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An Introduction to a Multi-dimensional Hénon-like Family

Shin Kiriki (桐木 紳)

Department of Mathematical Sciences, Tokyo Denki University,

Hatoyama, Hiki, Saitama, 350-0349, Japan

E-mail : ged@r.dendai.ac.jp

Masaki Nakajima (中島 正樹)

Hitachi Software Engineering Co., Ltd.,

9-12 Minato-cyo, Kanagawa-ku, Yokohama, 221-0056, Japan

E-mail : nmasaki@pyt.hitachi-sk.co.jp

1 Introduction

Hénon map $f_{a,b}(x,\ y) = (a-by-x^2, x)$ is one of the most important objects in the chaotic dynamical systems. Through its study, many mathematicians have confirmed some essential concepts in the dynamical systems presented by Smale [4], and encountered various phenomena and interesting new problems which contain many difficulties originating from its nonlinearity. Computer simulations are also indispensable for the progress of this study. There have been many works on this map. However, we here state briefly some essential works which motivated our study of dynamics with a high-dimensional Hénon-like diffeomorphisms. This planar map was first presented by Hénon [2] with its non-trivial phenomenon called a strange attractor which was discovered by computational experiment as shown in the first panel of Figure 1. Nitecki and Devany showed in [1] that, when a and b satisfy some conditions, it has the same structure as a horseshoe map, that is, it has a hyperbolic structure and its dynamics is coded perfectly by a topological conjugacy with the 2-shift. Yang moreover showed that this structure is preserved until the first homoclinic tangency occurs when a decreases and b takes very small positive values [6].

Benedicks and Carleson proved that the occurrence of the non-trivial attractors is a abundant phenomenon in the measure-theoretical sense. Mora and Viana extended their result to more generic one-parameter family of surface diffeomorphisms. In [5], Viana moreover extended their result to higher dimensions, that is, it assures the existence of codimension-1 Hénon-like non-hyperbolic attractors near the homoclinic tangency on high-dimensional manifolds.

