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Farey series and the Riemann hypothesis

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The aim of this paper is to consider the equivalent conditions to the Riemann hypothesis in terms of Farey series.

Farey series F_x of order x is the sequence of all irreducible fractions in (0, 1] with denominator not bigger than x, arranged in increasing order of magnitude;

$$F_x = F_{[x]} = \left\{ \rho_\nu = \frac{b_\nu}{c_\nu} \mid (b_\nu, c_\nu) = 1, 0 < b_\nu \le c_\nu \le x \right\}$$

and the cardinality of F_x is the summatory function of Euler's function

$$\#F_x = \Phi(x) = \sum_{n \le x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad \text{(Mertens)}.$$

$$\varphi(n) = \sum_{\substack{m \le n \\ (m,n)=1}} 1: \text{ Euler's function.}$$

This asymptotic formula is due to Mertens.

For example, from F_2 we form F_3:

$$F_2 = \left\{ \frac{1}{2}, \frac{1}{1} \right\} \to F_3 = \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\},$$

and so on.

And the Riemann hypothesis RH states that the Riemann zeta function does not vanish for real part σ of s bigger than $\frac{1}{2}$. It is well known that the RH is equivalent to each of the following asymptotic formulas, forms of the prime number theorem:

$$\begin{aligned}
\text{RH} \iff & \zeta(s) \ne 0 \text{ for } \sigma := \Re s > \frac{1}{2} \\
\iff & M(x) := \sum_{n \le x} \mu(n) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \\
& \mu(n)\text{: Möbius' function} \\
\iff & \psi(x) := \sum_{n \le x} \Lambda(n) = \sum_{p^m \le x} \log p \\
& \qquad = x + O\left(x^{\frac{1}{2}+\varepsilon}\right), \\
& \Lambda(n)\text{: von Mangoldt's function} \\
& \psi(x)\text{: Chebyshev's function}
\end{aligned}$$

Here the weak Riemann hypothesis RH(α) states that $\zeta(s)$ does not vanish for $\sigma > \alpha$:

$$\mathrm{RH}(\alpha) \Longleftrightarrow \zeta(s) \neq 0 \text{ for } \sigma > \alpha.$$

In this paper I'd like to state main results of Part V [9] and Part VI [6] of this series of papers. In Part V we aim at the implications of the RH(α) on the estimates of error terms associated to Farey series, with occasional acquisition of equivalent conditions.

Principle.
Suppose f has a bounded derivative and consider the error term $E_f(x)$ defined by

$$E_f(x) := \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) - \Phi(x) \int_0^1 f(u)du.$$

Suppose the RH implies the estimate:

$$\mathrm{RH} \Longrightarrow E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

and that the Mellin transform $F(s)$ defined by

$$F(s) = s\zeta(s) \int_1^\infty E_f(x) x^{-s-1} dx \quad \text{for } \sigma > 1$$

satisfies following conditions:
(i) $F(s)$ is regular for $\sigma > \frac{1}{2}$, $s \neq 1$,
(ii) $F(s) \neq 0$ for $\frac{1}{2} < \sigma < 1$.
Then

$$\mathrm{RH} \Longleftrightarrow E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

We note that if we define the arithmetic function $a(n)$ by

(1)
$$a(n) = \sum_{k=1}^{n}{}' f\left(\frac{k}{n}\right) - n \int_0^1 f(u)du,$$

then $E_f(x)$ can be written as

$$E_f(x) = \sum_{n \le x} (\mu * a)(n) = \sum_{n \le x} M\left(\frac{x}{n}\right) a(n),$$

where $*$ denotes the Dirichlet convolution, and $F(s)$ becomes the generating function of $a(n)$:

