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INTEGRAL MEANS OF THE FRACTIONAL DERIVATIVE OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. By using the definition of fractional derivative (cf., [2]), we investigate the sharp integral means inequalities of the fractional derivatives of univalent functions with negative coefficients and extend the sharp results of H. Silverman [5, Theorem 2.2].

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Then a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be in the class $\mathcal{K}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ whose functions may be represented by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.3}$$

Silverman [4] showed that f of the form (1.3) is in $\mathcal{T}$ if and only if $\sum_{k=2}^{\infty} k a_k \leq 1$, and that the extreme points of $\mathcal{T}$ are

$$f_1(z) = z \quad \text{and} \quad f_m(z) = z - z^m/m, \; m = 2, 3, \cdots. \tag{1.4}$$

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Further a function f of the form (1.3) is in $\mathcal{C} = \mathcal{T} \cap \mathcal{K}$ if and only if $\sum_{k=2}^{\infty} k^2 a_k \leq 1$, and that the extreme points of $\mathcal{C}$ are $g_1(z) = z$ and $g_2(z) = z - z^m/m^2$ $(m = 2, 3, \cdots)$.

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ so that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$ and $g(z) = h(w(z))$, we denote this subordition by $g(z) \prec h(z)$.

Many essentially equivalent definition of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (*cf., e.g.*, [3], [6, p 45] and [7]). We find it to be convenient to recall here the following definition which were used recently by Owa [2] (and by Srivastava and Owa [7]).

Definition 1. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(z)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \leq \lambda < 1), \tag{1.5}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring for $\log(z-\zeta)$ to be real for $z > \zeta$.

Definition 2. Under the hypotheses of Definition 1, the fractional derivative of order $n + \lambda$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}). \tag{1.6}$$

In [5] it is proven that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\beta} d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^{\beta} d\theta \tag{1.7}$$

for all $f \in \mathcal{T}$, $\beta > 0$ and $0 < r < 1$. In this paper, by using the fractional derivative, we prove that

$$\int_0^{2\pi} \left|\mathcal{D}_z^{\lambda} f(re^{i\theta})\right|^{\beta} d\theta \leq \int_0^{2\pi} \left|\mathcal{D}_z^{\lambda} f_2(re^{i\theta})\right|^{\beta} d\theta \tag{1.8}$$

for all $f \in \mathcal{T}$, $\beta > 0$, $0 < r < 1$ and $0 \leq \lambda < 1$. We also obtain the integral means inequality for $\mathcal{D}_z^{n+\lambda} f(z)$ $(n = 1, 2)$ if $f \in \mathcal{C}$(or $\mathcal{T}$).

2. Main Results

The following result will be required in our investigation.

Lemma. (Littlewood [1]) *If f and g are analytic in $\mathcal{U}$ with $g \prec f$, then, for $\beta > 0$ and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\beta d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\beta d\theta. \tag{2.1}$$

Applying the above lemma, we prove

Theorem 1. *Let $\beta > 0$ and $f_2(z)$ is defined by (1.4). If $f \in \mathcal{T}$, then for $z = re^{i\theta}$ and* $0 < r < 1$,

(i) $\int_0^{2\pi} \left|\mathcal{D}_z^\lambda f(z)\right|^\beta d\theta \le \int_0^{2\pi} \left|\mathcal{D}_z^\lambda f_2(z)\right|^\beta d\theta \qquad (0 \le \lambda < 1)$

(ii) $\int_0^{2\pi} \left|\mathcal{D}_z^{2+\lambda} f(z)\right|^\beta d\theta \le \int_0^{2\pi} \left|\mathcal{D}_z^{2+\lambda} f_2(z)\right|^\beta d\theta \qquad (0 < \lambda < 1)$.

Proof. We prove (i). The proof of (ii) is similar and will be omitted. If $f(z) = z - \sum_{k=2}^\infty a_k z^k$ $(a_k \ge 0)$, then

$$\mathcal{D}_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \sum_{k=2}^\infty \Phi(k) k a_k z^{k-1}\right),$$

where

$$\Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \qquad (k \ge 2). \tag{2.2}$$

Note that $\Phi(k)$ is a non-increasing function of k,

$$0 < \Phi(k) \le \Phi(2) = \frac{1}{2-\lambda}. \tag{2.3}$$

Since

$$\mathcal{D}_z^\lambda f_2(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{1}{2-\lambda} z\right),$$

we must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^\infty \Phi(k) k a_k z^{k-1}\right|^\beta d\theta \le \int_0^{2\pi} \left|1 - \frac{1}{2-\lambda} z\right|^\beta d\theta.$$

By Lemma, it sufficies to prove that

$$1 - \sum_{k=2}^\infty \Phi(k) k a_k z^{k-1} \prec 1 - \frac{1}{2-\lambda} z.$$

Setting

$$1-\sum_{k=2}^{\infty}\Phi(k)ka_kz^{k-1}=1-\frac{w(z)}{2-\lambda}. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$|w(z)|\le\left|\sum_{k=2}^{\infty}(2-\lambda)\Phi(2)ka_kz^{k-1}\right|\le|z|\sum_{k=2}^{\infty}ka_k\le|z|.$$

This completes the proof of (i).

Remark. If $\lambda=0$ in (i) of Theorem 1, then it would immediately yield the result of Silverman [5, Theorem 2.2].

