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ON AN ω-INCOMPLETE DUCK AND ITS APPLICATION

KIYOYUKI TCHIZAWA
知沢 清之
Dept. of mathematics, Musashi Institute of Technology

Abstract. Introducing a parameter newly to the singular perturbation problem, an ω-incomplete duck solution; a singular ω-limit of the duck solution for the parameter will be discussed. It will become clear that the winding number for one of the solutions goes to infinity at that limit. Furthermore, the problem :when certain coefficients have different infinitesimals, what happens in the equation? will be solved.

1.INTRODUCTION.

The explicit duck solutions (or simply ducks) were constructed by Benoit[4] but not include a parameter in the differential equations. He showed, in his paper, that if the difference of each the winding numbers associated with the ducks is more than 3/2, there exists a duck which is not S^1 (S^1 is a smoothness class in nonstandard analysis). It is important to remark that this solution could not be constructed explicitly yet. The explicit ω-incomplete solution in the local model of the FitzHugh-Nagumo(FHN) equation will be discussed, however, this solution is the exact solution in the first approximation of this model. The solutions very near by this exact solution are winding so many times; the winding number goes to infinity as the parameter tends to infinity. In Section2 and Section3, introducing a parameter in the differential equations, the ω-incomplete ducks will be discussed as a singular limit of the ducks for the parameter. In Section4 and Section5, this paper will treat of the FitzHugh-Nagumo equation and, in Section6, analyze its ω-incomplete duck in the first approximation of the "local model" ([4], [18]).

2. PRELIMINARIES.

Let consider a constrained system(2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ h(x,y,z,u) &= 0, \end{aligned} \tag{2.1}$$

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where u is a parameter (any fixed) and f,g,h are defined in $R^3 \times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ \epsilon dz/dt &= h(x,y,z,u), \end{aligned} \tag{2.2}$$

where ϵ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S = \{(x,y,z) \in R^3 : h(x,y,z,u) = 0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T = \{(x,y,z) \in R^3 : \partial h(x,y,z,u)/\partial z = 0\}$ transversely so that the set $PL = \{(x,y,z) \in S \cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p \in PL$.

Let $(x(t,u), y(t,u), z(t,u))$ be a solution of (2.1). By differentiating $h(x,y,z,u)$ with respect to the time t, the following equation holds:

$$h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u) + h_z(x,y,z,u)dz/dt = 0, \tag{2.3}$$

where $h_i(x,y,z,u) = \partial h(x,y,z,u)/\partial i$, $i = x,y,z$. The above system (2.1) becomes the following system:

$$\begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ dz/dt &= -\{h_x(x,y,z,u)f(x,y,z,u)+ \\ &\quad h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned} \tag{2.4}$$

where $(x,y,z) \in S \setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p \in S \setminus PL$. In order to study the system (2.4), let consider the following system:

$$\begin{aligned} dx/dt &= -h_z(x,y,z,u)f(x,y,z,u), \\ dy/dt &= -h_z(x,y,z,u)g(x,y,z,u), \\ dz/dt &= h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u). \end{aligned} \tag{2.5}$$

As the system(2.5) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of (2.4) coincide with those of (2.1) on $S \setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any $(x,y,z) \in S$, either of the following holds;

$$h_y(x,y,z,u) \neq 0, h_x(x,y,z,u) \neq 0, \tag{2.6}$$

that is, the surface S can be expressed as $y = \varphi(x,z,u)$ or $x = \psi(y,z,u)$ in the neighborhood of PL. Let $y = \varphi(x,z,u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

$$\begin{aligned} dx/dt &= -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u), \\ dz/dt &= h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+ \\ &\quad h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned} \tag{2.7}$$

$(A5)$ All the singular points of (2.7) are nondegenerate, the matrix induced from the linearized system of (2.6) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS = \{(x, y, z) \in PL : dz/dt = 0\}$, which is called *pseudo singular points* are the singular points of (2.7).

Definition2.1. Let $p \in PS$ and $\mu_1(u)$, $\mu_2(u)$ be two eigenvalues of the linealized system of (2.7), then the point p is called *pseudo singular saddle* if $\mu_1(u) < 0 < \mu_2(u)$ and called *pseudo singular node* if $\mu_1(u) < \mu_2(u) < 0$ or $\mu_1(u) > \mu_2(u) > 0$.