(2)
$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Theorem 1 (Part V, Theorem 2). *Suppose f is an integrable function on $(0,1)$. Let $a(n)$ be the arithmetic function defined by (1), let $F(s)$ be the generating function (2) of $a(n)$, and we suppose that $F(s)$ and $a(n)$ satisfy following conditions* (i)–(v):
(i) *$F(s)$ is absolutely convergent for $\sigma > \sigma_a$ with $\sigma_a \leq 1$,*
(ii) *$F(s)$ is continued to an analytic function with finitely many singularities in the half-plane $\sigma > \alpha$,*
(iii) *$F(s) \ll |t|^{\kappa+\varepsilon}$ for some $\kappa \geq 0$ and every $\varepsilon > 0$ uniformly in the region $\alpha < \sigma \leq 1$, $|t| \geq t_0 > 0$,*
(iv) *$(\mu * a)(n) \ll n^{\beta+\varepsilon}$ for some β, $0 \leq \beta$ $(\leq \sigma_a)$,*
(v) *there exists a non-negative number θ satisfying*

$$\sum_{n=1}^{\infty} \frac{|(\mu * a)(n)|}{n^{\sigma}} \ll (\sigma - 1)^{-\theta} \qquad \text{as } \sigma \to 1.$$

Then, on the RH(α), we have the (asymptotic) formula:

$$\sum_{n \leq x} (\mu * a)(n) = \frac{1}{2\pi i} \int_C \frac{F(s)}{s\zeta(s)} x^s ds + O\left(x^{\omega+\varepsilon}\right),$$

where

$$\omega = \min_{0 \leq \xi \leq 1} \{\max\{\beta + 1 - \xi, 1 + (\kappa - 1)\xi, \alpha + \kappa\xi\}\},$$

and the contour C encircles all singularities of $F(s)/\zeta(s)$ in the strip $\alpha < \sigma < 1$.

In particular, in the special cases of $\kappa = 0$ and $\beta = 0$ we have $\omega = \max\{\alpha, \beta\}$, and $\omega = \alpha + \kappa(1 - \alpha)$, respectively.

Corollary 1 (Codecà-Perelli [2], Theorem 1). (i) *Let $f(u)$ be absolutely continuous and let $f' \in L^p[0,1]$ for some $p \in (1,2]$. Then, on the RH(η), we have*

$$E_f(x) = O\left(x^{\max\{\eta, \frac{1}{p}\}+\varepsilon}\right).$$

(This covers the Main result of Codecà-Perelli, Theorem 1.)

(ii) *Moreover, if $F(s)$ satisfies the conditions* (i) – (iii) *of Theorem 1 and* $0 \leq \kappa \leq \frac{2}{p} - 1$. *Then, on the RH($\alpha$),*

$$E_f(x) = O\left(x^{\alpha+\kappa(1-\alpha)+\varepsilon}\right).$$

Corollary 2. *For any rational number $\frac{r}{q} \in (0,1)$ other than $\frac{1}{2}$, the RH implies*

$$E\left(\frac{r}{q}; x\right) := \sum_{\rho_\nu \leq \frac{r}{q}} 1 - \frac{r}{q}\Phi(x) = O\left(x^{\frac{1}{2}+\frac{35}{432}+\varepsilon}\right),$$

by the result of Kolesnik.

Moreover, if we assume the GRH (on some Dirichlet L-functions mod q) and the RH, we have the estimate:

$$E\left(\frac{r}{q}; x\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right). \qquad \textit{(Codecà [1])}$$

We can not only cover the strong result of Codecà-Perelli [2] (Corollary 1), some results of Codecà [1] and their developments as above, but also we can widen the width of validity of the parameter by $\frac{1}{2}$ of some theorems proved earlier.

