数理解析研究所講究録
1063 巻 1998 年 104-110

Weakly Q-polynomial distance-regular graphs

TOMIYAMA, Masato
冨山正人

Department of General Education
Ishikawa National College of Technology
Tsubata, Kahoku-gun, Ishikawa 929-0392, JAPAN

tomiyama@ishikawa-nct.ac.jp

1 Results

This note is based on the author's paper [10]. In this section, we will state our results. In the next section, we will review some definitions and basic concepts. For more background information, the reader may refer to Bannai and Ito [1], Brouwer, Cohen and Neumaier [2] or Godsil [3].

Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.

Let E and F denote nontrivial primitive idempotents of Γ.

In [6], Pascasio investigated the situation that $E \circ F$ is a scalar multiple of a primitive idempotent H of Γ.

She showed this occurs exactly when Γ is either bipartite or tight (in the sense of Jurišić, Koolen and Terwilliger [4]). Moreover, she showed that at least one of E and F is equal to E_d if Γ is bipartite, and that E and F are a permutation of E_1 and E_d if Γ is tight. See [6, Theorems 3,5, 3.6 and 3.7]. If Γ is bipartite, Lang obtained an inequality involving the cosines of E, and showed that equality is closely related to Γ being Q-polynomial with respect to E. See [5]. If Γ is tight, Pascasio obtained similar inequalities involving the cosines of E, and showed that again equality is closely related to Γ being Q-polynomial with respect to E. See [7].

In this note, we investigate a slightly more general situation. Let E denote a nontrivial primitive idempotent of Γ and let $F \in \{E_1, E_d\}$. Our situation is that there exists a primitive idempotent H of Γ such that $E \circ F$ is a linear combination of F and H. Our main purpose is to obtain the above inequalities under our general assumption, and to show that again equality is closely related to Γ being Q-polynomial with respect to E.

Theorem 1.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E denote a nontrivial primitive idempotent of Γ and let $F \in \{E_1, E_d\}$. Assume there exists a primitive idempotent H of Γ such that $E \circ F$ is a linear combination of F and H. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with E.*

(i) *Assume $F = E_1$. Then*

$$(\sigma - \sigma_{i+1})(\sigma - \sigma_{i-1}) \leq (\sigma_2 - \sigma_i)(1 - \sigma_i) \qquad (1 \leq i \leq d-1).$$

(ii) *Assume $F = E_d$. Then*

$$(\sigma - \sigma_{i+1})(\sigma - \sigma_{i-1}) \geq (\sigma_2 - \sigma_i)(1 - \sigma_i) \qquad (1 \leq i \leq d-1).$$

Our assumption $F \in \{E_1, E_d\}$ is quite natural. As we reviewed before, Pascasio proved $F \in \{E_1, E_d\}$ always holds if Γ is either bipartite or tight.

Theorem 1.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E denote a nontrivial primitive idempotent of Γ and let $F \in \{E_1, E_d\}$. Assume there exists a primitive idempotent H of Γ such that $E \circ F$ is a linear combination of F and H. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with E. Then the following are equivalent.*

(i) *Γ is Q-polynomial with respect to E.*

(ii) *$\sigma_i \neq 1$ $(1 \leq i \leq d)$, and $(\sigma - \sigma_{i+1})(\sigma - \sigma_{i-1}) = (\sigma_2 - \sigma_i)(1 - \sigma_i) \qquad (3 \leq i \leq d-1)$.*

(iii) *$\sigma_i \neq 1$ $(1 \leq i \leq d)$, and $(\sigma - \sigma_4)(\sigma - \sigma_2) = (\sigma_2 - \sigma_3)(1 - \sigma_3)$.*

Suppose (i)–(iii) *hold. Then Γ is Q-polynomial with respect to the ordering $E_0, E, \ldots, H, F$ of primitive idempotents.*

Assume Γ is primitive. Then we can eliminate the condition $\sigma_i \neq 1$ $(1 \leq i \leq d)$.

Theorem 1.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let E denote a nontrivial primitive idempotent of Γ and let $F \in \{E_1, E_d\}$. Assume there exists a primitive idempotent H of Γ such that $E \circ F$ is a linear combination of F and H. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with E. Then the following are equivalent.*

(i) *Γ is Q-polynomial with respect to E.*

(ii) $(\sigma - \sigma_{i+1})(\sigma - \sigma_{i-1}) = (\sigma_2 - \sigma_i)(1 - \sigma_i) \qquad (3 \leq i \leq d-1)$.

(iii) $(\sigma - \sigma_4)(\sigma - \sigma_2) = (\sigma_2 - \sigma_3)(1 - \sigma_3)$.

