数理解析研究所講究録
1060 巻 1998 年 249-256

The space of Hilbert cusp forms and the representation of $SL_2(\mathbb{F}_q)$

東京理大理工　浜畑芳紀　(Yoshinori Hamahata)

1. Introduction.

In this paper, we would like to report our result about the representation of $SL_2(\mathbb{F}_q)$ on the space of Hilbert modular cusp forms.

In his paper [2], Hecke considered the representation π of $SL_2(\mathbb{F}_p)$ on the space of elliptic cusp forms of weight 2 for $\Gamma(p)$, and he determined how tr π decomposes into irreducible characters. Above all, he showed that the difference of the multiplicities of certain two irreducible characters yields the Dirichlet expression for $h(\mathbb{Q}(\sqrt{-p}))$, the class number of $\mathbb{Q}(\sqrt{-p})$. The result was generalized to cusp forms of several variables, i.e., Hilbert cusp forms by H. Yoshida and H. Saito, and Siegel cusp forms of degree 2 by K. Hashimoto.

Using his trace formula, Eichler [1] obtained another expression for the difference of the multiplicities above. This expression can be rewritten as the Dirichlet expression for $h(\mathbb{Q}(\sqrt{-p}))$. This Eichler's result was generalized to Hilbert cusp forms for real quadratic fields by H. Saito.

The purpose of this paper is to report that Eichler's result can be generalized to Hilbert cusp forms for totally real cubic fields. The plan of this paper is as follows. In section 2 we review the definiton of Hilbert cusp forms and then recall the results of Hecke, Eichler, and Yoshida-Saito. In section 3, we recall Saito's result on Hilbert cusp forms for real quadratic fields. In section 4, our result is stated. In the last section, the sketch of proof for our result is given.

The author is grateful to Professor S. Kanemitsu for giving him an opportunity to write this report.

Notation. Let $\mathbb{R}, \mathbb{Q}$ be the field of real, and rational numbers, respectively, and $\mathbb{F}_q$ the finite field of q-elements. For a number field K, let $h(K)$ denote the class number of K. Put $\mathbf{e}[\bullet] = \exp(2\pi i \bullet)$. By $\#(S)$, we mean the cardinality of the set S.

2. Hilbert modular forms.

In this section we first review the definition of Hilbert cusp forms. Then we recall the results of Hecke, Eichler, Yoshida, and Saito.

Let K be a totally real number field of degree n, and $\mathfrak{o}_K$ the ring of integers of K. There exist n different embeddings of K into $\mathbb{R}$. Denote them by $K \hookrightarrow \mathbb{R}$, $\quad x \mapsto x^{(i)}$ $(x \in K)$. Let $\mathfrak{H}$ be the upper half plane of all complex numbers with positive imaginary part. The group $SL_2(\mathfrak{o}_K)$ acts on $\mathfrak{H}^n$, the n-th fold product of $\mathfrak{H}$, as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_K)$ and $z = (z_1, \cdots, z_n) \in \mathfrak{H}^n$ we have

$$\gamma \cdot z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \cdots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right).$$

Let $\mathfrak{p}$ be a prime ideal of K, and set

$$\Gamma(\mathfrak{p}) = \{\gamma \in SL_2(\mathfrak{o}_K) \mid \gamma \equiv 1_2 \pmod{\mathfrak{p}}\}.$$

Then $\Gamma(\mathfrak{p})$ also acts on $\mathfrak{H}^n$. Let k be an even positive integer. For any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_K)$, put $j_k(\gamma, z) = \prod_{i=1}^{n} (c^{(i)} z_i + d^{(i)})^{-k}$. We now define Hilbert modular cusp forms.

Definition 2.1. A holomorphic function f on $\mathfrak{H}^n$ is called *Hilbert cusp form* of weight k for $\Gamma(\mathfrak{p})$ if it satisfies

i) $f(\gamma z) j_k(\gamma, z) = f(z)$ for any $\gamma \in \Gamma(\mathfrak{p})$,

ii) f is holomorphic at each cusp of $\Gamma(\mathfrak{p})$, and its Fourier expansion at each cusp has no the constant term.

