay estimates of solutions to the Airy equation and the following L^∞ time decay estimate $\left\|\left(-\partial_x^2\right)^{1/4}U(t)u_0\right\|_\infty \leq Ct^{-\frac{1}{2}}\|u_0\|_1$ of the half derivative of solutions to the Airy equation.

Finally in [4] Christ and Weinstein extended the result of Ponce and Vega to the powers $\rho > (23-\sqrt{57})/4 \approx 3.86$.

Proposition 1.4. *Assume that* $\rho > (23-\sqrt{57})/4 \approx 3.86$, *the initial data* $u_0 \in L^1 \cap H^{2,0}$, $u_0' \in L^1$ *and the norm* $\epsilon = \|u_0\|_1 + \|\partial_x u_0\|_1 + \|u_0\|_{2,0}$ *is sufficiently small. Then the same result as in Proposition 1.3 holds. Furthermore the solution* $u(t)$ *satisfies the time decay estimate* $\|u(t)\|_p \leq C\epsilon(1+t)^{-\frac{1}{3}(1-\frac{1}{p})}$ *for* $p > 4$.

The proof of Proposition 1.4 is based on the previous methods. Also it uses the L^p decay estimates of solutions to the Airy equation

$$\|U(t)u_0\|_p \leq Ct^{-\frac{1}{3}(1-\frac{1}{p})}\|u_0\|_1. \tag{1.3}$$

for all $p > 4$.

Thus we do not know a character of the large time asymptotic behavior of the solutions to the Cauchy problem for the generalized Korteweg-de Vries equation (1.1) with $\rho < 3.86$. Our result in Theorem 1.1 fills a gap $3 < \rho \leq 3.86$. The asymptotic expansion of the solutions to the Cauchy problem (1.1) was obtained in [21] for the integer values of $\rho \geq 4$. The evaluation of the asymptotics in [21] is based on the perturbation theory and essentially uses the explicit representation of the Fourier transform of the nonlinearity and therefore does work only for the integer values of ρ.

The Airy free evolution group is defined by

$$\begin{aligned} U(t)\phi &= \mathcal{F}^{-1} e^{it\xi^3/3}\hat{\phi}(\xi) \\ &= \frac{1}{2\pi}\int dy\phi(y)\int d\xi e^{i\xi(x-y)+it\xi^3/3} = \frac{1}{\sqrt[3]{t}}\mathrm{Re}\int \mathrm{Ai}\left(\frac{x-y}{\sqrt[3]{t}}\right)\phi(y)dy, \end{aligned}$$

where $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty e^{ixz+iz^3/3}dz$ is the Airy function (we take a slightly different definition of the Airy function, usually the real part of our function Ai is called by the Airy function). The Airy function has the following asymptotics: $\mathrm{Ai}(\eta) = \frac{C}{\sqrt[4]{|\eta|}}\exp\left(-\frac{2}{3}i\sqrt{|\eta|^3}+i\frac{\pi}{4}\right) + O\left(|\eta|^{-7/4}\right)$ as $\eta = \frac{x}{\sqrt[3]{t}} \to -\infty$ and $\mathrm{Ai}(\eta) = \frac{C}{\sqrt[4]{\eta}}e^{-\frac{2}{3}\sqrt{\eta^3}} + O\left(\eta^{-7/4}e^{-\frac{2}{3}\sqrt{\eta^3}}\right)$ as $\eta = \frac{x}{\sqrt[3]{t}} \to +\infty$ (see, e.g., [8]). In [12, Theorem 1.3] we showed that the solution of (1.1) has the same asymptotics as that of the Airy function when the function u_0 decays as $x \to \infty$ faster than any exponent.

§2 Key linear estimates.

Our method uses the estimate (1.3) and the following time decay estimate of solutions to the Airy equation

$$\|(U(t)u_0)(U(t)u_0)_x\|_\infty \leq Ct^{-2/3}(1+t)^{-1/3}|||u_0|||_{X_0}, \tag{2.1}$$

where

$$|||u_0|||_{X_0} = \|u_0\|_{1,0} + \|D^\alpha x u_0\| + \|\partial_x x u_0\|,$$

and $\alpha = 1/2-\gamma, \gamma \in \left(0, \min\left(\frac{1}{2}, \frac{\rho-3}{3}\right)\right)$. The inequality (2.1) is obtained from the estimates

$$|U(t)u_0(x)| \leq C(1+t)^{-1/3}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{-1/4}|||u_0|||_{X_0}$$

and

$$|\partial_x U(t)u_0(x)| \leq Ct^{-2/3}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{1/4}|||u_0|||_{X_0}.$$