For the fractional derivative of order $1+\lambda$, we have

Theorem 2. *If $f\in\mathcal{C}$ and $\beta>0$, then for $z=re^{i\theta}$ and $0<r<1$,*

(i) $\int_0^{2\pi}\left|\mathcal{D}_z^{1+\lambda}f(z)\right|^\beta d\theta\le\int_0^{2\pi}\left|\mathcal{D}_z^{1+\lambda}f_2(z)\right|^\beta d\theta \qquad (0\le\lambda<1)$

(ii) $\int_0^{2\pi}\left|\mathcal{D}_z^{1+\lambda}f(z)\right|^\beta d\theta\le\int_0^{2\pi}\left|\mathcal{D}_z^{2+\lambda}g_2(z)\right|^\beta d\theta \qquad (0\le\lambda\le 2/3).$

Proof. (i) From the definition (1.6), we have

$$\mathcal{D}_z^{1+\lambda}f(z)=\frac{z^{-\lambda}}{\Gamma(1-\lambda)}\left(1-\sum_{k=2}^{\infty}\Psi(k)k(k-1)a_kz^{k-1}\right), \tag{2.5}$$

where

$$\Psi(k)=\frac{\Gamma(k-1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} \qquad (k\ge 2).$$

Note that $0<\Psi(k)\le\Psi(2)=1/(1-\lambda)$.
Since

$$\mathcal{D}_z^{1+\lambda}f_2(z)=\frac{z^{-\lambda}}{\Gamma(1-\lambda)}\left(1-\frac{1}{1-\lambda}z\right),$$

it suffices to show that

$$1-\sum_{k=2}^{\infty}\Psi(k)k(k-1)a_kz^{k-1}\prec 1-\frac{1}{1-\lambda}z.$$

Setting

$$1-\sum_{k=2}^{\infty}\Psi(k)k(k-1)a_k z^{k-1}=1-\frac{w(z)}{1-\lambda},$$

$$|w(z)|\le\left|\sum_{k=2}^{\infty}k(k-1)a_k z^{k-1}\right|\le|z|\sum_{k=2}^{\infty}k^2 a_k\le|z|.$$

By Lemma, the proof of (i) is completed.

(ii) Making use of (1.6) and (2.5), we obtain

$$\mathcal{D}_z^{1+\lambda}f(z)=\frac{z^{-\lambda}}{\Gamma(1-\lambda)}\left(1-\sum_{k=2}^{\infty}\Theta(k)k^2 a_k z^{k-1}\right),$$

where

$$\Theta(k)=\frac{\Gamma(k)\Gamma(1-\lambda)}{k\Gamma(k-\lambda)}\qquad(k\ge 2).$$

We note that $0<\Theta(k)\le\Theta(2)=1/2(1-\lambda)$ for $0\le\lambda\le 2/3$. Thus the proof of (ii) is much akin to that of (i), and we omit the details involved.

Denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, $0\le\alpha<1$, the subclasses of $\mathcal{T}$ that are, respectively, starlike of order α and convex of order α. In [4], Silverman showed that $f\in\mathcal{T}^*(\alpha)$ if and only if $\sum_{k=2}^{\infty}((k-\alpha)/(1-\alpha))a_k\le 1$ and $f\in\mathcal{C}(\alpha)$ if and only if $\sum_{k=2}^{\infty}(k(k-\alpha)/(1-\alpha))a_k\le 1$. In addition, the extreme points of $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are $h_m(z)=z-((1-\alpha)/(m-\alpha))z^m$ and $k_m(z)=z-((1-\alpha)/m(m-\alpha))z^m$ for $m\ge 2$.
For the cases of $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, the proof is much akin to that of Theorem 1 and Theorem 2, and we omit the details involved.

Theorem 3. (i) *If $f\in\mathcal{T}^*(\alpha)$ and $\beta>0$, then for $0<r<1$,*

$$\int_0^{2\pi}\left|\mathcal{D}_z^{\lambda}f(re^{i\theta})\right|^{\beta}d\theta\le\int_0^{2\pi}\left|\mathcal{D}_z^{\lambda}h_2(re^{i\theta})\right|^{\beta}d\theta\qquad(0\le\lambda<1)$$

and

$$\int_0^{2\pi}\left|\mathcal{D}_z^{2+\lambda}f(re^{i\theta})\right|^{\beta}d\theta\le\int_0^{2\pi}\left|\mathcal{D}_z^{2+\lambda}h_2(re^{i\theta})\right|^{\beta}d\theta\qquad(0<\lambda<1).$$

(ii) *If $f\in\mathcal{C}(\alpha)$ and $\beta>0$, then for $0<r<1$,*

$$\int_0^{2\pi}\left|\mathcal{D}_z^{\lambda}f(re^{i\theta})\right|^{\beta}d\theta\le\int_0^{2\pi}\left|\mathcal{D}_z^{\lambda}k_2(re^{i\theta})\right|^{\beta}d\theta\qquad(0\le\lambda<1),$$

$$\int_0^{2\pi} \left|\mathcal{D}_z^{1+\lambda} f(re^{i\theta})\right|^\beta d\theta \leq \int_0^{2\pi} \left|\mathcal{D}_z^{1+\lambda} h_2(re^{i\theta})\right|^\beta d\theta \qquad (0 \leq \lambda < 1),$$

$$\int_0^{2\pi} \left|\mathcal{D}_z^{1+\lambda} f(re^{i\theta})\right|^\beta d\theta \leq \int_0^{2\pi} \left|\mathcal{D}_z^{1+\lambda} k_2(re^{i\theta})\right|^\beta d\theta \qquad (0 \leq \lambda \leq 2/3)$$

and

$$\int_0^{2\pi} \left|\mathcal{D}_z^{2+\lambda} f(re^{i\theta})\right|^\beta d\theta \leq \int_0^{2\pi} \left|\mathcal{D}_z^{2+\lambda} k_2(re^{i\theta})\right|^\beta d\theta \qquad (0 < \lambda < 1).$$

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