Let $S_k(\Gamma(\mathfrak{p}))$ be the set of Hilbert cusp forms with weight k for $\Gamma(\mathfrak{p})$. Put $f|_k[\gamma] = f(\gamma z) j_k(\gamma, z)$ for $\gamma \in SL_2(\mathfrak{o}_K)$. Then $SL_2(\mathfrak{o}_K)$ acts on $S_k(\Gamma(\mathfrak{p}))$ by $(\gamma, f) \mapsto f|_k[\gamma]$. Since $\Gamma(\mathfrak{p})$ acts on it trivially, $SL_2(\mathbb{F}_q) \cong SL_2(\mathfrak{o}_K)/\Gamma(\mathfrak{p})$ acts on it ($q = N\mathfrak{p}$). Let π be the representation associated to this action. We are interested in the representation π. Let q be a power of an odd prime. Then, there are two pairs of irreducible characters of $SL_2(\mathbb{F}_q)$ whose values are conjugate each other. We give a list of values at $\epsilon = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\epsilon' = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$ (η is a nonsquare element of $\mathbb{F}_q^*$) of such pairs (α_1, α_2) and (β_1, β_2) as follows:

	ϵ	ϵ'
α_1	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$
α_2	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$
β_1	$\frac{-1+\sqrt{-q}}{2}$	$\frac{-1-\sqrt{-q}}{2}$
β_2	$\frac{-1-\sqrt{-q}}{2}$	$\frac{-1+\sqrt{-q}}{2}$

Note that each pair has the same values on other conjugacy classes. If $q \equiv 1 \pmod 4$, then β_1 and β_2 do not appear. If $q \equiv 3 \pmod 4$, then α_1 and α_2 do not appear. Let y_1 be the multiplicity of α_1 (resp. β_1) in tr π when $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$), and y_2 the multiplicity of α_2 (resp. β_2) in tr π when $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$). For the multiplicities y_1 and y_2, Hecke proved the following result.

Theorem 2.2(Hecke [2]). *If $n = 1$ and $k = 2$, then*

$$y_1 - y_2 = \begin{cases} 0 & (q \equiv 1 \bmod 4), \\ h(\mathbb{Q}(\sqrt{-q})) & (q \equiv 3 \bmod 4), \end{cases}$$

Eichler got the following result.

Theorem 2.3(Eichler [1]). *If $n = 1$ and $k = 2$, then*

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2} q}} \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \nu(i),$$

where $\left(\frac{\cdot}{q}\right)$ is the quadratic residue symbol mod $\mathfrak{p}$, and $\nu(i) = \mathbf{e}[i/p]/(1-\mathbf{e}[i/p])$.

Using the Selberg trace formula, H. Yoshida and H. Saito generalized Theorem 2.2 to Hilbert cusp forms:

Theorem 2.4(Yoshida and Saito, cf [3]). If $k \geq 4$, then we have

$$|y_1 - y_2| = 2^{n-1} \sum_{K_j} \frac{h(K_j)}{h(K)},$$

where K_j runs over totally imaginary quadratic extensions of K with the relative discriminant $\mathfrak{p}$.

3. The result of Saito.

In this section we review Hilbert modular varieties, and recall the result of Saito, which is an analogue of Eichler's formula (Theorem 2.3).

Let the notation be as above. Since $\Gamma(\mathfrak{p})$ acts on $\mathfrak{H}^n$, we have the quotient space $\mathfrak{H}^n/\Gamma(\mathfrak{p})$. One can compactify it by adding all cusps of $\Gamma(\mathfrak{p})$. We denote by $\overline{\mathfrak{H}^n/\Gamma(\mathfrak{p})}$ the resulting surface. The space $\overline{\mathfrak{H}^n/\Gamma(\mathfrak{p})}$ has two kinds of singularities, i.e., quotient singularities and cusp singularities. Let $X(\mathfrak{p})$ be the desingularization of $\overline{\mathfrak{H}^n/\Gamma(\mathfrak{p})}$. If we assume that $\overline{\mathfrak{H}^n/\Gamma(\mathfrak{p})}$ has no quotient singularities and that $h(K)=1$, then the resolution of singularities can be described by a complex Σ obtained from the pair $(\mathfrak{o}_K, U(\mathfrak{p}))$. Here $U(\mathfrak{p})$ denotes the group of units of K congruent to 1 modulo $\mathfrak{p}$. Let γ be any element of $SL_2(\mathfrak{o}_K)$. Since $\Gamma(\mathfrak{p})$ is a normal subgroup, γ induces f_γ, the automorphism of $\mathfrak{H}^n/\Gamma(\mathfrak{p})$ defined by $(z_1,\cdots,z_n)\mapsto(\gamma^{(1)}z_1,\cdots,\gamma^{(n)}z_n)$. Here $\gamma^{(i)}$ denotes the matrix defined by exchanging the components of γ for the images of them by the i-th embedding of K. The automorphism f_γ can be extended to that of $\overline{\mathfrak{H}^n/\Gamma(\mathfrak{p})}$, and moreover that of $X(\mathfrak{p})$, which is also denoted by f_γ.