Suppose (i)–(iii) *hold. Then Γ is Q-polynomial with respect to the ordering $E_0, E, \ldots, H, F$ of primitive idempotents.*

Let Γ denote a distance-regular graph with diameter $d \geq 3$. Assume Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Then Pascasio showed that $a_d = 0$ if and only if $q^d_{1d} = 0$, that is, $E_1 \circ E_d$ is a scalar multiple of E_{d-1}. See [7, Corollary 4.3]. We generalize her result.

Theorem 1.4 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E and F denote nontrivial primitive idempotents of Γ. Assume there exists a primitive idempotent H of Γ such that $E \circ F$ is a linear combination of F and H. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with E. Assume further $\sigma_d \neq 1$. Then the following are equivalent.*

(i) $a_d = 0$.

(ii) *$E \circ F$ is a scalar multiple of H.*

Suppose (i)(ii) *hold. Then Γ is either bipartite or tight.*

2 Definitions

Let X denote a nonempty, finite set. Then let $Mat_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of all matrices over $\mathbb{R}$ with rows and columns indexed by X, and let $V := \mathbb{R}^X$ denote the $\mathbb{R}$-vector space consisting of all column vectors over $\mathbb{R}$ with coordinates indexed by X. We observe $Mat_X(\mathbb{R})$ acts on V by left multiplication. We endow V with the inner product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V), \tag{1}$$

where ^{t} denotes transposition. We write $\|u\|^2 := \langle u, u\rangle$ $(u \in V)$. For each $x \in X$, let $\hat{x}$ denote the vector in V with a one in the x coordinate and zeros in all other coordinates. We note $\{\hat{x}|x \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, distance function ∂ and diameter $d := \max\{\partial(x,y)|x, y \in X\}$. We say Γ is *distance-regular* whenever, for all integers h, i, j $(0 \leq h, i, j \leq d)$ and for all vertices $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\{z \in X|\partial(x,z) = i, \partial(y,z) = j\}|$$

is independent of x and y. We refer to p^h_{ij} $(0 \leq h, i, j \leq d)$ as the *intersection numbers* of Γ. Observe $p^h_{ij} = p^h_{ji}$ $(0 \leq h, i, j \leq d)$. For notational convenience, set

$$c_i := p^i_{1i-1} \quad (1 \leq i \leq d), \qquad a_i := p^i_{1i} \quad (0 \leq i \leq d), \qquad b_i := p^i_{1i+1} \quad (0 \leq i \leq d-1),$$

and define $c_0 := 0$ and $b_d := 0$. Observe $b_i > 0$ $(0 \leq i \leq d-1)$, $c_i > 0$ $(1 \leq i \leq d)$ and $c_i + a_i + b_i = k$ $(0 \leq i \leq d)$, where $k := b_0$.

In the following, let Γ denote a distance-regular graph with diameter $d \geq 3$. For each integer i $(0 \leq i \leq d)$, let A_i denote the matrix in $Mat_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We refer to A_i as the *ith distance matrix* of Γ. We abbreviate $A = A_1$. Observe

$$\begin{aligned} A_0 &= I, \\ A_0 + A_1 + \cdots + A_d &= J, \\ A_i^t &= A_i \qquad\qquad (0 \leq i \leq d), \\ A_iA_j &= \sum_{h=0}^{d} p^h_{ij} A_h \qquad (0 \leq i, j \leq d), \end{aligned}$$

where I and J denote the identity matrix and the all ones matrix, respectively. Therefore, the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative subalgebra M of $Mat_X(\mathbb{R})$. We refer to M as the *Bose-Mesner algebra of* Γ. By [1, pp. 59, 64], M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$\begin{aligned} E_0 &= |X|^{-1}J, \\ E_0 + E_1 + \cdots + E_d &= I, && (2) \\ E_i^t &= E_i \qquad (0 \leq i \leq d), && (3) \\ E_iE_j &= \delta_{ij}E_i \qquad (0 \leq i, j \leq d). && (4) \end{aligned}$$

We refer to $E_0, E_1, \ldots, E_d$ as the *primitive idempotents* of Γ, and E_0 is called the *trivial idempotent*. Observe from (1), (3) and (4), for all primitive idempotents E,

$$(E)_{xy} = \langle E\hat{x}, E\hat{y} \rangle \quad (x, y \in X). \tag{5}$$

Since $E_0, E_1, \ldots, E_d$ is a basis for the Bose-Mesner algebra M, there exist $\theta_0, \theta_1, \ldots, \theta_d \in \mathbb{R}$ such that

$$A = \sum_{i=0}^{d} \theta_i E_i. \tag{6}$$

It is known $\theta_0, \theta_1, \ldots, \theta_d$ are distinct and $\theta_0 = k > \theta_i \geq -k$ $(1 \leq i \leq d)$. See [1, p. 197]. Observe from (4) and (6),

$$E_i A = A E_i = \theta_i E_i \qquad (0 \leq i \leq d). \tag{7}$$

We refer to θ_i as the *eigenvalue of* Γ *associated with* E_i, and θ_0 is called the *trivial eigenvalue of* Γ. For each integer i $(0 \leq i \leq d)$, let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i).

