数理解析研究所講究録
1066 巻 1998 年 201-204

On the Number States and the Annihilation/Creation Operators Related to the Continuous Wavelet Transformation

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Key Words: Continuous wavelet transform, Cauchy wavelet, Eigenfunction system of linear operator, Annihilation and Creation operators, Associated Laguerre polynomial, Number states

1 Introduction

As is well-known, Wavelet systems used in the continuous wavelet transformation [1-4] are non-orthogonal over-complete wava-packet systems with continuous-valued parameters a and b. One of the over-complete wavelet systems of this kind is made of Cauchy wavelets, whose basic wavelet function is $\frac{(Const.)}{(t-i)^{k+1}}$ (k is a positive integer). This Cauchy wavelet system is equivalent to the eigenfunction system of the linear operator $Q-ikP^{-1}$[5-8] where Q and P are the position-coordinate operator and the momentum operator used in quantum mechanics. This operator is just corresponding mathematically to the operator $T+kJ$ where T denotes the multiplication operator by the time t and J denotes the integral operator in signel processing. Since this operator is not self-adjoint (and this operator and its adjoint do not commute), the eigenfunction system is not orthogonal, and the over-complete-ness of Cauchy wavelet system is hence derived.

This relation is very analogous to the relation between the short-time Fourier transform(STFT) or Gabor transform with the Gaussian window function and the eigenfunction system of the boson annihilation operator $(Q+iP)/\sqrt{2}$ in quantum mechanics. The eigenstates of this operator are the coherent states. In this case, it is well known that this operator is the step-down operator (down-ladder) of the eigenfunction of the boson number operator $n\equiv(Q^2+P^2-I)/2$ (I: identity operator). In quantum mechanics, this important relation is interpreted as the 'annihilation' of bosons. Similarly, the adjoint of the annihilation is interpreted as the 'creation'. The wavefunction system of the eigenstates of the number operator is an orthogonal function system, which is equivalent to the orthogonal system of Gaussian-weighted Hermite polynomials, as is well known.

In this paper, the similar relation to this will be investigated for the Cauchy wavelet case where the operator $Q+ikP^{-1}$ itself is not but a function of $Q+ikP^{-1}$ is the step-down operator. What is the analogue to the 'number operator' will be shown there. The eigenfunctions of this analogue are closely related to the associated Laguerre polynomials, as will be shown later. Possibility of some applications in signal processing will be discussed also.

2 Eigenfunction system of $Q - ikP^{-1}$

Let Q and P be the position-coodinate operator and the momentum operator which satisfy $[Q, P] = iI$ (I: identity op.). For a fixed positive integer k, define the operator

$$A_k \equiv Q - ikP^{-1}. \qquad (1)$$

Because A_k has complex eigenvalues, and the eigenvectors are not orthogonal. In fact, it is easily shown that the eigenfunction in the position-coordinate representation of this operator with the eigenvalue α is

$$h_k^{(\alpha)}(x) \equiv {}_Q\langle x|\alpha\rangle_{A_k} = \frac{G_k^{(\alpha)}}{(x-\alpha)^{k+1}} \qquad (2)$$

with appropriate constant $G_k^{(\alpha)}$, and the eigenfunction system is over-complete[8]. For non-real α, this function is a complex-valued square-integrable wavepacket localized almost around $t = Re\ \alpha$.

Let b be the real part of the eigenvalue α and a be the imaginary part of α. Then, by choosing

$$G_k^{(\alpha)} = i^{k+1} 2^k \sqrt{\frac{(k!)^2 |a|^{2k+1}}{\pi (2k)!}} \qquad (3)$$

so that the eigenfunctions may be normalized, we have

$$h_k^{(b+ia)}(x) = \frac{1}{|a|^{1/2}} h_k^{(i)}(\frac{x-b}{a}) , \qquad (4)$$

because $x - \alpha = a\ (\frac{x-b}{a} - i)$. This relation shows that the eigenfunction system of the operator A_k defined in (2) is just a wavelet system [1-4] with continuous parameters, and that the real part of the eigenvalue is corresponding to the scale parameter a and the imaginary part is corresponding to the time-shift parameter b. This wavelet system is often called Cauchy wavelet sustem. For real-valued signals, we can use the real part or the imaginary part of $h_k^{(b+ia)}$.

It is well known that the Fourier tranform of the Cauchy wavelet has the support only in the positive-frequency part (if $a < 0$) or only in the negative-frequency part (if $a > 0$). Using this property, we can separate the eigenvalue domain in two part by the real axis, and discuss the analytic signal component (or positive-frequency component) of a signal and the negative-frequency component saparately. In quantum mechanins, these are interpreted as the positive/negative-momentum components.