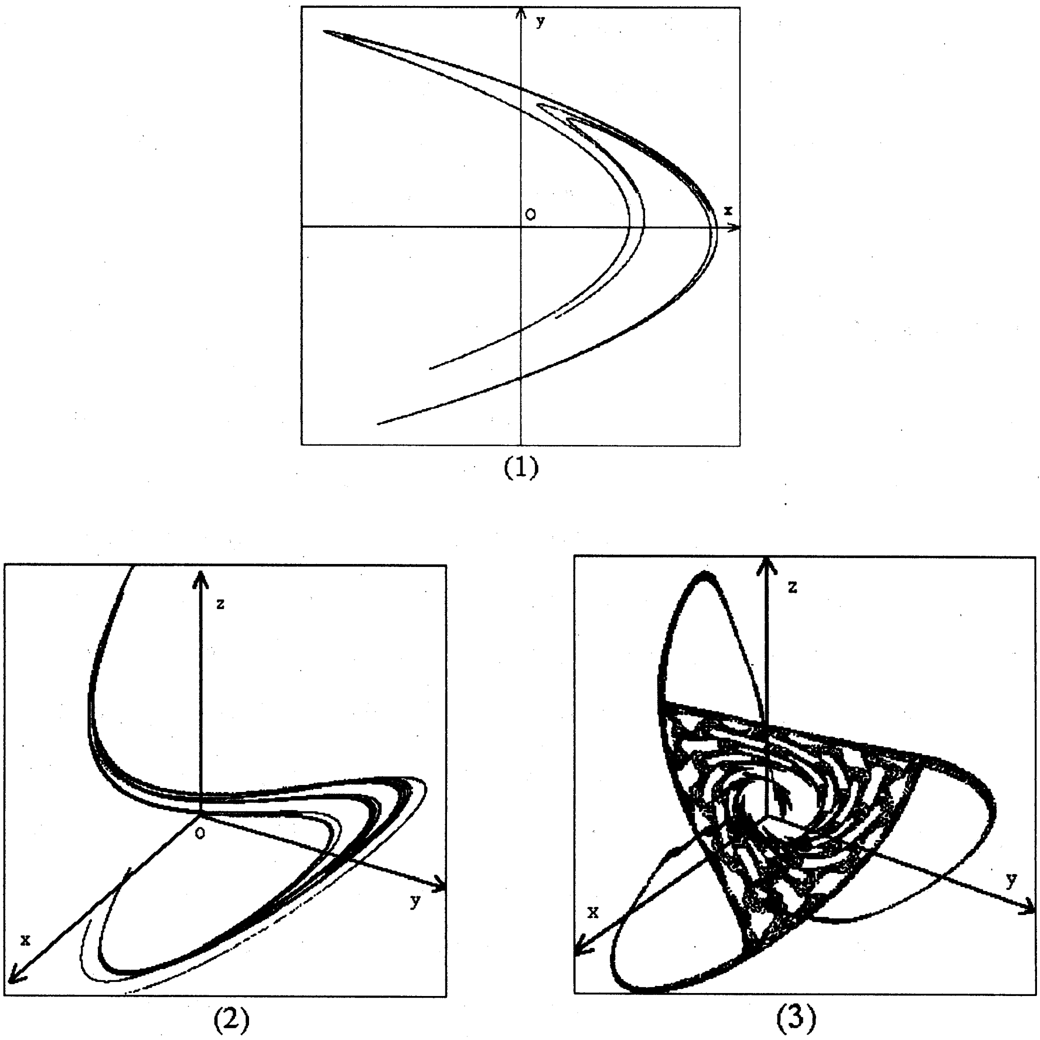

Figure 1: (1) A 2-dimensional Heńon attractor, for $a = 1.4$ and $b = 0.3$. (2) and (3) The 3-dimensional Heńon attractors for (1.7, −0.1) and (0.2, 0.8), respectively.

In our recent paper [3], on a somewhat different ground from that of Viana, we extend the Hénon map to a high-dimensional space. One of the merits of our method is that we can actually have a formula for this extended map which has some properties similar to the 2-dimensional Hénon map. Therefore, one can observe globally its dynamics and its progress of bifurcations by numerical computations. Actually, in the 3-dimensional Euclidean space, we define a family of diffeomorphisms of Hénon type as

$$F_{a,b}(x,\ y,\ z) = (a + bz - x^2,\ x,\ y).$$

As in the 2-dimensional case, some non-trivial attractors are observed easily in $\mathbb{R}^3$ by numerical experiments, for example for $(a,\ b)$ close to $(1.7,\ -0.1)$, $(0.2,\ 0.8)$, as shown in the second, the third panel of Figure 1, respectively. However, it is not yet clear whether these attractors are non-hyperbolic and persistently strange.

As the results of the paper [3], we present a two-parameter family $F_{a,b}$ of diffeomorphisms of Hénon type on the Euclidean space of an arbitrary dimension m, and show that there is an open set $\mathcal{H}_m$ in the product space of parameters such that, for every $(a,\ b) \in \mathcal{H}_m$, $F_{a,b}$ has a hyperbolic structure. The following article is a brief summary of the results in [3].

2 Definitions and the main result

Let $m \geq 3$ be an integer, and $\mathbb{R}^m$ be the m-dimensional Euclidean space. We define the two-parameter family $F_{a,b}$ of diffeomorphisms on $\mathbb{R}^m$ as

$$F_{a,b}(x_1,\ x_2,\ \ldots,\ x_m) = (a + bx_m - x_1^2,\ x_1,\ x_2,\ \ldots,\ x_{m-1}),$$

where a and b are real parameters. One shall see why we call this family a *m-dimensional special Hénon family* by the following fundamental properties which are similar to those of the 2-dimensional Hénon map. When $b \neq 0$, it's inverse is

$$F_{a,b}^{-1}(x_1,\ x_2,\ \ldots,\ x_m) = (x_2,\ x_3,\ \ldots,\ x_m,\ b^{-1}(-a + x_1 + x_2^2)),$$

and it is a bijection. Moreover, $F_{a,b}$ is a diffeomorphism on $\mathbb{R}^m$. When $b = 0$, $F_{a,0}(x_1, x_2, \ldots, x_m) = (a - x_1^2, x_1, \ldots, x_{m-1})$ which maps all points of $\mathbb{R}^m$ to the parabolic surface. The absolute value of the Jacobian of $F_{a,b}$ is equal to $|b|$ for every $\mathbf{x} \in \mathbb{R}^m$, that is,

$$\det(DF_{a,b})_{\mathbf{x}} = \det \begin{pmatrix} -2x_1 & 0 & \cdots & 0 & b \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} = (-1)^{m-1} b.$$