By [1, p. 59], the Bose-Mesner algebra M is closed under the entry wise matrix product $\circ$. Therefore, for all integers i, j $(0 \leq i, j \leq d)$, there exist $q_{ij}^h \in \mathbb{R}$ $(0 \leq h \leq d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q_{ij}^h E_h. \tag{8}$$

We refer to q_{ij}^h $(0 \leq h, i, j \leq d)$ as the *Krein parameters* of Γ. Observe $q_{ij}^h = q_{ji}^h$ $(0 \leq h, i, j \leq d)$. By [1, p. 67], it is known that

$$q_{ij}^h = 0 \quad \text{if and only if} \quad q_{ih}^j = 0 \qquad (0 \leq h, i, j \leq d). \tag{9}$$

Let E denote a primitive idempotent of Γ with corresponding eigenvalue θ, and let m denote the multiplicity of E. Since $A_0, A_1, \ldots, A_d$ form a basis for the Bose-Mesner algebra M of Γ, there exist $\sigma_0, \sigma_1, \ldots, \sigma_d \in \mathbb{R}$ such that

$$E = |X|^{-1} m \sum_{i=0}^{d} \sigma_i A_i. \tag{10}$$

We refer to $\sigma_0, \sigma_1, \ldots, \sigma_d$ as the *cosine sequence of* Γ *associated with* E (or θ). We abbreviate $\sigma = \sigma_1$. Observe from (5) and (10) that, for all vertices $y, z \in X$ with $\partial(y, z) = i$ $(0 \leq i \leq d)$,

$$\langle E\hat{y}, E\hat{z} \rangle = m|X|^{-1} \sigma_i. \tag{11}$$

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Fix a vertex $x \in X$. For each integer i $(0 \leq i \leq d)$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{R})$ with y, y entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad (y \in X). \tag{12}$$

We recall the bipartite graphs, antipodal graphs and primitive graphs. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. We say Γ is said to be *bipartite* whenever $a_i = 0$ for all integers i $(0 \leq i \leq d)$. Define a binary relation $\sim$ on X as follows:

$$x \sim y \quad \text{if and only if} \quad \partial(x, y) \in \{0, d\} \qquad (x, y \in X).$$

Then we say Γ is *antipodal* whenever $\sim$ is an equivalence relation. We say Γ is *primitive* whenever Γ is neither bipartite nor antipodal.

Let Γ denote a distance-regular graph with diameter $d \geq 3$. An ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of Γ is said to be *Q-polynomial* whenever, for all integers h, j $(0 \leq h, j \leq d)$, the Krein parameters of Γ satisfy

$$q_{1j}^h = 0 \quad \text{if} \quad |h - j| > 1 \qquad \text{and} \qquad q_{1j}^h \neq 0 \quad \text{if} \quad |h - j| = 1.$$

Observe from (8), $E_1 \circ E_d \in \operatorname{Span}(E_d, E_{d-1})$ in this case. Let E denote a nontrivial idempotent of Γ with associated eigenvalue θ. We say Γ is *Q-polynomial with respect to E (or θ)* if there exists a Q-polynomial ordering $E_0, E_1, E_2, \ldots, E_d$ of the primitive idempotents such that $E_1 = E$.

References

[1] Bannai, E. and Ito, T.: Algebraic Combinatorics I: Association Schemes. (Benjamin-Cummings Lecture Note Ser. 58) Benjamin-Cummings Menlo Park CA 1984.

[2] Brouwer, A.E., Cohen, A.M. and Neumaier, A.: Distance-Regular Graphs. Springer New York 1989.

[3] Godsil, C.D.: Algebraic Combinatorics. Chapman and Hall New York 1993.

[4] Jurišić, A., Koolen, J., and Terwilliger, P.M.: Tight distance-regular graphs (preprint).

[5] Lang, M.: A new inequality for bipartite distance-regular graphs (preprint).

[6] Pascasio, A.A.: Tight distance-regular graphs and their primitive idempotents (preprint).

[7] Pascasio, A.A.: Tight Distance-regular graphs and the Q-polynomial property (preprint).

[8] Terwilliger, P.M.: The subconstituent algebra of an association scheme, I. Journal of Combin. 1(4). 363-388 (1992).

[9] Terwilliger, P.M., A new inequality for distance-regular graphs, Discrete Math. 37. 319-332 (1995).

[10] Tomiyama, M., A note on the primitive idempotents of distance-regular graphs (preprint).