数理解析研究所講究録
1062 巻 1998 年 140-149

On the t distribution and some formula of Bessel functions

高野 勝男

(Katsuo TAKANO)
Department of Mathematics
Faculty of Science, Ibaraki University
Mito-city, Ibaraki 310, JAPAN
e-mail: ktaka@mito.ipc.ibaraki.ac.jp

1 INTRODUCTION

It is well-known that a Cauchy distribution is infinitely divisible. This is clear from the following equality:

$$\int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}dx = e^{-|t|} = (e^{-|t|/n})^n = (\int_{-\infty}^{\infty} e^{ity}\frac{n}{\pi(1+n^2y^2)}dy)^n$$

for every positive integer n. That is, a Cauchy distribution can be expressed as the n-fold convolution of a Cauchy distribution with itself. In this paper we consider a probability distribution whose density is the normed product of Cauchy densities such as a following form,

$$f(a_1,...,a_n;x) = \frac{c}{\prod_{j=1}^{n}(x^2+a_j^2)}, \quad -\infty < x < \infty$$

where $0 < a_1 < a_2 < \cdots < a_n$ and c is a normalized constant. As an interesting example we can raise a formula of Gamma function, $x^2|\Gamma(ix)|^2 = 1/\prod_{n=1}^{\infty}(1+x^2/n^2)$. The density function $f(a_1,...,a_n;x)$ is an approximation of the above right hand side in the sense of weak limit, and we can look on the Student t distribution with degree of freedom $2n-1$ as the degenerate case of the above desity function since it holds that $f(a_1,...,a_n;x)\rightarrow c/(1+x^2)^n$ as $(a_1,...,a_n)\rightarrow(1,1,...,1)$. In Section 2 we show that a distribution with density $f(a_1,a_2;x)$ is infinite divisible. If we let a_1 and a_2 go to 1, we get the Student t

distribution of degree of freedom 3 and from the Lévy measure of the probability distribution with the density $f(a_1, a_2; x)$, we can obtain the Lévy measure of the Student t distribution of degree of freedom 3 with no use of the Bessel functions. In Section 3 we will show the infinite divisibility of a probability distribution whose density is the normed product of three Cauchy densities $f(a_1, a_2, a_3; x)$. In this case we can obtain the Lévy measure of the Student t distribution of degree of freedom 5 with no use of the Bessel functions but general case is unsolved. The Lévy measure of the Student t distribution with any degree of freedom is obtained by E. Grosswald [4] using the Bessel functions. Works related to this paper are [2], [3], [4], [5], [6], [8]. The author was motivated by Bondesson's paper [2], which obtained original analytic results of the infinite divisibility which are connected to Steutel's integral equation and also inspired by Prof. H. M. Srivastava's lecture.

2 A PROBABILITY DISTRIBUTION WHOSE DENSITY IS THE NORMED PRODUCT OF TWO CAUCHY DENSITIES

Let us set $a_1 = b, a_2 = a$ in $f(a_1, a_2; x)$. From $a > b > 0$, we see that $f(a, b; x)$ has a nice property to make calculation simple. That is, $f(a, b; x)$ can be written as

$$\begin{aligned} f(a,b;x) &= \frac{c}{a^2-b^2}\{\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\} \\ &= \frac{c}{a^2-b^2}(\int_0^\infty e^{-(x^2+b^2)t}dt - \int_0^\infty e^{-(x^2+a^2)t}dt) \\ &= \int_0^\infty e^{-tx^2}\frac{c}{a^2-b^2}(e^{-b^2t} - e^{-a^2t})dt \\ &= \int_0^\infty \frac{1}{\sqrt{\pi v}}e^{-x^2/v}\frac{c\sqrt{\pi}}{a^2-b^2}(e^{-b^2/v} - e^{-a^2/v})v^{-3/2}dv. \qquad (1) \end{aligned}$$

From the fact that the last integral is a mixture of the normal distribution, it is sufficient to show the infinite divisibility of the mixture distribution whose density is

$$g(a,b;v) = \frac{c\sqrt{\pi}}{a^2-b^2}(e^{-b^2/v} - e^{-a^2/v})v^{-3/2}, \ v > 0. \qquad (2)$$

The Laplace transform of $g(a, b; v)$ is as follows:

$$\zeta(s) = \int_0^\infty e^{-sv}g(a,b;v)dv = \frac{c\pi}{a^2-b^2}\{\frac{e^{-2b\sqrt{s}}}{b} - \frac{e^{-2a\sqrt{s}}}{a}\}. \qquad (3)$$

By analytic continuation we can extend $\zeta(s)$ to the whole complex plane with cut along the nonpositive real line.

