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# Examples of Starlike Functions and Convex Functions of order $\alpha$

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**Abstract**

H.Silverman determines certain coefficient inequalities and distortion theorems for univalent functions with negative coefficients that are starlike of order $\alpha$ and convex of order $\alpha$. The same coefficient inequalities and the similar distortion theorems are obtained for such univalent functions with not always negative coefficients. We give some examples of those univalent functions and illustrate the images of the examples by *Mathematica*. Further we estimate those univalent functions.

## 1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_kz^k \quad (n\in\mathbb{N}=\{1,2,3,\cdots\}) \tag{1}$$

that are analytic in the unit disk $U=\{z:|z|<1\}$.

We first consider the so-called subclasses of analytic functions with negative coefficients. Let $A(n)$ denote the subclass of $A$ consisting of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k \quad (a_k\geqq 0,\ n\in\mathbb{N}=\{1,2,3,\cdots\}). \tag{2}$$

Let $T(n)$ denote the subclass of $A(n)$ consisting of functions which are univalent in U. Further a function in $T(n)$ is said to be starlike of order $\alpha(0\leqq\alpha<1)$ if and only if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (z\in U) \tag{3}$$

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and such a subclass of $A(n)$ consisting of all the starlike functions of order $\alpha$ is denoted by $T_\alpha(n)$. Also, $f(z) \in T(n)$ is said to be convex of order $\alpha(0 \leqq \alpha < 1)$ if and only if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U) \tag{4}$$

and the subclass by $C_\alpha(n)$. The classes $T(n)$, $T_\alpha(n)$ and $C_\alpha(n)$ were introduced by Chatterjea[1] and these classes have been studied by Srivastava, Owa and Chatterjea[9], Kiryakova, Saigo and Owa [2], and Sekine[4].

For $n = 1$, these notations are usually used as $T_\alpha(1) = T^*(\alpha)$, $C_\alpha(1) = C(\alpha)$ which were introduced by Silverman[8].

Using the same way of Silverman[8], Chatterjea[1] determined a necessary and sufficients conditions for a function in $A(n)$ belongs to $T_\alpha(n)$ and $C_\alpha(n)$.

**Theorem A**(Chatterjea [1]). *A function $f(z)$ in $A(n)$ is in $T_\alpha(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k-\alpha)a_k \leqq 1-\alpha \quad (0 \leqq \alpha < 1). \tag{5}$$

**Theorem B**(Chatterjea [1]). *A function $f(z)$ in $A(n)$ is in $C_\alpha(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leqq 1-\alpha \quad (0 \leqq \alpha < 1). \tag{6}$$

Recently, in [5] we introduced the subclass $A(n,\theta)$ of $A$, and the subclasses $T^*_\alpha(n,\theta)$ and $C_\alpha(n,\theta)$ of $A(n,\theta)$ in the following manner.

Let $A(n,\theta)$ denote the subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (a_k \geqq 0,\ n \in \mathbb{N}). \tag{7}$$

We note that $A(n,0) = A(n)$, that is, $A(n,0)$ is the subclass of analytic functions with negative coefficients. We denote by $T^*_\alpha(n,\theta)$ and $C_\alpha(n,\theta)$ the subclasses of $A(n,\theta)$ of starlike and convex functions of order $\alpha$ in $U$, respectively.

We proved the same coefficients inequalities as Chatterjea showed for $f(z)$ in $A(n,\theta)$ as follows.

**Theorem C**(Sekine and Owa [5]). *A function $f(z)$ in $A(n,\theta)$ is in $T^*_\alpha(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k-\alpha)a_k \leqq 1-\alpha \quad (0 \leqq \alpha < 1). \tag{8}$$

**Theorem D**(Sekine and Owa [5]). *A function $f(z)$ in $A(n,\theta)$ is in $C_\alpha(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leqq 1-\alpha \quad (0 \leqq \alpha < 1). \tag{9}$$

Further we determined the following Distortion theorems.