Definition2.2. A solution $(x(t,u), y(t,u), z(t,u))$ of the system(2.2) is called *a duck*, if there exist standard $t_1 < t_0 < t_2$ such that
(1) $^*(x(t_0,u), y(t_0,u), z(t_0,u)) \in S$, where $^*(X)$ denotes the standard part of X,
(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,u), y(t,u), z(t,u))$ is infinitesimally close to the attracting part of the slow curves,
(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and
(4) the attracting and repelling parts of the trajectory are not infinitesimally small.

Definition2.3. Let E be a set in R^3. We call a point p is a δ-micro-galaxy of E when the distance from p to E is less than $exp(-n/\delta)$, where n is some positive integer and $\delta = \epsilon/\alpha^2$ (α is infinitesimally small).

Theorem2.1(Benoit). In the system(2.1), if the following two conditions at a pseudo singular saddle or node point;
(1) $f(O,u) \simeq h(O,u) \simeq h_y(O,u) \simeq h_z(O,u) \simeq 0$,
(2) $g(O,u) \not\simeq 0, h_x(O,u) \not\simeq 0, h_{zz}(O,u) \not\simeq 0$, where $O = (0,0,0) \in PS$, are satisfied, the first approximation of the "local model"

$$\begin{aligned} dX/dt &= pY + qZ, \\ dY/dt &= 1, \\ \delta dZ/dt &= -(Z^2 + X), \end{aligned} \tag{2.8}$$

is obtained, where p, q are constants and $\delta \simeq 0$. Then, the explicit duck solutions $\gamma_{\mu_i(u)}$ in the first approximation of the system(2.2) can be constructed:

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2 t^2 - \delta\mu_i(u), t, \mu_i(u)t)(i = 1, 2). \tag{2.9}$$

Proceeding the following coordinate transformations,

$$\begin{aligned} u &= X + Z^2 + \delta\mu, \\ v &= Y - Z/\mu, \\ z &= Z, \end{aligned} \tag{2.10}$$

$$\begin{aligned} u &= r cos\theta, \\ v &= r sin\theta, \end{aligned} \tag{2.11}$$

the solution $\gamma_\mu(t)$ is transformed to $(0, 0, \mu t)$. Moreover, proceeding the coordinate transformation

$$\rho = \delta \log r, \tag{2.12}$$

the system ((2.13) is obtained:

$$\begin{aligned} \delta d\theta/dt &= 2z\cos\theta\sin\theta + \cos^2\theta/\mu - \delta p \sin^2\theta, \\ d\rho/dt &= -2z\cos^2\theta + (1/\mu + \delta p)\cos\theta\sin\theta, \\ dz/dt &= -\exp(\rho/\delta)\cos\theta + \delta\mu, \end{aligned} \tag{2.13}$$

and proceeding

$$w = -\cot\theta, \tag{2.14}$$

the first equation in the equations (2.13) becomes the following Riccati equation,

$$\delta dw/dt = w(w - ct)/\mu - \delta p, (c : \text{some constant}) \tag{2.15}$$

is obtained. Furthermore, applying time scaling and applying the certain cordinate transformation w to W, the Hermite equations associated with $\gamma_{\mu_i(u)}$ $(i = 1, 2)$ are obtained as the following:

$$\delta \ddot{W} - \tau \dot{W} + K_i z = 0, t = \tau/\alpha, (\alpha : \text{any constant})(i = 1, 2), \tag{2.16}$$

where K_i is a positive integer and $K_1 = 1 + \mu_2(u)/\mu_1(u)$, $K_2 = 1 + \mu_1(u)/\mu_2(u)$. See [4].

Definition2.4. The winding number $N(\psi)$ of a duck ψ is defined as follows:

$$N(\psi) = (1/2\pi)\int_\psi d\theta, \tag{2.17}$$

where ψ is contained partially in the δ-micro-galaxy of γ_μ.

The above Definition2.3 is based on the following fact. If ϵ is fixed arbitrarily and $\gamma(t)$ is a duck near $\gamma_{\mu(u)}(t)$, then the distance from $\gamma(t)$ to $\gamma_{\mu(u)}(t)$ is within $exp(-n/\delta)$ in some neighborhood of the pseudo singular point. See [19].
It is said that a duck $\psi(t)$ has *a jump* if the shadow of it contains a vertical segment and that $\psi(t)$ is *long* if it is in an infinitesimally small neighborhood at the pseudo singular point. It can be proved that if ψ is not long, the standard part of the winding number $N(\psi_i)$ associated with μ_i is an integer. If the pseudo singular point is node, it is positive. If the point is saddle, it needs some conditions such as K_i is poitive. The relation between $N(\psi_i)$ and K_i $(i = 1, 2)$ is as follows.