3 Annihilation Operator and Number Operator in Wavalet Version

Define the operators

$$a_{k\pm} \equiv (A_k \mp iI)^{-1} (A_k \pm iI) \qquad (+,-). \ (5)$$

Because the eigenfunction of a operator with the eigenvalue λ is also the eigenfunction of the operator obtained by substituting that operator into a function $f(\cdot)$ with the eigenvalue $f(\lambda)$, the eigenfunctions (in the position representation) of the operator $a_{k\pm}$ with the eigenvalue $(\alpha \pm 1)/(\alpha \mp 1)$ is the Cauchy wavelet $h_k^{(\alpha)}(t)$. (NB. a_{k+}/a_{k-} is not singular for the positive/negative-momentum component of a signal.)

In this section, we will show that $a_{k\pm}$ whose eigenfunction system is the Cauchy wavelet system is the step-up operator of the orthogonal eigenfunction system of a hermitian operator and $a_{k,\pm}^\dagger$ is its step-up operator.

First, the analogue to the number operator is defined in wavelet version as follows; Define, by using A_k,

$$N_{k\pm} \equiv \mp\frac{1}{2}(A_k \pm iI)^\dagger\ P\ (A_k \pm iI) \qquad (6)$$

We restrict the domain of N_{k+} to the positive-momentum components and the negative-momentum components. $N_{k\pm}$ is hermitian, and, as will be shown below, $N_{k\pm}$ has the eigenvalues $0, 1, 2, 3, ...$

In the special case with $k = 1$, the operator $\pm 2N_{k\pm} \pm 3I$ is mathematically equivalent to the Hamiltonian given in p.41 of

Daubechies' textbook[3]. For general k, the corresponding 'Hamiltonian' in our notation is defined by

$$H_k \equiv QPQ + k^2 P^{-1} + P \ , \tag{7}$$

and the relation to $N_{k\pm}$ is

$$N_{k\pm} = \mp\frac{1}{2}H_k - (k+\frac{1}{2})I \ . \tag{8}$$

In the momentum representation, the characteristic equation of the operator $K_{k\pm} \equiv e^P \ P^{-k} \ N_{k\pm} \ P^k \ e^{-P}$ is

$$\mp \frac{1}{2}[p\frac{d^2}{dp^2} + (2k+1-2p)\frac{d}{dp}]\Phi(p) = \lambda\Phi(p) \tag{9}$$

where $\Psi_\lambda^{(k\pm)}(p) \ \equiv \ {}_P\langle p|\lambda\rangle_{K_{(k\pm)}}$.

Because this equation is rewritten into the associated Laguerre differential equation with orders λ, $2k$ by the change of variable $\eta = \pm 2p$, this equation has polynomial solutions

$$\Phi_\lambda(p) = \begin{cases} (const.) \ L_\lambda^{2k}(\pm 2p) & (\text{if } \pm p \geq 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{10}$$

only when $\lambda = 0, 1, 2, 3, ..$, where $L_n^m(x)$ denotes the associated Laguerre polynomial (or Sonine polynomial) with orders n, m. Since the momentum operator P is the scalar p in the momentum representation, the above result shows that $N_{k\pm}$ has the eigenvalues $0, 1, 2, 3, ...$ and the eigenfunction $\Psi_\lambda^{(k\pm)}(p)$ $(\equiv \ {}_P\langle p|n\rangle_{N_{(k\pm)}})$ with the eigenvalue $\lambda = n$ is

$$\Psi_n^{(k\pm)}(p) = \begin{cases} C_n^{(k\pm)} \ e^{\mp p}(\pm p)^k L_n^{2k}(\pm 2p) & (\text{if } \pm p \geq 0) \\ 0 & (\text{otherwise}). \end{cases} \tag{11}$$

($C_n^{(k\pm)}$ is the normalization constant.) The eigenfunction in the position-coodinate representation $\psi_n^{(k\pm)}(x) \ \equiv \ {}_Q\langle x|n\rangle_{N_{(k\pm)}}$ is the inverse Fourier transform of $\Psi_n^{(k\pm)}(p)$. It is easily shown that the eigenfunction in the position representation is expanded by the power series of $(x \pm i)^{-1}$ as

$$\psi_n^{(k\pm)}(x) = (const.)\sum_{r=0}^{m}(-2i)^{m-r}m \cdot \frac{(n+2m-r)!}{r!(m-r)!(n-m-r+1)!}(x \pm i)^{-(n+m-r+1)}. \tag{12}$$

It is interesting this expansion is made of the cauchy wavelets in (2) with varions k's. Since $N_{k\pm}$ is hermitian and the eigenvalues are not degenerated, the eigenfunction system $\{\psi_n^{(k\pm)}(x)|n = 0, 1, 2, ..\}$ in the position representation is also orthogonal.