**Theorem E**(Sekine and Owa [5]). If $f(z)$ is in $T^*_\alpha(n,\theta)$, then

$$|z| - \frac{1-\alpha}{n+1-\alpha}|z|^{n+1} \leq |f(z)| \leq |z| + \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}. \tag{10}$$

The right-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad \left(z = re^{-i(\theta+\frac{\pi}{n})},\ r<1\right) \tag{11}$$

and the left-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad (z = re^{-i\theta},\ r<1). \tag{12}$$

**Theorem F**(Sekine and Owa [5]). If $f(z)$ is in $C_\alpha(n,\theta)$, then

$$|z| - \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1} \leq |f(z)| \leq |z| + \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1}. \tag{13}$$

The right-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{(n+1)(n+1-\alpha)}z^{n+1} \quad \left(z = re^{-i(\theta+\frac{\pi}{n})},\ r<1\right) \tag{14}$$

and the left-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{(n+1)(n+1-\alpha)}z^{n+1} \quad (z = re^{-i\theta},\ r<1). \tag{15}$$

## 2 Examples

Let $A_\alpha(n,\theta,h)$ denote the subclass of $A(n,\theta)$ consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta}a_{k,h}z^k \quad (h \geqq -n), \tag{16}$$

*where* $a_{k,h} = \dfrac{(1-\alpha)^2}{(k+h-\alpha)(k+1+h-\alpha)(k-\alpha)}$ $(0 \leqq \alpha < 1)$.

Let $B_\alpha(n,\theta,h)$ denote the subclass of $A(n,\theta)$ consisting of functions of the form

$$g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta}b_{k,h}z^k \quad (h \geqq -n), \tag{17}$$

*where* $b_{k,h} = \dfrac{(1-\alpha)^2}{(k+h-\alpha)(k+1+h-\alpha)(k-\alpha)k}$ $(0 \leqq \alpha < 1)$.

Then we have the following theorems.

**Theorem 2.1.** *If $f(z) \in A_\alpha(n,\theta,h)$, then $f(z) \in T_\alpha^*(n,\theta)$.*

*Proof.*

$$
\begin{aligned}
\sum_{k=n+1}^{\infty}(k-\alpha)a_{k,h} &= \sum_{k=n+1}^{\infty}(k-\alpha)\frac{(1-\alpha)^2}{(k+h-\alpha)(k+1+h-\alpha)(k-\alpha)}\\
&= (1-\alpha)^2\sum_{k=n+1}^{\infty}\left(\frac{1}{k+h-\alpha}-\frac{1}{k+1+h-\alpha}\right)\\
&= (1-\alpha)^2\frac{1}{n+1+h-\alpha}\\
&= \begin{cases} \dfrac{(1-\alpha)^2}{1-\alpha} = 1-\alpha, & h=-n,\\[2mm] \dfrac{(1-\alpha)^2}{n+1+h-\alpha} < \dfrac{(1-\alpha)^2}{1-\alpha} = 1-\alpha, & h>-n. \end{cases}
\end{aligned}
$$

Hence we know that $f(z)$ is an element of $T_\alpha^*(n,\theta)$ by virtue of the theorem C. □

Using Theorem B, we can also prove the following theorem 2.2.

**Theorem 2.2.** *If $g(z) \in B_\alpha(n,\theta,h)$, then $g(z) \in C_\alpha(n,\theta)$.*

In the case of $\theta = 0$, these theorems were proved by Sekine and Yamanaka [6].

**Example 2.1.** *If $f(z) \in A_0(1,\frac{\pi}{4},-1)$, then we have*

$$
\begin{aligned}
f(z) &= z-\sum_{k=2}^{\infty}e^{i(k-1)\frac{\pi}{4}}\frac{1}{(k-1)k^2}z^k\\
&= z-\frac{1+i}{4\sqrt{2}}z^2-\frac{i}{18}z^3+\frac{1-i}{48\sqrt{2}}z^4+\frac{1}{100}z^5\\
&\quad +\frac{1+i}{180\sqrt{2}}z^6+\frac{i}{294}z^7-\frac{1-i}{448\sqrt{2}}z^8-\frac{1}{648}z^9-\cdots
\end{aligned}
\tag{18}
$$

