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SOLUTIONS OF DIFFUSION EQUATIONS WHOSE SPATIAL LEVEL SURFACES ARE INVARIANT WITH RESPECT TO THE TIME VARIABLE

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1. Introduction. There are some symmetry results due to Alessandrini [Al 1,2] some of which proved a conjecture of Klamkin [Kl] (see also [Zal]). We quote a theorem of [Al 2] (see [Al 2, Theorem 1.3, p. 254]).

Theorem A (Alessandrini). *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$ with boundary $\partial\Omega$ and let all of its boundary points be regular with respect to the Laplacian. Let $\varphi \in L^2(\Omega)$ satisfy $\varphi \not\equiv 0$ and let $u = u(x,t)$ be the unique solution of*

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{1.1}$$

If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot,t)$ is constant on every level surface $\{ x \in \Omega \ ; \ u(x,\tau) = \ const. \ \}$ of $u(\cdot,\tau)$ in Ω, then one of the following two cases occurs.

(i) *φ is an eigenfunction of $-\Delta$ under the homogeneous Dirichlet boundary condition.*
(ii) *Ω is a ball, $u(\cdot,t)$ is radially symmetric for each $t \geqq 0$, and u never vanishes in $\Omega \times [\tau,\infty)$.*

Klamkin's conjecture [Kl] was that if all the spatial level surfaces are invariant with respect to the time variable t for positive constant initial data under the homogeneous Dirichlet boundary condition, then the domain must be a ball. Therefore Theorem A proved the Klamkin's conjecture [Kl].

In the present paper we consider the similar problem under the homogeneous Neumann boundary condition or the problems for nonlinear diffusion equations such as the porous medium equation. Our first result is:

Theorem 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geqq 2)$ with boundary $\partial\Omega$, and let $\varphi \in L^2(\Omega)$ satisfy $\varphi \not\equiv 0$ and $\int_\Omega \varphi \, dx = 0$. Let $u = u(x,t)$ be the unique solution of the following initial-Neumann problem:*

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

where ν denotes the exterior normal unit vector to $\partial\Omega$. If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on every level surface $\{ x \in \Omega \ ; \ u(x, \tau) = \text{ const. } \}$ of $u(\cdot, \tau)$ in Ω, then one of the following two cases occurs.

(i) *φ is an eigenfunction of $-\Delta$ under the homogeneous Neumann boundary condition.*

(ii) *By a rotation and a translation of coordinates we have one of the following:*

(a) *There exists a finite interval (a, b) such that u is extended as a function of x_1 and t only, say $u = u(x_1, t)$ $((x_1, t) \in [a, b] \times (0, \infty))$, and $\frac{\partial u}{\partial x_1} = 0$ on $\{a, b\} \times (0, \infty)$. Furthermore, $\Omega \subset (a, b) \times \mathbb{R}^{N-1}$ with $\partial\Omega \cap (\{a\} \times \mathbb{R}^{N-1}) \neq \emptyset$ and $\partial\Omega \cap (\{b\} \times \mathbb{R}^{N-1}) \neq \emptyset$.*

(b) *There exist a finite interval (a, b) with $a \geqq 0$ and a natural number k with $2 \leqq k \leqq N$ such that u is extended as a function of $r = (x_1 + \cdots + x_k)^{\frac{1}{2}}$ and t only, say $u = u(r, t)$ $((r, t) \in [a, b] \times (0, \infty))$, whose derivative $\frac{\partial u}{\partial r}(r, t)$ does not vanish in $(a, b) \times (\tau, \infty)$ and it vanishes on $\{a, b\} \times (0, \infty)$. Furthermore, there exist a Lipschitz domain S in the standard $k-1$-dimensional unit sphere S^{k-1} in $\mathbb{R}^k$ (S can be the whole sphere S^{k-1}) and a bounded Lipschitz domain $\tilde{\Omega}$ in $\mathbb{R}^{N-k}$ such that $\Omega = \{r\omega \in \mathbb{R}^k ; r \in (a, b) \text{ and } \omega \in S\} \times \tilde{\Omega}$ when $a > 0$, and either $\Omega = \{r\omega \in \mathbb{R}^k ; r \in (0, b) \text{ and } \omega \in S\} \times \tilde{\Omega}$ with $S \neq S^{k-1}$ or $\Omega = \{(x_1, ..., x_k) \in \mathbb{R}^k ; r < b\} \times \tilde{\Omega}$ when $a = 0$. Here, when $k = N$, the domain $\tilde{\Omega}$ is disregarded.*

In particular, in case (ii), *if $\partial\Omega$ is C^1, then Ω must be either a ball or an annulus.*

We refer the reader to [Br] for existence and uniqueness of solutions of the initial-Neumann problem in Lipschitz cylinders. Since any constant function is a trivial solution of the initial-Neumann problem (1.2) with constant initial data, and since adding any constant function to the solution u in Theorem 1 does not have any influence on the invariance condition of spatial level surfaces of u, so for simplicity we assumed that $\varphi \not\equiv 0$ and $\int_\Omega \varphi \, dx = 0$ for initial data φ.