We now recall the result of Saito [3]. Let K be a real quadratic field, and $\mathfrak{p}$ a prime ideal of K such that $\mathfrak{p}$ is generated by a totally positive element μ, prime to $6\cdot d_K$ (d_K is the discriminant of K), and $q=\#(O_K/\mathfrak{p})$ is a power of an odd prime. Let U be the unit group of K, and $U(\mathfrak{p})$ the group of units congruent to 1 modulo $\mathfrak{p}$. Let $[U : U(\mathfrak{p})]=t$. There exists an element $w\in\mathfrak{o}_K$ such that $\mathfrak{o}_K=\mathbb{Z}+\mathbb{Z}w$ and $0<w'<1<w$. Here w' denotes the conjugate of w. We have the continued fraction

$$w=b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots-\cfrac{1}{b_r-\cfrac{1}{w}}}}.$$

Then we define positive integers p_k and q_k by

$$\frac{p_k}{q_k}=b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots-\cfrac{1}{b_{k-1}-\cfrac{1}{b_k}}}}$$

for a positive integer k $(1 \leq k \leq r)$. For any element $\alpha \in \mathfrak{o}_K$, H. Saito defines

$$\nu(\alpha) :=$$

$$\sum_i \frac{\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right] \cdot \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]}{\left(1 - \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right]\right)\left(1 - \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]\right)}$$
$$+ \sum_j \frac{\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]}{\left(1 - \mathbf{e}\left[-\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]\right)} \left\{-1 + \frac{b_j}{1 - \mathbf{e}\left[-\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]}\right\},$$

where i runs over such indices as $1 \leq i \leq rt$ and neither $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right]$ nor $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]$ equal 1, and j runs over such indices as $1 \leq j \leq rt$ and $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{j-1} + q_{j-1} w'}{w - w'}\right] = 1$. Note that each integer $-b_j$ is the selfintersection number of some irreducible curve arising from the cusp resolution of α/μ. Using the holomorphic Lefschetz formula of Atiyah-Singer, H. Saito proved the following:

Theorem 3.1(Saito [3]). *On $S_2(\Gamma(\mathfrak{p}))$ we have*

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2} q}} \cdot \frac{2}{[U : U(\mathfrak{p})]} \sum_{\alpha \in (O_K/\mathfrak{p})^\times} \left(\frac{\alpha}{\mathfrak{p}}\right) \nu(\alpha),$$

where $\left(\frac{\ }{\mathfrak{p}}\right)$ denotes the quadratic residue symbol mod $\mathfrak{p}$.

4. The main result.

Let K be a totally real cubic number field. Let $\mathfrak{p}$ be a prime ideal of K such that $\mathfrak{p}$ is generated by a totally positive element μ, prime to $6 \cdot d_K$, and $q = \#(O_K/\mathfrak{p})$ is a power of an odd prime. Let U be the unit group of K, and $U(\mathfrak{p})$ the group of units congruent to 1 modulo $\mathfrak{p}$. Let Σ be the complex which describes the cusp resolution of $\overline{\mathfrak{H}^3/\Gamma(\mathfrak{p})}$ attached to $X(\mathfrak{p})$. Let $\Sigma^{(r)}$ be the set of r-simplices in Σ.

Definition 4.1. For each $\alpha \in O_K$, f_α denotes the automorphism of $X(\mathfrak{p})$ induced by $\begin{pmatrix} 1 & \alpha/\mu \\ 0 & 1 \end{pmatrix}$. Put $\mathbf{e}[\ ] = \exp(2\pi i\ \ )$, and $d(a, b, c) =$

$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$. Then for each $\alpha \in O_K$, we define

$$\nu(\alpha) := \sum_{(1)} \frac{\mathbf{e}\left[\frac{d(\alpha/\mu,v,w)}{d(u,v,w)}\right] \cdot \mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right] \cdot \mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{\left(1-\mathbf{e}\left[\frac{d(\alpha/\mu,v,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]\right)}$$
$$+\sum_{(2)} \frac{\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right] \cdot \mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{\left(1-\mathbf{e}\left[-\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]\right)}$$
$$\times\left\{-1-\frac{a(v,w)}{1-\mathbf{e}\left[-\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]}-\frac{a(w,v)}{1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}\right\}$$
$$+\sum_{(3)} \frac{\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]} \cdot \left\{1-\frac{c(w)}{1-\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}\right\},$$

where $\sum_{(i)}$ runs over the elements of $\Sigma^{(4-i)}$ corresponding to the $i-1$ dimensional fixed subvarieties of f_α $(i = 1, 2, 3)$, $a(v, w) = F_{\langle v\rangle} \cdot F^2_{\langle w\rangle}$, and $c(w) = \sum_{\langle u\rangle \in \Sigma^{(1)}} F_{\langle u\rangle} \cdot F^2_{\langle w\rangle}$ ($F_{\langle v\rangle}$ denotes the divisor of $X(\mathfrak{p})$ corresponding to $\langle v\rangle$).