In particular, on the GRH, the RH is equivalent to each of estimates

$$E\left(\frac{1}{3};x\right)=O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

and

$$E\left(\frac{1}{4};x\right)=O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Theorem 2 (Part VI). *If $f(u)$ is the gap-Fourier series;*

$$f(u)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^{mk_1}(2n+1)^{k_2}}\cos 2\pi 2^{ml_1}(2n+1)^{l_2}u,$$

with k_1, $k_2\in\mathbb{C}$, l_1, $l_2\in\mathbb{N}$, $2\Re k_1\geq l_1+1$ and $2\Re k_2\geq l_2+2$, then we have the equivalence:

$$RH\Longleftrightarrow E_f(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Corollary 3. $k_1=l_1=l_2=1,\ k_2=2\Longrightarrow$
$f(u)$ is Takagi's function, and

$$F(s)=\frac{3}{2}\frac{1-2^{-s-1}}{1-2^{-s}}\zeta(s)\zeta(s+1)\neq 0 \text{ for } \sigma>\tfrac{1}{2}.$$

Hence

$$RH\Longleftrightarrow E_f(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

($k_1=k_2=l_1=l_2=2\Longrightarrow f$: Riemann's function)

Recall that if $E_f(x)=(M*a)(x)$ with suitable $a(n)$, then

$$F(s)=s\zeta(s)\int_1^{\infty}E_f(x)x^{-s-1}dx=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},$$

and vice versa.

Hence, when $f(u)$ has a Fourier expansion

$$f(u)=\sum_{n=1}^{\infty}c(n)\cos 2\pi nu$$

satisfying the condition;

$$\sum_{n=1}^{\infty}|c(n)|d(n)<\infty,$$

then, with $a(n) = n \sum_{m=1}^{\infty} c(mn)$, we have the Ramanujan expansion:

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} c(n)\sigma_{1-s}(n), \quad \sigma > 1$$

($E_f = M * a$ also holds).

Conversely, if $F(s)$ is the generating Dirichlet series;

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \sigma \geq 1,$$

the Fourier coefficient of the corresponding f is given by

$$c(n) = \frac{1}{n} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} a(kn).$$

This is a Hecke-like correspondence.

$f(\tau) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n \tau}$	$\longleftrightarrow$	$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$
dynamical system		generating Dirichlet series
generating Fourier series		$\approx$ zeta-function with
		Euler product if a is multiplicative

Example.

$f(\tau)$		$F(s)$		$E_f(x)$
$\sum_{n=1}^{\infty} c(n)e^{2\pi i n \tau}$	$\longleftrightarrow$	$\sum_{n=1}^{\infty} c(n)\sigma_{1-s}(n)$		
		$= \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$	$\longleftrightarrow$	$(M * a)(x)$
$\cos 2\pi\tau$		1		$M(x)$
$\log 2\lvert \sin \pi x \rvert$		$\zeta'(s)$		$\psi(x)$
$B_{2n}(\tau) \quad (n \in \mathbb{N})$		$\zeta(s + 2n - 1)$		$\sum_{m \leq x} M\left(\frac{x}{m}\right) \frac{1}{m^{2n-1}}$

Here $B_k(x)$ is the k-th Bernoulli polynomial.

Theorem 3 (Part VI). (i) *Let $f_{k,l}(u)$ be a gap Fourier series*

$$f_{k,l}(u) := \sum_{n=1}^{\infty} \frac{1}{n^k} \cos 2\pi n^l u \quad \textit{for } \Re k > 1,\ l \in \mathbb{N}.$$

Then $F_{k,l}$ can be decomposed with $G_{k,l}$ having an Euler product as follows:

$$F_{k,l}(s) = \zeta(k)\zeta(ls + k - l)G_{k,l}(s),$$

$$G_{k,l}(s) = \prod_p \left(1 + p^{-k} \sum_{n=1}^{l-1} p^{n(1-s)}\right).$$

(ii) *If* $2\Re k \geq l+2$, *we have*

$$RH \Longleftrightarrow E_{f_{k,l}}(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

(iii) *For* $k=2$, $l=3$

$$RH \Longleftrightarrow E_{f_{2,3}}(x) = \frac{\zeta(2)G_{2,3}\left(\frac{2}{3}\right)}{2\zeta\left(\frac{2}{3}\right)}x^{\frac{2}{3}} + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

$$G_{2,3}\left(\frac{2}{3}\right) = \prod_p\left(1+p^{-\frac{4}{3}}+p^{-\frac{5}{3}}\right).$$

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