Theorem 1 . *The distribution with density $g(a,b;v)$ is infinitely divisible. Furthermore it is a generalized Gamma convolution and the distribution with density $f(a,b;x)$ is infinitely divisible and self-decomposable.*

Proof. To show the infinite divisibility of the distribution with density $g(a,b;v)$, it suffices to show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty e^{-sx}k(x)dx$$

holds where $k(x)$ is a nonnegative function. By analytic continuation of the above $\zeta(s)$ to the whole complex plane with cut along the nonpositive real line and by the inverse Laplace transform of

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \frac{1}{e^{-2b\sqrt{s}}/b - e^{-2a\sqrt{s}}/a}\left\{-\frac{e^{-2b\sqrt{s}}}{\sqrt{s}} + \frac{e^{-2a\sqrt{s}}}{\sqrt{s}}\right\} \\ &= \int_0^\infty e^{-sx}k(x)dx \end{aligned} \tag{4}$$

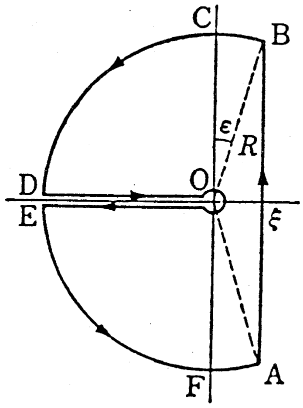
we can get the function of $k(x)$. For simplicity let

$$F(s) = -\zeta'(s)/\zeta(s) = \frac{1}{\sqrt{s}}\frac{1-e^{-2(a-b)\sqrt{s}}}{\frac{1}{b}-\frac{1}{a}e^{-2(a-b)\sqrt{s}}}.$$

Suppose that (4) holds. We will calculate the inverse Laplace transform,

$$k(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\xi-iR}^{\xi+iR} e^{ts}F(s)ds, \quad \xi > 0,\ t > 0. \tag{5}$$

We calculate a contour integration along a curve C like the following figure:



(A) Contour integral along a small circle at O.
From

$$e^{-2(a-b)\sqrt{s}} = e^{-2(a-b)\sqrt{\rho}e^{i\theta/2}}, \text{ for } -\pi < \theta < \pi,$$

we see that

$$\begin{aligned} \oint e^{st}F(s)ds &= \int_\pi^{-\pi} e^{st}\frac{(1-e^{-2(a-b)\sqrt{s}})i\rho e^{i\theta}}{\sqrt{\rho}e^{i\theta/2}(\frac{1}{b}-\frac{1}{a}e^{-2(a-b)\sqrt{s}})}d\theta \\ &= \int_\pi^{-\pi} e^{st}\frac{(1-e^{-2(a-b)\sqrt{s}})}{(\frac{1}{b}-\frac{1}{a}e^{-2(a-b)\sqrt{s}})}i\sqrt{\rho}e^{i\theta/2}d\theta. \end{aligned} \tag{6}$$

By $a > b > 0$ we have

$$|\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{s}}| > \frac{1}{b} - \frac{1}{a}, \ (-\pi \leq \theta \leq \pi),$$

and

$$\oint e^{st}F(s)ds \to 0 \text{ as } \rho \to +0.$$

(B) Integral along BD.
From $s = Re^{i\theta}$, $\sqrt{s} = \sqrt{R}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$, we see that

$$\begin{aligned}\int_{BD} e^{st}F(s)ds &= \int_{\frac{\pi}{2}-\epsilon}^{\pi} \frac{e^{st}(1 - e^{-2(a-b)\sqrt{s}})iRe^{i\theta}}{\sqrt{R}e^{i\theta/2}(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{s}})}d\theta \\ &= i\int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{i\theta/2}\frac{e^{Re^{i\theta}t}(1 - e^{-2(a-b)\sqrt{R}e^{i\theta/2}})}{(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{R}e^{i\theta/2}})}d\theta \\ &+ i\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{i\theta/2}\frac{e^{Re^{i\theta}t}(1 - e^{-2(a-b)\sqrt{R}e^{i\theta/2}})}{(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{R}e^{i\theta/2}})}d\theta. \end{aligned} \quad (7)$$