**Example 2.2.** *If $g(z) \in B_0(1,\frac{\pi}{4},-1)$, then we have*

$$
\begin{aligned}
g(z) &= z-\sum_{k=2}^{\infty}e^{i(k-1)\frac{\pi}{4}}\frac{1}{(k-1)k^3}z^k\\
&= z-\frac{1+i}{8\sqrt{2}}z^2-\frac{i}{54}z^3+\frac{1-i}{192\sqrt{2}}z^4+\frac{1}{500}z^5\\
&\quad +\frac{1+i}{1080\sqrt{2}}z^6+\frac{i}{2058}z^7-\frac{1-i}{3584\sqrt{2}}z^8-\frac{1}{5832}z^9-\cdots
\end{aligned}
\tag{19}
$$

We show the images of $|z| \leqq 1$ by the apporoximate expressions for the examples with *Mathematica*. In view of the figures, we can image that the functions of the examples 2.1 and 2.2 are starlike function and convex function, respectively.

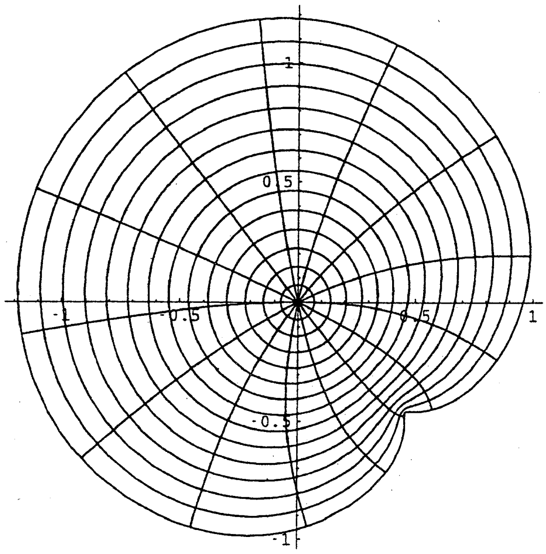


Figure 1: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{9} e^{i(k-1)\frac{\pi}{4}} \frac{1}{(k-1)k^2} x^k$

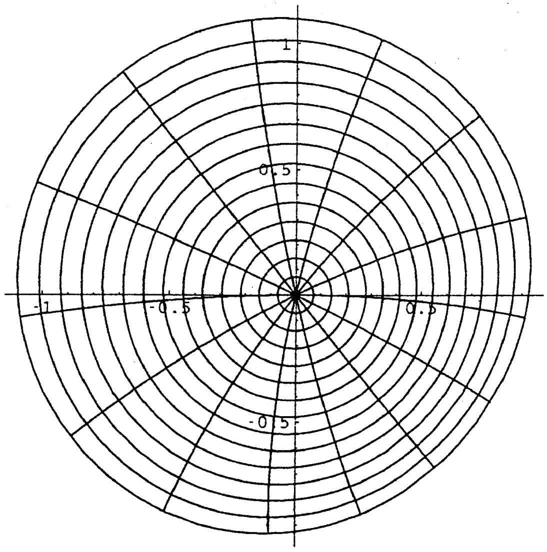


Figure 2: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{9} e^{i(k-1)\frac{\pi}{4}} \frac{1}{(k-1)k^3} x^k$

Further we estimate the functions in $T^*_\alpha(n,\theta)$ and $C_\alpha(n,\theta)$. Let $f(z) \in T^*_0(1,\frac{\pi}{4})$, then by the theorem E we have

$$|z| - \frac{1}{2}|z|^2 \leqq |f(z)| \leqq |z| + \frac{1}{2}|z|^2. \tag{20}$$

And the right-hand equality hold for the function $f(z) = z - \frac{1+i}{2\sqrt{2}}z^2$ on the harf line $z = re^{-(\pi+\frac{\pi}{4})}$, also the left-hand equality on $z = re^{-i(\frac{\pi}{4})}$. By letting $r \longrightarrow 1$ for the function $f(z) = z - \frac{1+i}{2\sqrt{2}}z^2$, we have $\frac{1}{2} \leqq |f(z)| \leqq \frac{3}{2}$.