Alessandrini used an eigenfunction expansion and a special case of a well-known theorem of symmetry for elliptic equations due to Serrin [Ser, Theorem 2, pp. 311-312] in order to prove Theorem A:

Theorem S (Serrin). *Let D be a bounded domain with C^2 boundary ∂D and let $v \in C^2(\overline{D})$ satisfy the following:*

$$\begin{cases} \Delta v = f(v) \ \ \text{and} \quad v > 0 \ \ \text{in} \ \ D, \\ v = 0 \qquad \text{and} \ \ \dfrac{\partial v}{\partial \nu} = c \ \ \text{on} \ \ \partial D, \end{cases}$$

where $f = f(s)$ is a C^1 function of s, c is a constant, and ν denotes the exterior normal unit vector to ∂D. Then D is a ball and v is radially symmetric and decreasing in D.

Under the hypothesis that case (i) of Theorem A does not hold, Alessandrini showed that there exists a level set $D = \{ x \in \Omega \ ; \ \psi(x) > s \}$ with $s > 0$ of an eigenfunction

$\psi = \psi(x)$ of $-\Delta$ under the homogeneous Dirichlet boundary condition such that the function $v = \psi - s$ satisfies the overdetermined boundary conditions as in Theorem S. Then applying Theorem S to v implies that D is a ball and v is radially symmetric and decreasing in D. By a little more argument one gets case (ii) of Theorem A. In this proof essential is the fact that the boundary of D does not touch the boundary $\partial\Omega$. This fact comes from the homogeneous Dirichlet boundary condition of the eigenfunction ψ. Therefore, in our problem (1.2) we can not use Theorem S because of the homogeneous Neumann boundary condition. We overcome this obstruction by using the invariance condition of spatial level surfaces much more with the help of the theory of *isoparametric surfaces in Euclidean space* (see [Lc, Seg]). Also, we can give another proof of Theorem A which does not depend on Theorem S.

Next we want to consider nonlinear diffusion equations. For the porous medium equation under the homogeneous Neumann boundary condition we have:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$ with smooth boundary $\partial\Omega$, and let $u = u(x,t) \in C^\infty(\overline{\Omega} \times (0,\infty))$ satisfy*

$$\begin{cases} \partial_t \beta(u) = \Delta u & \text{in } \Omega \times (0,\infty), \\ u > 0 & \text{in } \overline{\Omega} \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.3}$$

where $\beta(s) = s^{\frac{1}{m}} (m > 0, m \neq 1)$ and ν denotes the exterior normal unit vector to $\partial\Omega$. If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot,t)$ is constant on every level surface $\{ x \in \Omega \ ; \ u(x,\tau) = \text{ const. } \}$ of $u(\cdot,\tau)$ in Ω, then one of the following two cases occurs.

(i) *u is a positive constant for $t \geqq \tau$.*
(ii) *Ω is either a ball or an annulus, and for each $t \geqq \tau$ $u(\cdot,t)$ is radially symmetric with respect to the center and for $t > \tau$ the derivative with respect to the radial direction, say $\frac{\partial u}{\partial r}$, does not vanish in Ω except at the center of the ball.*

For the generalized porous medium equation under the homogeneous Dirichlet boundary condition we have:

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$ with smooth boundary $\partial\Omega$, and let $u = u(x,t) \in C(\overline{\Omega} \times (0,T)) \cap C^\infty(\Omega \times (0,T))$ satisfy*

$$\begin{cases} \partial_t \beta(u) = \Delta u \ \text{ and } \ u > 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.4}$$

where β is a continuous function on $[0,\infty)$ satisfying

(1) *β is real analytic on $(0,\infty)$,*
(2) *$\beta(0) = 0$ and $\beta'(s) > 0$ for any $s > 0$.*

If there exists $\tau \in (0,T)$ such that, for every $t > \tau$, $u(\cdot,t)$ is constant on every level surface $\{ x \in \Omega \; ; \; u(x,\tau) = \text{ const. } \}$ of $u(\cdot,\tau)$ in Ω, then one of the following two cases occurs.