Theorem2.2(Benoit). If the duck ψ_1, which is not long has 2 jumps, $N(\psi_1) \approx -[K_1/2]$, and if the duck ψ_2 has 2 jumps, $N(\psi_2) \approx 0$.

3. Definition of an ω-incomplete duck.

In the system(2.2), making the following coordinate transformations (3.1) and (3.2) successively;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha^2\tilde{x} \\ \alpha\tilde{y} \\ \alpha\tilde{z} \end{pmatrix}, (\alpha \simeq 0, \epsilon/\alpha^2 \simeq 0) \tag{3.1}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} h_x(0,u)h_{zz}(0,u)\tilde{x}/2 + (h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)\tilde{y}^2/4 \\ \tilde{y}/g(0,u) \\ -h_{yz}(0,u)\tilde{y}/2 - h_{zz}(0,u)\tilde{z}/2 \end{pmatrix}, \tag{3.2}$$

the following system(3.3) is obtained:

$$\begin{aligned} dX/dt &= pY + qZ + \xi(X,Y,Z,u), \\ dY/dt &= 1 + \eta(X,Y,Z,u), \\ \delta dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,u), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} p &= g(0,u)h_x(0,u)(f_y(0,u)h_{zz}(0,u) - f_z(0,u)h_{yz}(0,u))/2 \\ &\quad + g(0,u)^2(h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)/2, \\ q &= -h_x(0,u)f_z(0,u), \\ \delta &= \epsilon/\alpha^2. \end{aligned}$$

Here $\xi(X,Y,Z,u)$,$\eta(X,Y,Z,u)$ and $\zeta(X,Y,Z,u)$ are infinitesimal when X,Y and Z are limited.

Definition3.1. In the system(3.3), if the followings (1) and (2):
(1) for any limited parameter u,
it satisfies the conditions $(A1)$-$(A5)$ and has a duck,
(2) when the parameter u tends to infinity, one of the winding numbers
tends to infinity and the other tends to zero,
and the system does not have a duck as a singular limit,
are established, this solution is called an *ω-incomplete duck.*

Definition3.2. A solution $\psi(x,u)$ is called S^1 *at* a,
if there exists a real number b such that

$$\frac{\psi(x,u) - \psi(y,u)}{x - y} \approx b, \tag{3.4}$$

for any $x, y (x \approx a, y \approx a)$.
A duck is called an S^1 *duck* if it is S^1 in some neighborhood
of the pseudo singular point.

Theorem3.1(Benoit). In the first approximation of the system(3.3), if $\mu_1(u)/\mu_2(u)$ is positive (>3) but no an integer, then all the S^1 ducks are exponentially close to one of the two explicit ducks and there exists non S^1 ducks.

In the system(3.3), we assume that

$$f_y(0,u) = g_u(0,u) = h_{yz}(0,u) = h_{yyu}(0,u) = h_{zzu}(0,u) = 0, \tag{3.5}$$

and that the following (1) or (2):
(1) $h_x(0,u) = O(u)$ and $f_z(0,u) = O(1)$,
(2) $f_z(0,u) = O(u)$ and $h_x(0,u) = O(1)$,
where all the coefficients of higher order (more than 2) for u is negligible, that is, only the coefficient q can take an unlimited number ($q = c_1 u + o(1)$, a constant $c_1 \not\simeq 0$). Then, blowing up only the variable Z again;

$$Z = (1/u)\tilde{Z}, \tag{3.6}$$

the first approximation of the system(3.3) becomes the following:

$$\begin{aligned} &dX/dt = pY + c_1\tilde{Z},\\ &dY/dt = 1,\\ &(\delta/u)d\tilde{Z}/dt = -(\tilde{Z}^2/u^2 + X), \end{aligned} \tag{3.7}$$

where c_1 is limited (does not contain u) and $\delta/u \simeq 0$. The explicit solutions in the system(3.7) are