$F_{a,b}$ is obtained by the composed mapping of ϕ, ϕ_b and ϕ_a, that is, $F_{a,b} = \phi_a \circ \phi_b \circ \phi$, where

$$\phi(x_1, x_2, \cdots, x_m) = (x_m, x_1, \cdots, x_{m-1});$$

$$\phi_b(x_1, x_2, \cdots, x_m) = (bx_1, x_2, \cdots, x_m);$$

$$\phi_a(x_1, x_2, \cdots, x_m) = (a + x_1 - x_2^2, x_2, \cdots, x_m).$$

These above constructions of this family are similar to those of 2-dimensional Hénon family. Moreover, in the following main theorem, we get an essential property of this family, which corresponds to that of the 2-dimensional Hénon family.

Main Theorem. *Let $F_{a,b}$ be the m-dimensional special Hénon family. Let $\mathcal{H}_m$ be an open set in the product set of parameter spaces such that*

$$\mathcal{H}_m = \left\{ (a,\ b) \in \mathbb{R} \times \mathbb{R}\ :\ a > \frac{(1+|b|)^2}{4} \left(m + 4 + 2\sqrt{m+4} \right),\ b \neq 0 \right\}.$$

For any $(a,\ b) \in \mathcal{H}_m$, the nonwandering set of $F_{a,b}$ is a horseshoe, that is, it is structurally stable.

We can prove the main theorem using the following two theorems. See Figure 2. The open set $\mathcal{H}_m$ occupies a large area in the product space of parameters meaning that the structural stability is easily acquired by this high-dimensional family. However, numerical experiments showed characteristic division of the area outside of $\mathcal{H}_m$ by some typical bifurcations. Moreover, the combinations of parameters causing the non-trivial attractors are located outside $\mathcal{H}_m$. It may be very important to make a systematic investigation of all the types of bifurcations occurring in this family.

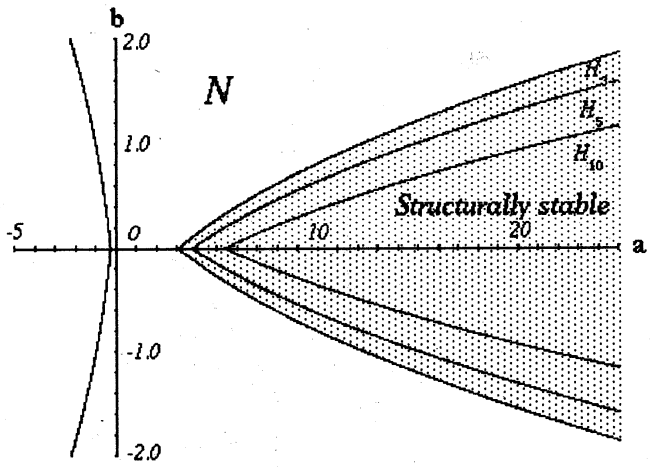

Figure 2: The open set $\mathcal{H}_m$ for m=3, 5, 10 and $\mathcal{N}$

3 Nonwandering set

Let $F_{a,b}$ be the m-dimensional special Hénon family. We assume $b \neq 0$. We denote the nonwandering set of $F_{a,b}$ by $\Omega_{a,b}$. In this section, we detect the location of $\Omega_{a,b}$. Let

$$r = \frac{1 + |b| + \sqrt{(1 + |b|)^2 + 4a}}{2},$$

which satisfies the quadratic equation $r^2 - (1+|b|)r - a = 0$. We assume that $(1+|b|)^2 + 4a \geq 0$. Using this r, partition $\mathbb{R}^m$ into $m+1$ areas as follows: for each $i = 2, 3, \cdots, m$,

$$\begin{aligned}
\mathcal{D} &= \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : |x_j| \leq r, \ j = 1, 2, \cdots, m\} \\
\mathcal{T}_1 &= \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : |x_1| > r, \ |x_1| > |x_j|, \ j = 2, 3, \cdots, m\} \\
\mathcal{T}_i &= \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : |x_i| > r, \ |x_i| \geq |x_j|, \ j = 1, 2, \cdots, m, \ j \neq i\}.
\end{aligned}$$

When $m = 3$, these partitions are shown in Figure 3.