In the following, it will be shown that $a_{k\pm}$ defined in (5) and its adjoint are the step-down and step-up operators of this eigenfunction system;

From (1),(5),(6) and the relations

$$[Q, P^{-1}] = -iP^{-2} \tag{13}$$

$$[P, A_k] = -iI \tag{14}$$

$$[A_k, A_k^\dagger] = 2kP^{-2} \ , \tag{15}$$

we have

$$\begin{aligned} [A_k, N_{k\pm}] &= \pm kP^{-2}P(A_k \pm iI) \\ &\quad \pm\tfrac{i}{2}(A_k \pm iI)^\dagger(A_k \pm iI) \pm \tfrac{i}{2}iI \\ &= \pm\tfrac{i}{2}(A_k^2 + I) \end{aligned} \tag{16}$$

$$\begin{aligned} [a_{k\pm}, N_{k\pm}] &= \pm\tfrac{i}{2}(A_k \mp iI)^{-1}(A_k^2 + I) \\ &\quad \mp\tfrac{i}{2}(A_k \mp iI)^{-2}(A_k^2 + I)(A_k \pm iI) \\ &= a_{k\pm} \ . \end{aligned} \tag{17}$$

By operating this relation on the eigenvector $|n\rangle_{N_{(k\pm)}}$ of the operator $N_{k\pm}$ (with the eigenvalue n), we have

$$\begin{aligned} &N_{k\pm}a_{k\pm}|n\rangle_{N_{(k\pm)}} \\ &\quad = a_{k\pm}N_{k\pm}|n\rangle_{N_{(k\pm)}} - a_{k\pm}|n\rangle_{N_{(k\pm)}} \\ &\quad = (n-1)a_{k\pm}|n\rangle_{N_{(k\pm)}}. \end{aligned} \tag{18}$$

This relation shows that $a_{k\pm}|n\rangle_{N_{(k\pm)}}$ is the eigenvector of $N_{k,\pm}$ with the eigenvalue $n-1$. Hence, the step-down/up relations

$$a_{k\pm}|n\rangle_{N_{(k\pm)}} = \gamma_n^{(k\pm)}|n-1\rangle_{N_{(k\pm)}} \tag{19}$$

$$a_{k\pm}^\dagger|n\rangle_{N_{(k\pm)}} = \gamma_{n+1}^{(k\pm)*}|n+1\rangle_{N_{(k\pm)}} \tag{20}$$

($\gamma_n^{(k\pm)}$ is constant and the superscript $*$ denotes complex conjugete) are derived.

The constant $\gamma_n^{(k\pm)}$ is determined (except for the phase factor) as follows; From (5) and (6), we have

$$a_{k\pm}^\dagger N_{k\pm} a_{k\pm} = -N_{k\mp} \tag{21}$$

$$N_{k\pm} = -N_{k\mp} - (2k+1)I \tag{22}$$

From these, we have

$$a_{k\pm}^{\dagger}\{N_{k\pm}+(2k+1)I\}a_{k\pm}=N_{k\pm}. \quad (23)$$

Because the eigenfunctions are normalized, from (19),

$$_{N_{(k\pm)}}\langle n|a_{k\pm}^{\dagger}\{N_{k\pm}+(2k+1)I\}a_{k\pm}|n\rangle_{N_{(k\pm)}} = \{(n-1)+(2k+1)\}|\gamma_n^{(k\pm)}|^2. \quad (24)$$

However, from (23), we have

$$|\gamma_n^{(k\pm)}|^2=\frac{n}{n+2k}. \quad (25)$$

By using the orthonormality of the eigenfunction system of $N_{k\pm}$ and the relation (25), we can show

$$a_{k\pm}^{\dagger}a_{k\pm}=N_{k\pm}\{N_{k\mp}+2kI\}^{-1} \quad (26)$$

$$a_{k\pm}a_{k\pm}^{\dagger}=(N_{k\pm}+I)\{N_{k\mp}+(2k+1)I\}^{-1}, \quad (27)$$

and hence the relation

$$(N_{k\pm}+I)\{N_{k\mp}+2kI\}a_{k\pm}^{\dagger}a_{k\pm} = N_{k\pm}\{N_{k\mp}+(2k+1)I\}a_{k\pm}a_{k\pm}^{\dagger} \quad (28)$$

is derived. These relations are different from those of boson number operator ($a^{\dagger}a = n,\ \ aa^{\dagger}=n+I$).

4 Possibility of Applications

The proposed relations between the Cauchy wavelets and the analogue of number operator may be applied to the signal processing and communication enginnering, in a similar manner to the operator method used in Wiener-Hermite expansion of signals which is mathematically equivalent the boson annihilation/creation relation used in quantum mechanics.

By using the above relations for Cauchy wavelet, we can treat the positive-frequency component and the negativa-frequency component saparately. This property is suitable especially for the processing of analytic signals.

5 Conclusions

Some relations between the Cauchy wavelets and the step-up/down operator of the orthogonal eigenfunction systems of a type of hermitian operator have been discussed. These relations are the analogue of the relation between boson creation/annihilation operators and the boson number states, but different from them.

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