(i) *There exists a positive C^∞ function $\lambda = \lambda(t)$ on $[\tau,T)$ such that $u(x,t) = \lambda(t)u(x,\tau)$ for any $(x,t) \in \overline{\Omega} \times [\tau,T)$.*

(ii) *Ω is a ball, for each $t \in [\tau,T)$ $u(\cdot,t)$ is radially symmetric with respect to the center, and for each $t \in (\tau,T)$ the derivative with respect to the radial direction, say $\frac{\partial u}{\partial r}$, is negative in Ω except at the center of Ω.*

See [BP, Sac 2, AMT] for existence and uniqueness of weak solutions of the initial-boundary value problems for $\partial_t\beta(u) = \Delta u$, and see [Sac 1] for continuity of bounded weak solutions. When $\beta(s) = s^{\frac{1}{m}}$ with $0 < m < 1$, if the initial data $u(x,0) \in L^\infty(\Omega)$ for the initial-Dirichlet problem, then there exists a finite extinction time T^* such that $u \equiv 0$ for $t \geqq T^*$ (see for example [BéC, p. 176]). Therefore in Theorem 3 we consider the finite time interval $(0,T)$. Concerning case (i) see [ArP, BerH] for separable solutions of (1.4) when $\beta(s) = s^{\frac{1}{m}}$ with $m > 0$.

In Section 2 we prove Theorems 1, 2, and 3 simultaneously.

2. Proofs of theorems. First of all, let us quote the classification theorem of isoparametric hypersurfaces in Euclidean space $\mathbb{R}^N$, which was proved by Levi-Civita [Lc] for $N = 3$, and by Segre [Seg] for arbitrary N. See [No, PaT] for a survey of isoparametric surfaces.

Theorem LcS (Levi-Civita and Segre). *Let D be a bounded domain in $\mathbb{R}^N$ ($N \geqq 2$) and let f be a real-valued smooth function on D satisfying $\nabla f \neq 0$ on D. Suppose that there exist two real-valued functions $g = g(\cdot)$ and $h = h(\cdot)$ of a real variable such that*

$$|\nabla f|^2 = g(f) \;\text{ and }\; \Delta f = h(f) \;\text{ on }\; D. \tag{2.1}$$

Then the family of level surfaces $\{x \in D \; ; \; f(x) = s\}$ $(s \in f(D))$ of f must be either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. In particular, by a rotation and a translation of coordinates one of the following holds:

(a) *There exists a finite interval (a_1,b_1) such that f is extended as a function of x_1 only, say $f = f(x_1)$ $(x_1 \in [a_1,b_1])$, and $D \subset (a_1,b_1) \times \mathbb{R}^{N-1}$ with $\partial D \cap (\{a_1\} \times \mathbb{R}^{N-1}) \neq \emptyset$ and $\partial D \cap (\{b_1\} \times \mathbb{R}^{N-1}) \neq \emptyset$.*

(b) *There exist a finite interval (a_1,b_1) with $a_1 \geqq 0$ and a natural number k with $2 \leqq k \leqq N$ such that f is extended as a function of $r = (x_1 + \cdots + x_k)^{\frac{1}{2}}$ only, say $f = f(r)$ $(r \in [a_1,b_1])$, and furthermore when $a_1 > 0$, $D \subset \{(x_1,\ldots,x_k) \in \mathbb{R}^k; a_1 < r < b_1\} \times \mathbb{R}^{N-k}$ with $\partial D \cap (\{(x_1,\ldots,x_k) \in \mathbb{R}^k; r = a_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial D \cap (\{(x_1,\ldots,x_k) \in \mathbb{R}^k; r = b_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$, and when $a_1 = 0$, $D \subset \{(x_1,\ldots,x_k) \in \mathbb{R}^k; 0 \leqq r < b_1\} \times \mathbb{R}^{N-k}$ with $\overline{D} \cap (\{0\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial D \cap (\{(x_1,\ldots,x_k) \in \mathbb{R}^k; r = b_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$. Here, when $k = N$, $\mathbb{R}^{N-k}$ is disregarded.*

In this theorem the function f is called an isoparametric function, and the level surfaces of f are called isoparametric surfaces. For our use we assumed that the domain D is bounded.

Let us put $u(x,\tau) = \psi(x)$ for $x \in \overline{\Omega}$. By the common assumption of Theorems 1, 2, and 3 (the invariance condition of spatial level surfaces) as in [Al 1, (2.2), p. 231] we have:

$$u(x,t) = \mu(\psi(x),t) \text{ for any } (x,t) \in \overline{\Omega} \times [\tau,\infty) \ ([\tau,T) \text{ in Theorem 3}) \tag{2.2}$$

for some function $\mu = \mu(s,t) : \mathbb{R} \times [\tau,\infty) \to \mathbb{R}$ satisfying

$$\mu(s,\tau) = s \text{ for any } s \in \mathbb{R}. \tag{2.3}$$

Although the time interval is $[\tau, T)$ in Theorem 3, for simplicity let us use the time interval $[\tau,\infty)$. In Theorems 1 and 3 ψ is not constant, and in Theorem 2 if ψ is constant, then we have case (i) and we have nothing to prove. Therefore we may assume that ψ is not constant. Hence there exist a point $x_0 \in \Omega$ and an open ball centered at x_0 with radius $r > 0$, say $B = B_r(x_0)$, such that

$$\nabla\psi \neq 0 \text{ on } \overline{B}(\subset \Omega). \tag{2.4}$$

Then by a standard difference quotient argument (see [Al 1, Lemma 1, p.232] and [Al 2, Lemma 2.1, p. 255]) we have