We see that

$$\begin{aligned}&\int_{\frac{\pi}{2}}^{\pi} \sqrt{R}|e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{tR\cos\theta}d\theta = \int_0^{\frac{\pi}{2}} \sqrt{R}e^{-tR\sin\phi}d\phi \\ \leq \ &\int_0^{\frac{\pi}{2}} \sqrt{R}e^{-2tR\phi/\pi}d\phi = \sqrt{R}[-\frac{\pi}{2tR}e^{-2tR\phi/\pi}]_0^{\frac{\pi}{2}} \\ = \ &\sqrt{R}\{\frac{\pi}{2tR}(-e^{-tR} + 1)\} \to 0 \end{aligned} \quad (8)$$

as $R \to +\infty$. Next, we show that

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} |\sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}|d\theta \to 0 \quad (9)$$

as $R \to \infty$. From the fact that

$$\cos\theta = \cos(\phi + \frac{\pi}{2}) = \sin(-\phi), -\epsilon \leq \phi \leq 0,$$
$$\sin\epsilon = \frac{\xi}{R} \geq \sin(-\phi) \geq 0,$$

we see that

$$\begin{aligned}&\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}|e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{tR\cos\theta}d\theta \\ \leq \ &\int_{-\epsilon}^{0} \sqrt{R}e^{tR\xi/R}d\theta = \sqrt{R}e^{t\xi}\epsilon \\ = \ &e^{t\xi}(\sqrt{R}\sin\epsilon)\frac{\epsilon}{\sin\epsilon} = e^{t\xi}\sqrt{R}\cdot\frac{\xi}{R}\frac{\epsilon}{\sin\epsilon} \to 0 \end{aligned} \quad (10)$$

as $R \to \infty$.
(C) Integrals along DO and OE.
From $s = \rho e^{i\pi}$, $\rho > 0$ on DO and $\sqrt{s} = \sqrt{\rho}e^{i\pi/2} = i\sqrt{\rho} = iy$, $\sqrt{\rho} = y$ we see that

$$\int_{DO} e^{st}F(s)ds = \int_{DO} \frac{e^{st}}{\sqrt{s}}\left(\frac{1 - e^{-2(a-b)\sqrt{s}}}{\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{s}}}\right)ds$$
$$= \int_{\sqrt{R}}^{0} \frac{e^{-ty^2}}{iy} \frac{1 - e^{-2(a-b)iy}}{\frac{1}{b} - \frac{1}{a}e^{-2(a-b)iy}}(-2y)dy = \frac{2}{i}\int_0^{\sqrt{R}} e^{-ty^2} \frac{1 - e^{-2(a-b)iy}}{\frac{1}{b} - \frac{1}{a}e^{-2(a-b)iy}}dy. \quad (11)$$

From $s = \rho e^{-i\pi} = -\rho$, $\rho > 0$, $\sqrt{s} = -i\sqrt{\rho} = -iy$, $\sqrt{\rho} = y$ on OE we see that

$$\int_{OE} e^{st}F(s)ds = \int_0^{\sqrt{R}} \frac{e^{-ty^2}}{-iy}\left(\frac{1 - e^{2(a-b)iy}}{\frac{1}{b} - \frac{1}{a}e^{2(a-b)iy}}\right)(-2y)dy$$
$$= \frac{2}{i}\int_0^{\sqrt{R}} e^{-ty^2} \frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a}e^{i2(a-b)y}}dy, \quad (12)$$

and

$$\frac{1}{2\pi i}\int_{DO+OE} \frac{e^{st}}{\sqrt{s}}\left(\frac{1 - e^{-2(a-b)\sqrt{s}}}{\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{s}}}ds\right.$$
$$\to -\frac{1}{\pi}\int_0^{\infty} e^{-ty^2} \frac{1 - e^{-i2(a-b)y}}{\frac{1}{b} - \frac{1}{a}e^{-i2(a-b)y}}dy - \frac{1}{\pi}\int_0^{\infty} e^{-ty^2} \frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a}e^{i2(a-b)y}}dy \quad (13)$$