Then our result is as follows:

Theorem 4.2. *On $S_2(\Gamma(\mathfrak{p}))$ we have*

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}} \cdot \frac{2}{[U : U(\mathfrak{p})]} \sum_{\alpha \in (O_K/\mathfrak{p})^\times} \left(\frac{\alpha}{\mathfrak{p}}\right) \nu(\alpha),$$

where $\left(\frac{\ }{\mathfrak{p}}\right)$ denotes the quadratic residue symbol mod $\mathfrak{p}$.

Remark 4.3. Though we only considered in the case $k = 2$ in Theorem 4.2, the theorem holds for any even positive integer k. Indeed, put $D := X(\mathfrak{p}) - \mathfrak{H}^3/\Gamma(\mathfrak{p})$. Let Ω^3 be the sheaf of germs of holomorphic 3-forms on $X(\mathfrak{p})$, and $L := \Omega^3(\log D)$ the sheaf of germs of 3-forms with logarithmic poles along D on $X(\mathfrak{p})$. Then we have $S_k(\Gamma(\mathfrak{p})) = H^0(X(\mathfrak{p}), L^{k/2-1} \otimes \Omega^3)$. Here L is trivial around D, and our theorem can be described in terms of the cusp resolution. Hence the claim follows with the use of the Kodaira vanishing theorem and the holomorphic Lefschetz formula.

We give an example to Theorem 4.2.

Example 4.4. Let K be the field $\mathbb{Q}(w)$ defined by $w^3 + 2w^2 - w - 1 = 0$. Then K is a totally real Galois cubic field with $h(K) = 1$ and $d_K = 7^2$. If we put $\mu := 2 - w$, then μ is totally positive. We find that $\mathfrak{p} := (\mu)$ is a prime ideal of K lying over 13. Then on $S_2(\Gamma(\mathfrak{p}))$ we have $y_1 - y_2 = 0$. This result agrees with the fact that there does not exist a totally imaginary quadratic extension of K with the relative discriminant $\mathfrak{p}$.

5. Sketch for the proof of Theorem 4.2.

The difference $y_1 - y_2$ is expressed as

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}}(\mathrm{tr}\ \pi(\epsilon) - \mathrm{tr}\ \pi(\epsilon')).$$

Since $S_2(\Gamma(\mathfrak{p})) = H^0(X(\mathfrak{p}), \Omega^3)$, we have

$$\mathrm{tr}\ \pi(\epsilon) = \mathrm{tr}(f_\epsilon | H^0(X(\mathfrak{p}), \Omega^3)).$$

The same thing holds for ϵ'. Put

$$\tau(\epsilon) := \sum_{i=0}^{3} (-1)^i \mathrm{tr}(f_\epsilon | H^i(X(\mathfrak{p}), \Omega^3)).$$

We define $\tau(\epsilon')$ in the same way. Since

$$H^1(X(\mathfrak{p}), \Omega^3) = H^2(X(\mathfrak{p}), \Omega^3) = 0, \quad H^3(X(\mathfrak{p}), \Omega^3) = \mathbb{C},$$

we have

$$\mathrm{tr}(f_\epsilon | H^0(X(\mathfrak{p}), \Omega^3) - \mathrm{tr}(f_{\epsilon'} | H^0(X(\mathfrak{p}), \Omega^3) = \tau(\epsilon) - \tau(\epsilon').$$

We apply the holomorphic Lefschetz formula to $\tau(\epsilon)$ and $\tau(\epsilon')$.

References

1. M. Eichler, *Einige Anwendung der Spurformel in Bereich der Modulkorrespondenzen*, Math. Ann., **168** (1967), 128-137.

2. E. Hecke, *Über das Verhalten der Integrale I Gattung bei beliebigen, insbesondere in der Theorie der elliptischen Modulfunktionen*, Abh. Math. Sem. Ham. Univ., **8** (1930), 271-281.

3. H. Saito, *On the representation of $SL_2(\mathbb{F}_q)$ in the space of Hilbert modular forms*, J. Math. Kyoto Univ., **15** (1975), 101-128.

Yoshinori HAMAHATA
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba, 278, JAPAN