Lemma 2.1. *There exists an interval $I = [\psi(x_0) - \delta, \psi(x_0) + \delta]$ with some $\delta > 0$ such that $I \subset \psi(B)$ and $\mu \in C^\infty(I \times [\tau,\infty))$.*

Proof. For convenience let us give a proof. The partial differentiability of μ with respect to t is a straightforward consequence of (2.2). It follows from (2.4) that there exists an interval $I = [\psi(x_0) - \delta, \psi(x_0) + \delta]$ with some $\delta > 0$ such that $I \subset \psi(B)$. Let $s \in I$. Then there exists a point $y \in B$ such that $\psi(y) = s$ and $\nabla\psi(y) \neq 0$. For $h \in \mathbb{R}$ sufficiently small, put $x(h) = y + h\nabla\psi(y) \in B$. Hence $\psi(x(h)) = s + h|\nabla\psi(y)|^2 + O(h^2)$ as $h \to 0$. Thus for every sufficiently small $k \in \mathbb{R}$ there exists a unique $h \in \mathbb{R}$ such that $\psi(x(h)) = s + k$, and $h = k|\nabla\psi(y)|^{-2} + O(k^2)$ as $k \to 0$. Consequently we have for each $t \in [\tau,\infty)$

$$\mu(s+k,t) - \mu(s,t) = u(x(h),t) - u(y,t) = k\frac{\nabla u(y,t)\cdot\nabla\psi(y)}{|\nabla\psi(y)|^2} + O(k^2) \text{ as } k \to 0. \tag{2.5}$$

This means that there exists a derivative $\frac{\partial\mu}{\partial s}$ given by

$$\frac{\partial\mu}{\partial s}(s,t) = \frac{\partial\mu}{\partial s}(\psi(y),t) = \frac{\nabla u(y,t)\cdot\nabla\psi(y)}{|\nabla\psi(y)|^2}. \tag{2.6}$$

On the other hand we have from (2.2)

$$\frac{\partial\mu}{\partial t}(s,t) = \frac{\partial\mu}{\partial t}(\psi(y),t) = \partial_t u(y,t). \tag{2.7}$$

In view of (2.4), since the right hand sides of both (2.6) and (2.7) are bounded on $\overline{B}\times[\tau,\tilde{t}]$ for each $\tilde{t} > \tau$, by using the mean value theorem we get $\mu \in C^0(I \times [\tau, \infty))$. Because of (2.4) the right hand side of (2.6) is smooth in $B \times [\tau, \infty)$, we can repeat the process as many times as we want and prove the existence of all the partial derivatives of μ with respect to s. Also, we can start the same process from (2.7) as many times as we want. Therefore, with the help of the mean value theorem we can get $\mu \in C^\infty(I \times [\tau, \infty))$. □

In view of Lemma 2.1 we can substitute (2.2) into the differential equation and get

$$\beta'(\mu)\mu_t = \text{ div } (\mu_s \nabla\psi) = \mu_s \Delta\psi + \mu_{ss}|\nabla\psi|^2 \ \text{ on } \psi^{-1}(I) \times [\tau, \infty), \tag{2.8}$$

where $\psi^{-1}(I) = \{ \ x \in \Omega \ ; \ \psi(x) \in I \ \}$ and in Theorem 1 we recognize that $\beta(s) \equiv s$. Differentiating (2.8) with respect to t yields

$$\beta''(\mu)(\mu_t)^2 + \beta'(\mu)\mu_{tt} = \mu_{st}\Delta\psi + \mu_{sst}|\nabla\psi|^2 \ \text{ on } \psi^{-1}(I) \times [\tau, \infty). \tag{2.9}$$

Let us introduce the function $\mathfrak{D}$ by

$$\mathfrak{D} = \det \begin{pmatrix} \mu_s & \mu_{ss} \\ \mu_{st} & \mu_{sst} \end{pmatrix} \equiv \mu_s \mu_{sst} - \mu_{ss}\mu_{st}. \tag{2.10}$$

We distinguish the following two cases:

(1) $\mathfrak{D} \equiv 0$ on $I \times [\tau, \infty)$,

(2) $\mathfrak{D} \not\equiv 0$ on $I \times [\tau, \infty)$.

Remark that these cases are slightly different from the cases in the paper [Al 1] where the time is fixed, that is, $t = \tau$. This modification is useful in dealing with nonlinear diffusion equations (Theorems 2 and 3).