as $R \to \infty$. From the Cauchy theorem we see that

$$\text{(the integral along AB)}/(2\pi i) = \frac{1}{2\pi i}\int_{\xi - iR_1}^{\xi + iR_1} e^{st} \frac{1 - e^{-2(a-b)\sqrt{s}}}{\sqrt{s}(1 - e^{-2(a-b)\sqrt{s}})}ds$$
$$\to \frac{1}{\pi}\int_0^{\infty} e^{-ty^2}\left\{\frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a}e^{i2(a-b)y}} + \frac{1 - e^{-i2(a-b)y}}{\frac{1}{b} - \frac{1}{a}e^{-i2(a-b)y}}\right\}dy, \quad (14)$$

where $R_1 = R\cos\epsilon$. Write the expression in the above { } to a fraction. Then

$$numerator = (1 - e^{i2(a-b)y})(\frac{1}{b} - \frac{1}{a}e^{-i2(a-b)y}) + (1 - e^{-i2(a-b)})(\frac{1}{b} - \frac{1}{a}e^{i2a-b)y})$$
$$= \frac{1}{b} - \frac{1}{b}e^{i2(a-b)y} - \frac{1}{a}e^{-i2(a-b)y} + \frac{1}{a} + \frac{1}{b} - \frac{1}{b}e^{-i2(a-b)y} - \frac{1}{a}e^{i2(a-b)y} + \frac{1}{a}$$
$$= 2(\frac{1}{a} + \frac{1}{b})(1 - \cos 2(a-b)y), \quad (15)$$

and

$$denominator = (\frac{1}{b} - \frac{1}{a}e^{i2(a-b)y})(\frac{1}{b} - \frac{1}{a}e^{-i2(a-b)y})$$
$$= \frac{1}{b^2} - \frac{1}{ab}e^{i2(a-b)y} - \frac{1}{ab}e^{-i2(a-b)y} + \frac{1}{a^2}$$
$$= \frac{1}{a^2} + \frac{1}{b^2} - \frac{2}{ab}\cos 2(a-b)y \geq (\frac{1}{b} - \frac{1}{a})^2. \quad (16)$$

Finally we obtain

$$k(t)=\frac{1}{\pi}\int_0^\infty e^{-ty^2}\frac{2(\frac{1}{a}+\frac{1}{b})(1-\cos 2(a-b)y)}{(\frac{1}{a}-\frac{1}{b})^2+\frac{2}{ab}(1-\cos 2(a-b)y)}dy,\ t>0. \tag{17}$$

Since the distribution with density $g(a,b;v)$ is a generalized Gamma convolution, the mixture distribution with density $f(a,b;x)$ is infinitely divisible and self-decomposable. □

Let

$$\pi u(y)=\frac{2(\frac{1}{a}+\frac{1}{b})(1-\cos 2(a-b)y)}{(\frac{1}{a}-\frac{1}{b})^2+\frac{2}{ab}(1-\cos 2(a-b)y)},\ y>0. \tag{18}$$

By the l'Hospital theorem we obtain

$$\lim_{b\to a}\pi u(y)=\lim_{b\to a}\frac{2ab(a+b)(1-\cos 2(a-b)y)}{(a^2+b^2)-2ab\cos 2(a-b)y}=\frac{2^3a^3y^2}{1+2^2a^2y^2}, \tag{19}$$

If $a=\frac{1}{2}$, we have

$$u(y)=\frac{y^2}{\pi(1+y^2)},y>0 \tag{20}$$

and $u(y)$ corresponds to

$$\frac{\pi^2 v}{2}M^2_{1+1/2}(v)=\frac{\pi^2 v}{2}(J^2_{3/2}(v)+Y^2_{3/2}(v))=\frac{\pi(1+v^2)}{v^2},$$

where $J_{3/2}(v)$, $Y_{3/2}(v)$ are the Bessel functions of the first kind and of second kind, respectively, with order 3/2 (cf. [1. 9.2.17]).