Proposition 1 *Let $\mathcal{N}$ be a subset of the direct product set of parameter spaces such that*

$$\mathcal{N} = \left\{(a,\ b) \in \mathbb{R} \times \mathbb{R}\ :\ a \geq -(1+|b|)^2/4,\ b \neq 0\right\}.$$

For any $(a,\ b) \in \mathcal{N}$, the nonwandering set $\Omega_{a,b}$ for $F_{a,b}$ is contained in $\mathcal{D}$.

Proof. See [3]. ■

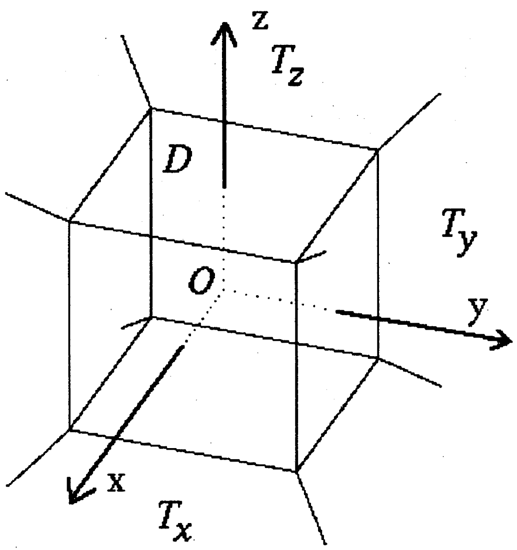

Figure 3: The partitions of $\mathbb{R}^3$.

From the following lemmas 1 and 2, we get directly the proof of this theorem. For any $\mathbf{x} = (x_1,\ x_2,\ \cdots, x_m) \in \mathbb{R}^m$, we write

$$\|\mathbf{x}\| = \max\{|x_1|,\ |x_2|,\ \cdots |x_m|\}.$$

Lemma 1 *For $(a,\ b) \in \mathcal{N}$,*

(1) $F_{a,b}(\mathcal{T}_1) \subset \mathcal{T}_1$;

(2) For every $\mathbf{x} \in \mathcal{T}_1$, $\|F^n_{a,b}(\mathbf{x})\| \to \infty$ *as* $n \to \infty$;

(3) $F_{a,b}(\mathcal{D}) \subset \mathcal{D} \cup \mathcal{T}_1$.

Proof. See [3]. ■

Lemma 2 *For any* $(a,\ b) \in \mathcal{N}$,

(1) $F_{a,b}^{-1}(\mathcal{T}_2) \subset \mathcal{T}_m$;

(2) $F_{a,b}^{-1}(\mathcal{T}_i) \subset \mathcal{T}_{i-1} \cup \mathcal{T}_m$ *for each* $i = 3,\ 4,\ \cdots,\ m$

(3) For every $\mathbf{x} \in \bigcup_{i=2}^{m} \mathcal{T}_i$, $\|F_{a,b}^{-n}(\mathbf{x})\| \to \infty$ *as* $n \to \infty$.

Proof. See [3]. ■

4 Hyperbolicity

From the previous section, we understand that the nonwandering set of $F_{a,b}$ is contained in $\mathcal{D}$. Therefore, in this section, we concentrate on the maximal invariant set of $\mathcal{D}$. Let $\Lambda_{a,b}$ be the maximal invariant set of $\mathcal{D}$ under $F_{a,b}$, i.e. $\Lambda_{a,b} = \bigcap_{i \in \mathbf{Z}} F_{a,b}^{i}(\mathcal{D})$. The following proposition gives a result on its hyperbolicity.