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - \delta\mu_i(u), t, u\mu_i(u)t)(i = 1,2), \tag{3.8}$$

where $\mu_1(u)$, $\mu_2(u)$ ($\mu_1(u) > \mu_2(u)$) are the solutions of the characteristic equation of the system(3.7) in case $\delta/u \simeq 0$.
The above system satifies the conditions $(A1)$-$(A5)$ and the solutions(3.8) satisfy the condition (1) and satisfies the condition (2) when $u \to \omega$ in Definition3.1.
In fact, if $q = \epsilon^{-1/3}$, then the existence of such a duck is ensured. We choose $\epsilon = 1/n$ ($n = 2,3,\dots$) ($u = 1/n^{1/3}$), then $1/n^{1/3} \gg exp(-n/\delta)$ for any n ($n \geq 2$). In the system for each any fixed n, let $J = [AB]$ be a connected segment in R^3, where the solution which starts at A(or B) belongs to the family of the duck γ_{μ_1} (or γ_{μ_2}). It can be proved that if any solution starting at $p \in J$ is not long, then it has the same winding number. From Theorem3.1, a duck passing through the pseudo singular node point belongs to one of two families of the above ducks. On the other hand, there exists a segment $[CD] \subset J$ such that any solution starting at $p \in [CD]$ is not long and the solutions passing through C or D are ducks. This fact ensures the existence of a non S^1 duck. Note that $\mu_1(u)/\mu_2(u)$ is positive but no an integer. If it is a positive integer k, it indicates the fact that the slow vector field has two C^1 trajectories but only one of them is C^k. Then, it is not possible to have an asymptotic expansion in powers of ϵ with the coefficients analytic in t. Furthermore, one of the solutions(3.8) may tend tangent to the X-axis, since $\mu_2(u) \to -\omega/2$ as $u \to \omega$ and for the first component of (3.8), the following

$$\frac{-(\omega/2)^2(2/\omega)^2 + (\omega/2)^2(1/\omega)^2}{2/\omega - 1/\omega} \to -3\omega/4, \tag{3.9}$$

establishes. In this state, the winding number $N(\psi_2)$ associated with μ_2 tends to infinity and the other $N(\psi_1)$ associated with μ_1 tends to infinitesimal. When $u \to \omega = \epsilon^{-1/3}$, the eigen space of the linear part of the slow vector field for $\mu_2 \simeq -\epsilon^{-1/3}$ is $z \simeq \epsilon^{1/3}y$ ($z \simeq -\epsilon^{1/3}y$ for $\mu_1 \simeq -\epsilon^{1/3}$). The ducks are almost tangent to the eigen spaces and therefore the ω-limit of the duck with respect to the parameter u (ω-incomplete duck) is not S^1.

Let $v = 1/u$, then $\partial_u = -v^2\partial_v$ holds and then the following conditions are assumed; $f(x,y,z,u) = \tilde{f}(x,y,z,v) \in C^3$, $g(x,y,z,u) = \tilde{g}(x,y,z,v) \in C^1$ and $h(x,y,z,u) = \tilde{h}(x,y,z,v) \in C^3$ at almost every where but $v = v_0 = 0$. From the assumptions, the relation $q = -h_x(0,u)f_z(0,u) = c_1 u$ holds. Differentiating the both side of this equation by the parameter v, we can lead to the following theorem.

Theorem3.2. In the first approximation of the system(3.3), if $\mu_1(u)/\mu_2(u)$ is positive but no integer under the condition (3.5) and if $\tilde{h}_{xv}(0,v)\tilde{f}_{zv}(0,v) = 0$ when either the condition (1) or (2) ;
(1) $\tilde{f}_z(0,v) = 0$, and $\tilde{h}_x(0,v)\tilde{f}_{zvv}(0,v) = 0$,
(2) $\tilde{h}_x(0,v) = 0$, and $\tilde{h}_{xvv}(0,v)\tilde{f}_z(0,v) = 0$,
where all the coefficients of higher order (more than 2) for u is negligible are satisfied, then this system has an ω-incomplete duck.

Remark. In the system(3.3), if the coefficient q satisfies $q = c_1 u + O(1)$, that is, $q = c_1 u + c_2$ where $c_1, c_2 \not\simeq 0$ and $p > 0$ or $0 > p \geq -1/32$, then there exists a finite value u_0 which makes the winding number infinite when u tends to u_0.

4. The FHN equation.

We consider the space-clamped FitzHugh-Nagumo(FHN) equation (4.1) with slowly varing in the time dependent parameter I;

$$(4.1)\qquad \begin{aligned} dv/dt &= -\rho(v) - w + I \\ dw/dt &= b(v - \gamma w), \rho(v) = v(v-1)(v-a) \end{aligned}$$

where $0 < a < 1/2$, b and γ are positive constants. Each variable is denoted as follows;
$v(t)$: the potential difference at the time t across the membrane of the axon,
$w(t)$: a recovery current which is often taken to be the sum of all ion flows,
I: an injected electric current treated as a control or bifurcation parameter depending on the time t on the membrane such that

$$(4.2)\qquad I = I_0 + \epsilon_1 t,$$

where ϵ_1 is infinitesimally small and I_0 is an initially given constant. The first of (4.1) expresses Kirchhoff's law applied to the membrane. The second relates the recovery current with the potential. The constant γ is restricted from biophysical considerations so that

$$(4.3)\qquad 0 < \gamma < 3/(1 - a + a^2)$$

standard form of a slow-fast system with two slow variables and one fast variable, that is,

$$b = c\epsilon_2, \tag{4.4}$$

where $\epsilon_1=\epsilon_2=\epsilon$ and c is any constant.