For the proofs of the above estimates, see [12, Lemma 2.2]. Our method is close to that of [11] in the point that here we also use the following operator $I = x+3t\int_{-\infty}^x \partial_t dy$ for obtaining estimates of the solution in the weighted Sobolev spaces. The operator

I almost commutes with the linear part $L = \partial_t + \frac{1}{3}\partial_x^3$ of equation (1.1) and acts on the nonlinear term $(|u|^{\rho-1}u)_x$ as a first order differential operator. Note that the operator I is related with the operator $J = U(-t)xU(t) = (x - t\partial_x^2)$ since we have $I - J = 3t\int_{-\infty}^{x} Ldx$.

In what follows we consider the positive time only. We define the function space X_T as follows

$$X_T = \{\phi \in C\left([0,T];L^2\right); |||\phi|||_{X_T} = \sup_{t\in[0,T]} \|\phi(t)\|_X < \infty\},$$

where $\|\phi(t)\|_X = \|\phi(t)\|_{1,0} + \|D^\alpha J\phi(t)\| + \|\partial J\phi(t)\|$. By virtue of (1.3) with $u_0 = U(-t)\phi(t)$ and by the Hölder's inequality we have for all $4 < p \leq \infty$

$$\begin{aligned}\|\phi(t)\|_p &\leq Ct^{-\frac{1}{3}(1-\frac{1}{p})}\|U(-t)\phi(t)\|_1\\ &\leq Ct^{-\frac{1}{3}(1-\frac{1}{p})}\left(\|\phi(t)\| + \|xU(-t)\phi(t)\|_{2/(1-2\alpha)}\right)\\ &\leq Ct^{-\frac{1}{3}(1-\frac{1}{p})}\left(\|\phi(t)\| + \|D^\alpha xU(-t)\phi(t)\|\right) \leq Ct^{-\frac{1}{3}(1-\frac{1}{p})}|||\phi|||_{X_T}.\end{aligned} \tag{2.2}$$

Via (2.1) we also get the estimate

$$\|\phi(t)\phi_x(t)\|_\infty \leq Ct^{-2/3}(1+t)^{-1/3}|||\phi|||_{X_T}^2. \tag{2.3}$$

Using estimates (2.2) and (2.3) we obtain in the next section the result of Theorem 1.1 by considering a-priori estimates of local solutions in the function space X_T.

§3 Proof of Theorem 1.1.

To clarify the idea of the proof of the Theorem 1.1 we only show a priori estimates of local solutions to gKdV equation. For that purpose we use the following local existence theorem.

Theorem 3.1. *We assume that* $u_0 \in H^{1,1}$, $\|u_0\|_{1,1} = \epsilon \leq \epsilon'$ *and* ϵ' *is sufficiently small. Then there exists a finite time interval* $[0,T]$ *with* $T > 1$ *and a unique solution* u *of (1.1) with* $\rho > 3$ *such that* $|||u|||_{X_T} \leq C\epsilon'$.

For the proof of Theorem 3.1, see, e.g., [9, 14, 15, 16, 20, 27].

Lemma 3.1. *Let* u *be the local solutions of the Cauchy problem (1.1) with* $\rho > 3$ *stated in Theorem 3.1. Then we have* $|||u|||_{X_T} \leq C\epsilon$, *where the constant* C *does not depend on the time* T *of existence of solutions.*

Proof. We write the gKdV equation in the form $Lu = -(|u|^{\rho-1}u)_x$. Differentiating it with respect to x we get

$$Lu_x = -(|u|^{\rho-1}u)_{xx}. \tag{3.1}$$

Multiplying both sides of (3.1) by u_x and integrating by parts we obtain

$$\frac{d}{dt}\|u_x\|^2 \le C\|u\|_\infty^{\rho-3}\|uu_x\|_\infty\|u_x\|^2. \tag{3.2}$$

Using estimates (2.2), (2.3) in the right hand side of (3.2), we get

$$\frac{d}{dt}\|u_x\|^2 \le C|||u|||_{X_T}^{\rho+1}t^{-\frac{2}{3}}(1+t)^{\frac{2}{3}-\frac{\rho}{3}}. \tag{3.3}$$