Case (1). In this case let us show that the solution u must be a separable solution, which implies case (i) of Theorems 1 and 3. It follows from (2.3) that $\mu_s(s, \tau) = 1$. Therefore there exists a time $T_1 > \tau$ such that

$$\mu_s > 0 \ \text{ on } I \times [\tau, T_1]. \tag{2.11}$$

Hence we have

$$(\log \mu_s)_{st} = \mathfrak{D}/(\mu_s)^2 = 0 \ \text{ on } I \times [\tau, T_1]. \tag{2.12}$$

Solving this equation yields

$$\mu(s, t) = \lambda(t)s + \eta(t) \ \text{ for any } (s, t) \in I \times [\tau, T_1] \tag{2.13}$$

for some C^∞ functions $\lambda = \lambda(t) \equiv \mu_s > 0$ and $\eta = \eta(t)$ on $[\tau, T_1]$ satisfying

$$\lambda(\tau) = 1 \text{ and } \eta(\tau) = 0. \tag{2.14}$$

On the other hand we know that for each time $t > 0$ $u(\cdot, t)$ is analytic in x (see Friedman [F 2]). Therefore by (2.13) and (2.2) we see that

$$u(x, t) = \lambda(t)\psi(x) + \eta(t) \ \text{ for any } (x, t) \in \overline{\Omega} \times [\tau, T_1]. \tag{2.15}$$

Now we distinguish Theorems 1, 2, and 3. Let us consider Theorem 3 first. The homogeneous Dirichlet boundary condition implies that $\eta \equiv 0$ on $[\tau, T_1]$. Namely, we have

$$u(x,t) = \lambda(t)\psi(x) \quad \text{for any } (x,t) \in \overline{\Omega} \times [\tau, T_1]. \tag{2.16}$$

Let $T^* = \sup\{T_1 \in (\tau, T); \mu_s > 0 \text{ on } I \times [\tau, T_1]\}$. Suppose that $T^* < T$. Since $u > 0$ in $\Omega \times (0,T)$, in view of (2.13) and (2.16) we have by continuity

$$\mu_s(s, T^*) = \lim_{t \uparrow T^*} \lambda(t) = u(x_0, T^*)/\psi(x_0) > 0 \text{ for any } s \in I. \tag{2.17}$$

This contradicts the definition of T^* and the continuity of μ_s. Therefore we get $T^* = T$ and have case (i) of Theorem 3.

Next we consider Theorem 1. Since $\int_\Omega \varphi \, dx = 0$, we have $\int_\Omega u(x,t) \, dx = 0$ for any $t > 0$. Therefore by integrating (2.15) we see that $\eta \equiv 0$ on $[\tau, T_1]$. Hence we get (2.16). By substituting (2.16) into the heat equation and letting $t = \tau$, we get from (2.14)

$$\Delta\psi = \lambda'(\tau)\psi \quad \text{in } \Omega. \tag{2.18}$$

Since ψ is not constant and satisfies the homogeneous Neumann boundary condition, by separating variables we have

$$u(x,t) = e^{-\lambda'(\tau)(t-\tau)}\psi(x) \quad \text{for any } (x,t) \in \Omega \times [0, \infty).$$

This implies case (i) of Theorem 1.

Finally, let us consider Theorem 2. Substituting (2.15) into the diffusion equation yields

$$\frac{1}{m}(\lambda(t)\psi(x) + \eta(t))^{\frac{1}{m}-1}(\lambda'(t)\psi(x) + \eta'(t)) = \lambda(t)\Delta\psi(x). \tag{2.19}$$

Dividing this by $\lambda(t)$ and differentiating the resulting equation with respect to t give

$$\begin{aligned}
&(\frac{1}{m} - 1)(\lambda'(t)\psi(x) + \eta'(t))^2 + (\lambda(t)\psi(x) + \eta(t))(\lambda''(t)\psi(x) + \eta''(t)) \\
&\quad - \frac{\lambda'(t)}{\lambda(t)}(\lambda(t)\psi(x) + \eta(t))(\lambda'(t)\psi(x) + \eta'(t)) = 0.
\end{aligned} \tag{2.20}$$

A further calculation gives

$$I(t)\psi^2(x) + II(t)\psi(x) + III(t) = 0, \tag{2.21}$$

where

$$\begin{cases}
I(t) = (\frac{1}{m} - 2)(\lambda'(t))^2 + \lambda(t)\lambda''(t), \\
II(t) = (\frac{2}{m} - 3)\lambda'(t)\eta'(t) + \lambda(t)\eta''(t) + \lambda''(t)\eta(t) - \dfrac{(\lambda'(t))^2}{\lambda(t)}\eta(t), \\
III(t) = (\frac{1}{m} - 1)(\eta'(t))^2 + \eta(t)\eta''(t) - \dfrac{\lambda'(t)}{\lambda(t)}\eta(t)\eta'(t).
\end{cases} \tag{2.22}$$