Though we first start at this condition, in Section6, we will refer to the case $\epsilon_1 \neq \epsilon_2$. Benoit[3] gave a sufficient condition for the existence of the ducks: a pseudo singular point is a saddle point. Under the assumptions (4.2) and (4.4), the system (4.1) has a pseudo singular saddle point [18], that is, this system has ducks. After using several coordinate transformations, the system (4.1) is transformed to the local model. This model has a pseudo singular node point under a certain condition. Then two duck solutions γ_{μ_i} $(i = 1, 2)$ at the pseudo singular point are obtained where μ_i $(i = 1, 2)$ are the eigenvalues associated with the linearized system. Furthermore, in a sufficiently small neighborhood of γ_{μ_i}, the winding number of the duck for μ_1, which depends on the Hermite equation derived by the several transformations becomes infinity when the constant γ in the system (4.1) tends to zero [11]. The winding number of the duck for μ_2 is limited and greater than 1 in this state. As mentioned in Section1, Benoit[4] has already pointed out that if the difference of each the ratio of the eigenvalues, which relates the winding number directly is greater than 3, there exists a duck without S^1. Therefore, this fact tells us the first approximation system of the local model has a duck without S^1 but this is not an explicit form.

In this back ground, there is the delay problem [1],[2],[14]. When $k \leq 1$ under $\epsilon = O(b^k)$, it had not been solved yet because of the difficulty: the uniformity of the solution is not ensured in the asymptotic expansion. Some authors ([6],[9],[12]) put $b = \epsilon$ directly in their papers and obtained various results, however, they did not touch ducks.

5. The ducks in the FHN.

Using the assumptions (4.2) and (4.4), the system(4.1) becomes the system(5.1):

$$\begin{aligned} \epsilon dv/dI &= -\rho(v) - w + I, \\ dw/dI &= c(v - \gamma w). \end{aligned} \tag{5.1}$$

By changing the coordinates, $w = X$, $I = Y$ and $v = Z$, the system(5.1) becomes the system(5.2):

$$\begin{aligned} dX/dI &= c(Z - \gamma X), \\ dY/dI &= 1, \\ \epsilon dZ/dI &= (-\rho(Z) - X + Y). \end{aligned} \tag{5.2}$$

Note that the system(5.2) satisfies the conditions $(A1)$-$(A5)$ in Section2 when $\epsilon = 0$, but that this system does not satisfy the conditons (1) and (2) in the Theorem2.1. Using the following coordinate transformation (5.3):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{pmatrix}, \tag{5.3}$$

the newly revised system on which the above conditions (i) and (ii) establish is obtained [18]:

$$(5.4) \qquad \begin{aligned} dx/dt &= -c\gamma(2x+y)+cz, \\ dy/dt &= c\gamma(2x+y)-cz+1, \\ \epsilon dz/dt &= -x-\rho(z+Z_0)+\rho(Z_0), \end{aligned}$$

where $P_0 = (X_0, Y_0, Z_0)$ is one of the two pseudo singular points of the system(5.2) in case $\epsilon = 0$. By the "local model theory", the explicit ducks of the system(5.5) which is the local model of the system(5.4) are obtained as follows.

$$(5.5) \qquad \begin{aligned} dX/dt &= pY+cZ, \\ dY/dt &= 1, \\ \delta dZ/dt &= -(Z^2+X), \end{aligned}$$

where $p = (-1)^i c\gamma\sqrt{(a^2-a+1)}$ $(i = 1, 2)$. The system (5.5) is the exact local model which effects (5.4) globally. In fact, the first and second equations are first order for the variables x, y and z. Furthermore, the third equation does not contain the variable y and co-variables are third order at most. So, it may be admissible for us to use the system (3.3) globally . Restricting the system(5.5) on the surface $-(Z^2+X) = 0$, the linearized system(5.6) is obtained:

$$(5.6) \qquad \begin{aligned} dY/dt &= Z, \\ dZ/dt &= -(pY+cZ)/2. \end{aligned}$$