Applying the operator I to the both sides of (3.1) and using the commutation relations $[L,I]=3\int_{-\infty}^{x}Ldx'$, $[I,\partial_x]=-1$ we have

$$LIu_x = -\rho\left(|u|^{\rho-1}(Iu_x)_x + (\rho-1)|u|^{\rho-3}uu_xIu_x + 2|u|^{\rho-1}u_x\right). \tag{3.4}$$

Multiplying (3.4) by Iu_x and integrating by parts we obtain

$$\frac{d}{dt}\|Iu_x\|^2 \le C|||u|||_{X_T}^{\rho-1}t^{-\frac{2}{3}}(1+t)^{\frac{2}{3}-\frac{\rho}{3}}(|||u|||_{X_T}^2 + \|Iu_x\|^2). \tag{3.5}$$

Since $(Iu)_x - (Ju)_x = 3t(|u|^{\rho-1}u)_x$ for the solution of equation (1.1) we have

$$\begin{aligned}\|Ju_x\| &\le C\left(\|u\| + \|(Iu)_x\| + 3t\|uu_x\|_\infty\|u\|\|u\|_\infty^{\rho-3}\right)\\ &\le C\|u_0\| + Ct^{1-\frac{\rho}{3}}|||u|||_{X_T}^{\rho-1}.\end{aligned} \tag{3.6}$$

Hence (3.5) and (3.6) yield

$$\frac{d}{dt}\|Iu_x\|^2 \le C|||u|||_{X_T}^{\rho+1}t^{-\frac{2}{3}}(1+t)^{\frac{2}{3}-\frac{\rho}{3}}. \tag{3.7}$$

Applying the operator $D^\alpha I$ to (1.1), multiplying the result by $D^\alpha Iu$ and using inequalities (see, [12, Lemma 2.3])

$$\|D^\alpha|u|^{\rho-1}u\|^2 \le C\left\||u|^{\rho-1}\right\|^2\left(\|uu_x\|_\infty + \|u\|_\infty^{6\gamma}\|uu_x\|_\infty^{1-3\gamma}\right),$$

$$\begin{aligned}|(D^\alpha h, D^\alpha|u|^{\rho-1}h_x)| &\le C\|D^\alpha h\|\left(\|D^\alpha h\| + \|\partial_x h\|\right)\left(\|u\|_\infty^{\rho-3}\|uu_x\|_\infty\right.\\ &\left.+ \|u\|_\infty^{\rho-3-2\gamma}\|u\|^{2\gamma}\|uu_x\|_\infty + \|u\|_\infty^{\rho-3}\|u\|_\infty^{2\gamma}\|uu_x\|_\infty^{1-\gamma}\right),\end{aligned}$$

where $\alpha = 1/2-\gamma$, $\gamma\in\left(0,\min\left(\frac{1}{2},\frac{\rho-3}{3}\right)\right)$, $h = Iu$ we get

$$\begin{aligned}\frac{d}{dt}\|D^\alpha Iu\|^2 &= -2\left(D^\alpha Iu, \rho D^\alpha|u|^{\rho-1}(Iu)_x + (3-\rho)D^\alpha\left(|u|^{\rho-1}u\right)\right)\\ &\le C\|D^\alpha Iu\|\Big(\left(\|D^\alpha Iu\| + \|\partial Iu\|\right)\left(\|u\|_\infty^{\rho-3}\|uu_x\|_\infty\right.\\ &\left.+ \|u\|_\infty^{\rho-3-2\gamma}\|u\|^{2\gamma}\|uu_x\|_\infty + \|u\|_\infty^{\rho-3+2\gamma}\|uu_x\|_\infty^{1-\gamma}\right)\\ &+ \left\||u|^{\rho-1}\right\|\left(\|uu_x\|_\infty^{1/2} + \|u\|_\infty^{3\gamma}\|uu_x\|_\infty^{\frac{1-3\gamma}{2}}\right)\Big).\end{aligned} \tag{3.8}$$