Therefore by (2.21) we have

$$I(t) \equiv II(t) \equiv III(t) \equiv 0. \tag{2.23}$$

Solving $I(t) \equiv 0$ gives

$$\lambda'(t) = \lambda^{2-\frac{1}{m}}(t)\lambda'(\tau). \tag{2.24}$$

By solving $II(t) \equiv 0$ with respect to $\eta''(t)$ we get

$$\eta''(t) = -(\frac{2}{m} - 3)\frac{\lambda'(t)}{\lambda(t)}\eta'(t) - (\frac{\lambda'(t)}{\lambda(t)})'\eta(t). \tag{2.25}$$

Substituting this into $III(t) \equiv 0$ gives

$$(\frac{1}{m} - 1)(\eta'(t))^2 - 2(\frac{1}{m} - 1)\frac{\lambda'(t)}{\lambda(t)}\eta(t)\eta'(t) - (\frac{\lambda'(t)}{\lambda(t)})'\eta^2(t) = 0. \tag{2.26}$$

Here by using (2.24) we have

$$\begin{cases} \dfrac{\lambda'(t)}{\lambda(t)} = \lambda^{1-\frac{1}{m}}(t)\lambda'(\tau), \\ (\dfrac{\lambda'(t)}{\lambda(t)})' = (1 - \dfrac{1}{m})\lambda^{2(1-\frac{1}{m})}(t)(\lambda'(\tau))^2. \end{cases} \tag{2.27}$$

By substituting these into (2.26) we get

$$\left(\eta'(t) - \lambda^{1-\frac{1}{m}}(t)\lambda'(\tau)\eta(t)\right)^2 = 0. \tag{2.28}$$

Therefore by using the first equation of (2.27) once more we conclude that

$$\left(\frac{\eta(t)}{\lambda(t)}\right)' = 0. \tag{2.29}$$

Since $\eta(\tau) = 0$(see (2.14)), this implies

$$\eta(t) \equiv 0 \quad \text{on } [\tau, T_1]. \tag{2.30}$$

Namely we get (2.16). Since $\int_\Omega u^{\frac{1}{m}}(x,t)\, dx = \int_\Omega \psi^{\frac{1}{m}}(x)\, dx > 0$ for any $t > 0$, we have from (2.16)

$$\lambda(t) \equiv 1 \quad \text{for any } t \in [\tau, \infty).$$

Then the diffusion equation implies that $\Delta\psi = 0$ in Ω. In view of the homogeneous Neumann boundary condition we see that ψ is a positive constant. This contradicts (2.4), that is, we can not have case (1) in the situation of Theorem 2.

Case (2). In this case, by supposing that each case (i) of Theorems 1, 2, and 3 does not hold, we show that each case (ii) of the theorems holds. It follows from the continuity of $\mathfrak{D}$ that there exist a nonempty open subinterval $J \subset I$ and a time $t_0 \geqq \tau$ such that $\mathfrak{D} \neq 0$ on $\overline{J} \times \{t_0\}$. Hence we can solve equations (2.8) and (2.9) with respect to $|\nabla\psi|^2$ and $\Delta\psi$ for $(x, t_0) \in \psi^{-1}(\overline{J}) \times \{t_0\}$. Namely, there exists a nonempty bounded domain $D \subset \psi^{-1}(\overline{J})(\subset \Omega)$ in $\mathbb{R}^N$ such that

$$|\nabla\psi|^2 = g(\psi) \ \text{ and } \ \Delta\psi = h(\psi) \ \text{ on } \ D \tag{2.31}$$

for some functions g and h as in (2.1). Then it follows from Theorem LcS that after a rotation and a translation of coordinates there exists a finite interval (a_1, b_1) such that either (a) or (b) of Theorem LcS holds for $f = \psi$ and (a_1, b_1). Consequently, since ψ is analytic in Ω, by (2.2) we have one of the following:

(a) There exists a finite interval $(a, b) \supset (a_1, b_1)$ such that u is extended as a function of x_1 and t only, say $u = u(x_1, t)$ $((x_1, t) \in [a, b] \times [\tau, \infty))$. Furthermore, $\Omega \subset (a, b) \times \mathbb{R}^{N-1}$ with $\partial\Omega \cap (\{a\} \times \mathbb{R}^{N-1}) \neq \emptyset$ and $\partial\Omega \cap (\{b\} \times \mathbb{R}^{N-1}) \neq \emptyset$.

(b) There exist a finite interval $(a, b) \supset (a_1, b_1)$ with $a \geqq 0$ and a natural number k with $2 \leqq k \leqq N$ such that u is extended as a function of $r = (x_1 + \cdots + x_k)^{\frac{1}{2}}$ and t only, say $u = u(r, t)$ $((r, t) \in [a, b] \times [\tau, \infty))$. Furthermore, when $a > 0$, $\Omega \subset \{(x_1, ..., x_k) \in \mathbb{R}^k; a < r < b\} \times \mathbb{R}^{N-k}$ with $\partial\Omega \cap (\{(x_1, ..., x_k) \in \mathbb{R}^k; r = a\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial\Omega \cap (\{(x_1, ..., x_k) \in \mathbb{R}^k; r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$, and when $a = 0$, $\Omega \subset \{(x_1, ..., x_k) \in \mathbb{R}^k; 0 \leqq r < b\} \times \mathbb{R}^{N-k}$ with $\overline{\Omega} \cap (\{0\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial\Omega \cap (\{(x_1, ..., x_k) \in \mathbb{R}^k; r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$. Here, when $k = N$, $\mathbb{R}^{N-k}$ is disregarded.