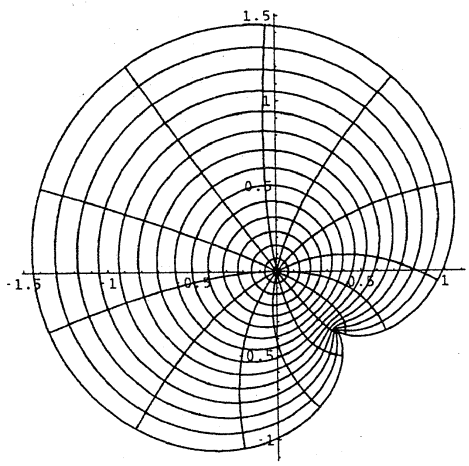


Figure 3: Image of $|z| \leqq 1$ by $f(z) = z - \frac{1+i}{2\sqrt{2}}z^2$

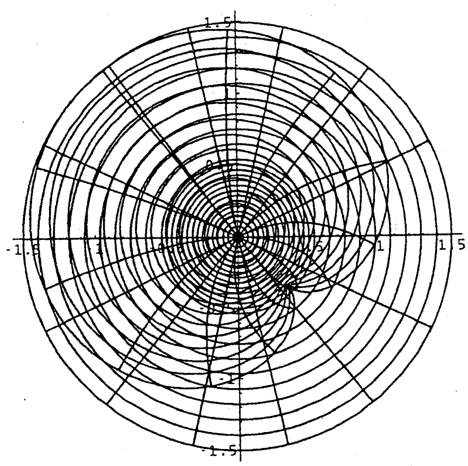


Figure 4: Images of $|z| \leqq 1$ by $z - \frac{1+i}{2\sqrt{2}}z^2$, $\frac{z}{2}$ and $\frac{3}{2}z$

Also, in case of $g(z) \in C_0^*(1, \frac{\pi}{4})$, we have $\frac{3}{4} \leqq |g(z)| \leqq \frac{5}{4}$.

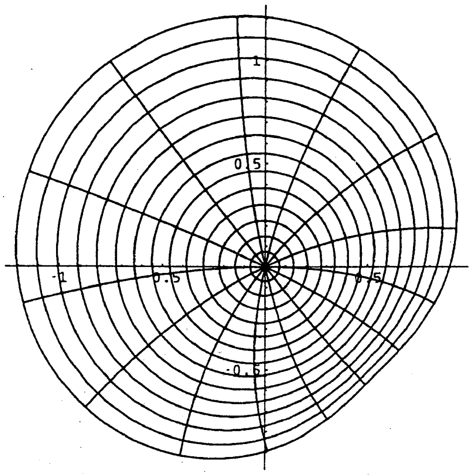


Figure 5: image of $|z| \leqq 1$ by $g(z) = z - \frac{1+i}{4\sqrt{2}}z^2$

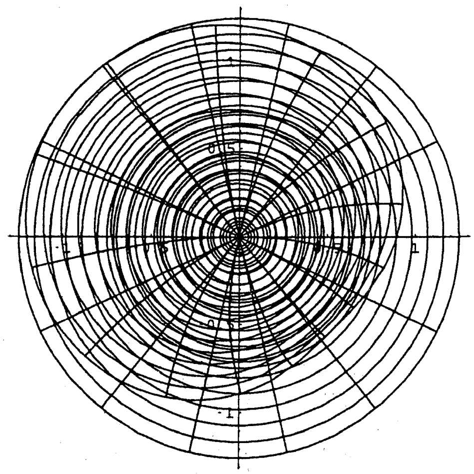


Figure 6: Images of $|z| \leqq 1$ by $z - \frac{1+i}{4\sqrt{2}}z^2$, $\frac{3}{4}z$ and $\frac{5}{4}z$

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