3 A PROBABILITY DISTRIBUTION WHOSE DENSITY IS THE NORMED PRODUCT OF THREE CAUCHY DENSITIES

Let us set $a_1=c,\ a_2=b,\ a_3=a,\ c=d$ in $f(a_1,a_2,a_3;x)$. We will show the infinite divisibility of the distribution with density $f(a,b,c;x)$ under the condition, $c=b-h, a=b+h, h>0$. In the same way as the proof of the infinite divisibility of the distribution with density $f(a,b;x)$ we have

$$f(a,b,c;x)=\int_0^\infty\frac{1}{\sqrt{\pi v}}e^{-x^2/v}g(a,b,c;v)dv,$$

where

$$g(a,b,c;v)=d\sqrt{\pi}\{\frac{1}{(b^2-c^2)(a^2-c^2)}e^{-c^2/v}$$
$$-\frac{1}{(a^2-b^2)(b^2-c^2)}e^{-b^2/v}+\frac{1}{(a^2-b^2)(a^2-c^2)}e^{-a^2/v}\}v^{-3/2},$$
$$v>0.$$

The function $f(a,b,c;x)$ can be expressed as the 3-fold integral,

$$\begin{aligned} f(a,b,c;x) &= d\int_0^\infty e^{-tx^2}dt \\ &\cdot \int\int_{u_1\geq 0,\ u_2\geq 0,\ u_1+u_2\leq t} e^{-c^2t-(a^2-c^2)u_1-(b^2-c^2)u_2}du_1du_2. \end{aligned}$$

Hence $g(a,b,c;v)$ is positive on the positive line. The Laplace transform of the mixture density $g(a,b,c;v)$ is as follows:

$$\begin{aligned} \zeta(s) &= d\pi[\frac{1}{(b^2-c^2)(a^2-c^2)c}e^{-2c\sqrt{s}} \\ &- \frac{1}{(a^2-b^2)(b^2-c^2)b}e^{-2b\sqrt{s}} + \frac{1}{(a^2-b^2)(a^2-c^2)a}e^{-2a\sqrt{s}}] \end{aligned}$$

From the above expression we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{numerator}{denominator},$$

$$\begin{aligned} numerator &= \frac{1}{(b^2-c^2)(a^2-c^2)\sqrt{s}}e^{-2c\sqrt{s}} \\ &- \frac{1}{(a^2-b^2)(b^2-c^2)\sqrt{s}}e^{-2b\sqrt{s}} + \frac{1}{(a^2-b^2)(a^2-c^2)\sqrt{s}}e^{-2a\sqrt{s}}, \end{aligned}$$

$$\begin{aligned} denominator &= \frac{1}{(b^2-c^2)(a^2-c^2)c}e^{-2c\sqrt{s}} \\ &- \frac{1}{(a^2-b^2)(b^2-c^2)b}e^{-2b\sqrt{s}} + \frac{1}{(a^2-b^2)(a^2-c^2)a}e^{-2a\sqrt{s}}. \end{aligned}$$

By the inverse Laplace transform of $-\zeta'(s)/\zeta(s)$ we can obtain the function $k(x)$ and the following

Theorem 2 . *The distribution with density $g(a,b,c;v)$ is infinitely divisible. Furthermore it is a generalized Gamma convolution and the distribution with density $f(a,b,c;x)$ is infinitely divisible and self-decomposable.*

We can show that

$$k(t) = \int_0^\infty e^{-ty^2}u(y)dy,$$

with $\pi u(y) = numerator/denominator$, where

$$\begin{aligned} numerator &= (1-\cos 2hy)\{\frac{4}{(b^2-c^2)(a^2-b^2)b} \\ &+ \frac{2}{(b^2-c^2)^2}(\frac{1}{b}+\frac{1}{c}) + \frac{2}{(a^2-b^2)^2}(\frac{1}{a}+\frac{1}{b})\} \\ &+ (\cos 4hy - \cos 2hy)(\frac{1}{a}+\frac{1}{c})\frac{2}{(a^2-b^2)(b^2-c^2)} \geq 0 \qquad (21) \end{aligned}$$

for $y \geq 0$,

$$denominator = \frac{1}{(b^2-c^2)^2}(\frac{1}{c}-\frac{1}{b})^2 + \frac{1}{(a^2-b^2)^2}(\frac{1}{b}-\frac{1}{a})^2$$
$$+\frac{2}{(b^2-c^2)(a^2-b^2)b^2} + (\frac{2}{bc(b^2-c^2)^2} + \frac{2}{ab(a^2-b^2)^2})(1-\cos 2hy)$$
$$-\frac{2}{b(b^2-c^2)(a^2-b^2)}(\frac{1}{c}+\frac{1}{a})\cos 2hy + \frac{2\cos 4hy}{ac(b^2-c^2)(a^2-b^2)}\} > 0, \quad (22)$$

for $y \geq 0$. As we let $h \to +0$, we have

$$\begin{aligned} \pi u(y) &= \lim_{h\to+0} \frac{numerator}{denominator} \\ &= \frac{2^5 b^5 y^4}{3^2 + 2^2 3 b^2 y^2 + 2^4 b^4 y^4}, \ y \geq 0 \end{aligned}$$