Since we take γ to be sufficiently small, we see that there exists a positive constant μ such that

$$\frac{d}{dt}\|D^\alpha Iu\|^2 \le C|||u|||_{X_T}^{\rho+1}(1+t)^{-1-\mu}, \tag{3.9}$$

where we have used the estimate $\|D^\alpha Iu\| \le C|||u|||_{X_T}$. Thus inequalities (3.2), (3.7), (3.8) and (3.9) yield

$$\|u(t)\|_{1,0} + \|D^\alpha Iu(t)\| + \|\partial_x Iu(t)\| \le C\|u_0\|_{1,1} + C|||u|||_{X_T}^{\rho+1}\int_0^t (1+s)^{-1-\mu}ds.$$

The last inequality with estimate $\|u(t)\|_X \le C(\|u(t)\| + \|D^\alpha Iu(t)\| + \|\partial_x Iu(t)\|)$ imply that if the initial data are sufficiently small then $|||u|||_{X_T} \le C\|u_0\|_{1,1}$ for any T. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.1 Via Lemma 3.1 we have $|||u|||_{X_T} \le C\epsilon$. We take ϵ satisfying $C\epsilon \le \epsilon'$. Then a standard continuation argument yields the a-priori estimate $|||u|||_{X_T} \le C\epsilon$ for any T because the constant C does not depend on the time T. Therefore it follows that there exists a unique global solution $u \in C(\mathbf{R}; H^{1,1})$, of the Cauchy problem (1.1) with $\rho > 3$ such that

$$\|u(t)\|_\beta \le \frac{C\epsilon}{(1+t)^{\frac{1}{3}-\frac{1}{3\beta}}}, \quad \|uu_x(t)\|_\infty \le \frac{C\epsilon^2}{t^{\frac{2}{3}}(1+t)^{\frac{1}{3}}},$$

for all $t > 0$ and for every $\beta \in (4, \infty]$. We next show the existence of the scattering state. Rewriting (1.1) in the form $(U(-t)u)_t = -U(-t)\partial_x(|u|^{\rho-1}u)$, we get

$$\begin{aligned}\|U(-t)u(t) - U(-s)u(s)\| &\le C\int_s^t \|u(\tau)\|_\infty^{\rho-3}\|uu_x(\tau)\|_\infty\|u(\tau)\|d\tau \\ &\le C\epsilon\int_s^t \tau^{-\rho/3}d\tau \le C\epsilon s^{-(\rho-3)/3} \end{aligned}\tag{3.10}$$

for $1 < s < t$. By virtue of (3.10) we find that there exists a unique function $u_+ \in L^2$ such that $\lim_{t\to\infty}\|u(t) - U(t)u_+\| = 0$. We let $t \to \infty$ in (3.10) to get (1.2). This completes the proof of Theorem 1.1. □

Finally we note that in paper [13] we studied the asymptotic behavior for large time of solutions to the Cauchy problem for the modified Korteweg-de Vries (mKdV) equation:

$$u_t + \partial_x\mathcal{N}(u) + \frac{1}{3}u_{xxx} = 0, \quad u(0,x) = u_0(x), \tag{mKdV}$$

where $x, t \in \mathbf{R}$, the nonlinear term is equal to $\mathcal{N}(u) = a(t)u^3$, $a(t) \in C^1(0,\infty)$ is real and bounded, and the initial data u_0 are small enough belonging to the weighted Sobolev space $H^{1,1}$ and the integral $\int u_0(x)dx = 0$.

Then we show that the solution $u(t)$ satisfies the decay estimates $\|u(t)\|_{L^6} \le C(1+t)^{-1/3}$, and $\|u(t)u_x(t)\|_{L^\infty} \le Ct^{-2/3}(1+t)^{-1/3}$ for all $t \ge 0$. Moreover we proved that there exist unique functions W and $\Phi \in L^\infty$ such that the following asymptotics for the solutions to the Cauchy problem for mKdV equation

$$u(t,x) = \frac{\sqrt{2\pi}}{\sqrt[3]{t}} \mathrm{ReAi}\left(\frac{x}{\sqrt[3]{t}}\right) W\left(\frac{x}{t}\right) \exp\left(- 3i\pi \left|W\left(\frac{x}{t}\right)\right|^2 \int_1^t a(\tau)\frac{d\tau}{\tau} \right.$$
$$\left. - 3i\pi\Phi\left(\frac{x}{t}\right)\right) + O\left(\epsilon t^{-1/2-\lambda}\right)$$

is valid for large time uniformly with respect to $x \in \mathbf{R}$, where $\lambda \in \left(0, \frac{1}{21}\right)$. Also in paper [13] we constructed the modified scattering states.

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