Here we have:

Lemma 2.2. *In case* (b) $u(r, \tau)(= \psi(r))$ *is monotone on* $[a, b]$ *(provided each case (i) of Theorems 1, 2, and 3 does not hold).*

Proof. Suppose that ψ is not monotone. Then ψ has either a local maximum point or a local minimum point. So suppose that ψ has a local maximum point. Since ψ is analytic and not constant, there exist three numbers in (a, b), say $r_1 < r_2 < r_3$, such that

$$\psi(r_1) = \psi(r_3) \ \text{ and } \ \psi'(r) \begin{cases} > 0 & \text{if } r_1 \leqq r < r_2, \\ < 0 & \text{if } r_2 < r \leqq r_3. \end{cases} \tag{2.32}$$

Hence by using Lemma 2.1 once more, if we put $\tilde{I} = [\psi(r_1), \frac{1}{2}(\psi(r_1) + \psi(r_2))]$, then $\tilde{I} \subset \psi((a, b))$ and $\mu \in C^\infty(\tilde{I} \times [\tau, \infty))$. Therefore we get (2.8) and (2.9), where I is replaced by $\tilde{I}$. If $\mathfrak{D} \equiv 0$ on $\tilde{I} \times [\tau, \infty)$, we have already proved that the cases (i)'s of both Theorem 1 and Theorem 3 hold as in Case (1) and in Theorem 2 this leads to a contradiction. Therefore we see that $\mathfrak{D} \not\equiv 0$ on $\tilde{I} \times [\tau, \infty)$. By proceeding as in the beginning of Case (2), we see that there exist a nonempty open subinterval $J \subset \tilde{I}$ and a time $t_0 \geqq \tau$ such that $\mathfrak{D} \neq 0$ on $\overline{J} \times \{t_0\}$. By solving equations (2.8) and (2.9) with respect to $|\nabla\psi|^2$ and $\Delta\psi$ for $(x, t_0) \in \psi^{-1}(\overline{J}) \times \{t_0\}$, we have in particular that

$$(\psi'(r))^2 = g(\psi(r)) \ \text{ on } \ \psi^{-1}(\overline{J}) \cap [r_1, r_3] \tag{2.33}$$

for some function $g = g(\cdot)$ of a real variable as in (2.31). In view of (2.32) we see that

$$\psi^{-1}(\overline{J}) \cap [r_1, r_3] = [r_4, r_5] \cup [r_6, r_7], \tag{2.34}$$

where $r_1 \leqq r_4 < r_5 < r_2 < r_6 < r_7 \leqq r_3$. Since $\psi(r_4) = \psi(r_7)$ and $\psi(r_5) = \psi(r_6)$, by using (2.33) we see that $r_5 - r_4 = r_7 - r_6 \left(= \int_{\psi(r_4)}^{\psi(r_5)} (g(s))^{-\frac{1}{2}}\, ds\right)$ and

$$\psi(r) = \psi(2r_* - r) \quad \text{for any} \quad r \in [r_4, r_5] \cup [r_6, r_7], \tag{2.35}$$

where $r_* = \frac{1}{2}(r_4 + r_7)$. Furthermore by (2.2)

$$u(r,t) = u(2r_* - r, t) \quad \text{for any} \quad (r,t) \in ([r_4, r_5] \cup [r_6, r_7]) \times [\tau, \infty). \tag{2.36}$$

On the other hand, since u satisfies the diffusion equation, we have

$$\partial_t \beta(u) = \partial_r^2 u + \frac{k-1}{r} \partial_r u \quad \text{in } (a,b) \times [\tau, \infty). \tag{2.37}$$

Since $k \geqq 2$, it follows from (2.36) and (2.37) that

$$\partial_r u \equiv 0 \quad \text{in } ([r_4, r_5] \cup [r_6, r_7]) \times [\tau, \infty). \tag{2.38}$$

In particular, this implies that $\psi' \equiv 0$ on $[r_4, r_5] \cup [r_6, r_7]$, which contradicts (2.32).
Similarly if we suppose that ψ has a local minimum point, then we get a contradiction. Consequently we have proved that $u(r, \tau)(= \psi(r))$ is monotone on $[a, b]$. □

We distinguish Theorems 1, 2, and 3. Let us consider Theorem 1 first. In view of (b) just before Lemma 2.2, from the boundary condition of (1.2) we see that in case (b)

$$\partial_r u(a,t) = \partial_r u(b,t) = 0 \quad \text{for any } t \in [\tau, \infty). \tag{2.39}$$