If $b = \frac{1}{2}$ the above $u(y)$ is

$$u(y) = \frac{y^4}{\pi(3^2 + 3y^2 + y^4)}, \ y \geq 0, \quad (23)$$

which corresponds to

$$\frac{\pi^2 v}{2} M^2_{2+1/2}(v) = \frac{\pi^2 v}{2}(J^2_{5/2}(v) + Y^2_{5/2}(v)) = \frac{\pi(3^2 + 3v^2 + v^4)}{v^4}.$$

4 AN ANALYTIC RESULT FROM THE t DISTRIBUTION

Lastly we will mention of an analytic result which comes from study of the infinite divisibility of the t distribution. Consider a $d-$dimensional probability distribution with density,

$$\frac{c_1}{(1+|x|^2)^{\alpha+d/2}}, \ x \in R^d,$$

where $\alpha > 0$ and c_1 is a normalized constant. Then we can obtain a following result.

Theorem 3 . *If $\alpha > \frac{1}{2}$, there exists a probability density function $h(\alpha; x)$ such that*

$$\frac{c_2}{(1+|x|^2)^{(d+1)/2}} = \int_{R^d} \frac{c_1}{(1+|x-y|^2)^{\alpha+d/2}} h(\alpha; y)dy,$$

holds where c_2 is a normalized constant. If $0 < \alpha < \frac{1}{2}$, there exists a probability density function such that

$$\frac{c_1}{(1+|x|^2)^{\alpha+d/2}} = \int_{R^d} \frac{c_2}{(1+|x-y|^2)^{(d+1)/2}} h(\alpha; y)dy$$

holds.

Proof is omitted. The characteristic function of $h(\alpha; x)$ is

$$\begin{aligned}\phi(t) &= \exp\{\int_{R^d-\{0\}} (e^{tx} - 1 - \frac{itx}{1+|x|^2}) \\ &(\frac{2}{|x|^d} \int_0^\infty \frac{1}{\pi}(1 - \frac{2}{\pi v(J_\alpha^2(v) + Y_\alpha^2(v))}) L_{d/2}(v|x|)dv)dx\end{aligned}$$

if $\alpha > \frac{1}{2}$ and the integrand in the above integral corresponding to the Levy measure changes sign if $0 < \alpha < \frac{1}{2}$, where $J_\alpha(v)$ and $Y_\alpha(v)$ are Bessel functions of first and second kinds, respectively, with order α and the function $L_{d/2}(v)$ is $L_{d/2}(v) = (2\pi)^{-d/2} v^{d/2} K_{d/2}(v)$, where $K_{d/2}(v)$ is the Bessel function of third kind with order $d/2$.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, New York, Dover, 1970.

[2] L. Bondesson, On the infinite divisibility of the half-Cauchy and other decreasing densities and probability functions on the nonnegative line, *Scand. Acturial J.*, (1985), 225-247.

[3] ——, Generalized Gamma Convolutions and Related Classes of Distributions and Densities, *Lecture Note in Statistics,* 76, Springer-Verlag, 1992

[4] E. Grosswald, The Student t-distribution of any degree of freedom is infinitely divisible, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* 36 (1976), 103–109.

[5] C. Halgreen, Self-decomposability of the generalized inverse Gaussian distribution and hyperbolic distributions, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 47 (1979), 13–17.

[6] D. H. Kelker, Infinite divisibility and variance mixtures of the normal distribution, *Ann. Math. Statist.*, 42 (1971), 802–808.

[7] G. K. Kristiansen, A proof of Steutel's conjecture, *The Annals of Probability,* 22 (1994), 442–452.

[8] F. W. Steutel, Preservation of infinite divisibility under mixing and related topics, *Math. Centre Tracts, Math. Centre, Amsterdam,* 33. 1970

[9] O. Thorin, On the infinite divisibility of the Pareto distribution, *Scand. Acturial. J.*, (1977), 31–40.

[10] G. N. Watson, A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, 1966