The characteristic equation for (5.6) is

$$(5.7) \qquad 2\mu^2 + c\mu + c\gamma\sqrt{a^2-a+1} = 0.$$

If we choose $p = c\gamma\sqrt{a^2-a+1}$ and $c > 8\gamma\sqrt{a^2-a+1}$ so that the system (5.5) has a pseudo singular node point, then the explicit ducks are

$$(5.8) \qquad \gamma_{\mu_i}(t) = (-\mu_i^2 t^2 - \delta\mu_i, t, \mu_i t)(i = 1, 2),$$

where μ_i $(i = 1, 2)$ are the solutions of (5.7).

6. An ω-incomplete duck in the FHN equation.

In this section, we will describe how to construct an ω-incomplete duck in the first approximation of the FHN local model. By the elementary calculations, the followings are obtained. See [11].

Lemma6.1. If γ tends to zero, the winding number $N(\psi_1)$ asssociated with μ_1 $(\mu_2 < \mu_1 < 0)$ tends to infinity.

Lemma6.2. If ψ_i $(i = 1, 2)$ are not long, then in the first approximation of the FHN local model the lower bound of $-N(\psi_1)$ is 1 and of $N(\psi_2)$ is infinitesimal.

As mentioned before, if $K_2 - K_1 > 3$, then there exists a duck which is not S^1 in the first approximation of the local model. So, these facts lead to the conclusion: there exists a duck which is not S^1 in the approximation model.

In this framework, we would try to consider the solution as a singular limit for the parameter of the duck. In the system(5.5), we choose the parameter v such as $c = 1/v$ and $\gamma{=}v$. Let $v = \beta$ (β is any fixed), then blowing up the variable Z again so that $Z = \beta\tilde{Z}$, the system(5.5) becomes the following;

$$\begin{aligned} &dX/dt = (-1)^i\sqrt{a^2 - a + 1}Y + \tilde{Z}, \\ &dY/dt = 1, \\ &(\delta\beta)d\tilde{Z}/dt = -(\beta^2\tilde{Z}^2 + X), \end{aligned} \tag{6.1}$$

where $\delta\beta \approx 0$. Under the above assumptions, the system(6.1) satisfies the conditions $(A1)$-$(A5)$ in Section2. Therefore, the explicit duck solutions $\gamma_{\mu_i(\beta)}$ of the system become

$$\gamma_{\mu_i(\beta)}(t) = (-\mu_i^2t^2 - \delta\mu_i, t, \mu_i t/\beta)(i = 1, 2), \tag{6.2}$$

where $\mu_1(\beta)$, $\mu_2(\beta)$ $(\mu_1(\beta) > \mu_2(\beta))$ are the solutions of the following equation:

$$2\mu^2 + \mu/\beta + \sqrt{a^2 - a + 1} = 0. \tag{6.3}$$

When $\gamma = \beta$ tends to zero, the eigenvalue μ_1 tends to zero and the other eigenvalue μ_2 tends to infinity. Then the first element of $\gamma_{\mu_2(\beta)}$ may tend to the solution which is not S^1. However, the conditions $(A1)$-$(A5)$ are satisfied. In this state, we can obtain an ω-incomplete duck $\tilde{\gamma_\omega}(t)$:

$$\tilde{\gamma_\omega}(t) \approx (-t^2/\beta^2 + \delta/\beta, t, -t/\beta^2), \tag{6.4}$$

since this solution does not satify the definition of the ducks when β tends to 0.

Theorem6.1. In the first approximation system(5.5) of the FHN local model, if the constant c tends to infinity when the constant γ tends to zero, there exists an ω-incomplete duck.

Corollary6.2. In the singular limit, when $\gamma = 1/\omega = \epsilon$ and $c = 1/\epsilon^{1/n}$ $(n \geq 2;$ integer), b satisfies $b = \epsilon(1/\epsilon^{1/n}){=}\epsilon^{1-1/n}$. If we put $I = I_0 + \epsilon^{1-1/n}t$ again, then the FHN equation still preserve the assumptions (4.2) and (4.4). This fact ensures that even in the case of $b{=}c\epsilon_1$ and $I{=}I_0 + \epsilon_2 t$ $(\epsilon_1 \neq \epsilon_2)$ the first approximation of the local model has an ω-incomplete duck at the singular limit of the parameter.

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1-28-1 Tamazutsumi Setagaya-Ku Tokyo, 158, Japan