Hence it follows from Lemma 2.2 and the strong maximum principle (see [F 1, Chapter 2] for the maximum principle) that $\partial_r u$ does not vanish in $(a,b) \times (\tau, \infty)$ in case (b). Consequently, this determines the domain Ω as in case (ii) of Theorem 1. Especially in Theorem 1, since problem (1.2) is solved by an eigenfunction expansion, we see that $u = u(x_1, t)((x_1, t) \in [a,b] \times (0, \infty))$ in case (a) and $u = u(r,t)((r,t) \in [a,b] \times (0,\infty))$ in case (b). This completes the proof of Theorem 1.

Next we consider Theorem 2. Since $u \in C^\infty(\overline{\Omega} \times (0, \infty))$, by using the boundary condition of (1.3) we have (2.39) in case (b) as in Theorem 1. Then it follows from Lemma 2.2 and the strong maximum principle that $\partial_r u$ does not vanish in $(a,b) \times (\tau, \infty)$ in case (b). Furthermore, since $\partial\Omega$ is smooth, in view of (a) and (b) we see that (ii) of Theorem 2 holds.

Finally, let us consider Theorem 3. In view of (a) and (b), it follows from Lemma 2.2 combined with the boundary condition of (1.4) that the domain Ω must be a ball. It remains to show that $\partial_r u$ is negative in $(0, b) \times (\tau, T)$. Remark that the equation

$\partial_t \beta(u) = \Delta u$ may be degenerate or singular parabolic depending on the behaviour of $\beta'(s)$ as $s \downarrow 0$, and therefore we can not always have the derivative $\partial_r u$ on $\partial\Omega \times (0, T)$. We can overcome this obstruction by using the standard approximating problem for $1 > \varepsilon > 0$:

$$\begin{cases} \partial_t \beta(v) = \Delta v & \text{in } \Omega \times (\tau, \infty), \\ v(x, \tau) = u(x, \tau) + \varepsilon & \text{in } \Omega, \\ v = \varepsilon & \text{on } \partial\Omega \times (\tau, \infty). \end{cases} \tag{2.40}$$

(In fact this problem is useful to show existence of solutions of the initial-Dirichlet problems for the degenerate or singular parabolic equation $\partial_t \beta(u) = \Delta u$.) Then, by the theory of quasilinear uniformly parabolic equations (see [LSU]) there exists a unique bounded classical solution $v = v_\varepsilon \in C^\infty(\overline{\Omega} \times (\tau, \infty)) \cap C^\infty(\Omega \times [\tau, \infty)) \cap C^0(\overline{\Omega} \times [\tau, \infty))$ of (2.40) satisfying

$$\varepsilon \leqq v_\varepsilon \leqq \max_{x\ in \Omega} u(x, \tau) + \varepsilon \quad \text{in } \overline{\Omega} \times (\tau, \infty).$$

It follows from this inequality combined with the regularity result of [Sac 1] that the family $\{v_\varepsilon\}_{0<\varepsilon<1}$ is equicontinuous on each compact subset of $\Omega \times (\tau, \infty)$. Since by the comparison principle we have $v_{\varepsilon_1} \leqq v_{\varepsilon_2}$ for $0 < \varepsilon_1 \leqq \varepsilon_2 < 1$, by a diagonalization argument, the Arzela-Ascoli theorem, and the uniqueness of the solution u we see that

$$v_\varepsilon \to u \quad \text{as } \varepsilon \to 0 \quad \text{uniformly on each compact subset of } \Omega \times (\tau, T).$$

Furthermore, since $v_\varepsilon \geqq u > 0$ in $\Omega \times (\tau, T)$, by the theory of uniformly parabolic equations ([LSU]) in particular this convergence implies

$$\partial_r v_\varepsilon \to \partial_r u \quad \text{as } \varepsilon \to 0 \quad \text{uniformly on each compact subset of } \Omega \times (\tau, T). \tag{2.41}$$

Observe that for $v_\varepsilon = v_\varepsilon(r, t)$

$$\partial_r v_\varepsilon(0, t) = 0 \text{ and } \partial_r v_\varepsilon(b, t) \geqq 0 \text{ for any } t > \tau.$$

Then it follows from Lemma 2.2 and the maximum principle that

$$\partial_r v_\varepsilon \leqq 0 \quad \text{in } (0, b) \times (\tau, \infty).$$

Therefore we get from (2.41)

$$\partial_r u \leqq 0 \quad \text{in } (0, b) \times (\tau, T). \tag{2.42}$$

Since $u > 0$ in $\Omega \times (\tau, T)$, we can apply the strong maximum principle to $\partial_r u$ and we see that $\partial_r u$ is negative in $(0, b) \times (\tau, T)$. This completes the proof of Theorem 3.

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