Proposition 2 *Let* $F_{a,b}$ *be the* m*-dimensional special Hénon family. Let* λ *be a real number such that* $\sqrt{m-1} \geq \lambda > 1$, *and let*

$$a_1(b,\ m) = \frac{(1+|b|)^2}{4}\left(m + 4 + 2\sqrt{m+4}\right).$$

For any $|b| > 0$, *if* $a > a_1(b,\ m)$, *then the maximal invariant set* $\Lambda_{a,b}$ *is a hyperbolic set.*

Proof. See [3]. ■

For $m \geq 3$, let us define

$$\mathcal{H}_m = \left\{(a,\ b) \in \mathbb{R} \times \mathbb{R}\ :\ a > \frac{(|b|+1)^2}{4}\left(m + 4 + 2\sqrt{m+4}\right),\ b \neq 0\right\}.$$

For any point $\mathbf{x} \in \Lambda_{a,b}$, we define the unstable cone :

$$C^u(\mathbf{x}) = \left\{(\xi_1,\ \xi_2,\ \cdots,\ \xi_m) \in T_{\mathbf{x}}\mathbb{R}^m\ :\ \lambda\sqrt{\xi_2^2 + \xi_3^2 + \cdots + \xi_m^2} \leq |\xi_1|\right\},$$

and take a stable cone in the complement of $C^u(\mathbf{x})$ in $T_\mathbf{x}\mathbb{R}^m$:

$$C^s(\mathbf{x}) = \left\{(\xi_1,\ \xi_2,\ \cdots,\ \xi_m) \in T_\mathbf{x}\mathbb{R}^m\ :\ \sqrt{\xi_2^2+\xi_3^2+\cdots+\xi_m^2} \geq \lambda|\xi_1|\right\},$$

where $\lambda > 1$ is a real number. The next lemma guarantees that these cones are invariant and expanded exponentially by $(DF_{a,b})_\mathbf{x}$ and $(DF_{a,b}^{-1})_\mathbf{x}$, respectively. Proposition 2 is given directly from this lemma.

Lemma 3 *Let $F_{a,b}$ be the m-dimensional special Hénon family such that the parameters a and b belong to $\mathcal{H}_m$. For each $\mathbf{x} \in \Lambda_{a,b}$, there exist the unstable and stable cones in $T_\mathbf{x}\mathbb{R}^m$ satisfying the following conditions:*

(i) $(DF_{a,b})_\mathbf{x}C^u(\mathbf{x}) \subset C^u(F_{a,b}(\mathbf{x}))$ *and* $\|(DF_{a,b})_\mathbf{x}\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ *for all* $\mathbf{v} \in C^u(\mathbf{x})$*;*

(ii) $(DF_{a,b}^{-1})_\mathbf{x}C^s(\mathbf{x}) \subset C^s(F_{a,b}^{-1}(\mathbf{x}))$ *and* $\|(DF_{a,b}^{-m+1})_\mathbf{x}\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ *for all* $\mathbf{v} \in C^s(\mathbf{x})$*,*

where $\|\mathbf{v}\| = \max\{|\xi_1|,\ |\xi_2|,\ \cdots,\ |\xi_m|\}$ *for* $\mathbf{v} = (\xi_1,\ \xi_2,\ \cdots,\ \xi_m)$*.*

Proof. See [3]. ■

5 Proof for the 3-dimensional special case

In [3], all proofs of the above statements are presented perfectly. In this last section, we give a proof of the above lemma for the special 3-dimensional case.

Let $\{F_{a,b}\}$ be the same family as the previous subsection. We take $m = 3$, $a = 5$ and $b = 0.1$. Then, we have

$$r = \frac{1+0.1+\sqrt{(1+0.1)^2+4\cdot 5}}{2} < 3.$$

$$\widetilde{\mathcal{D}} = \{(x,\ y,\ z) \in \mathbf{R}^3\ :\ |x| \leq 3,\ |y| \leq 3,\ |z| \leq 3\}.$$

Let Λ be the maximal invariant set of $\widetilde{\mathcal{D}}$ under F. For any point $\mathbf{x} \in \Lambda$, we define the unstable cone by

$$C^u(\mathbf{x}) = \left\{(\xi,\eta,\zeta) \in T_\mathbf{x}\mathbf{R}^3\ :\ \lambda\sqrt{\eta^2+\zeta^2} \leq |\xi|\right\},$$

and take a stable cone in the complement of $C^u(\mathbf{x})$ in $T_{\mathbf{x}}\mathbf{R}^3$:

$$C^s(\mathbf{x}) = \left\{(\xi, \eta, \zeta) \in T_{\mathbf{x}}\mathbf{R}^3 \;:\; \sqrt{\eta^2 + \zeta^2} \geq \lambda|\xi|\right\},$$

where λ is a positive real number. In the next lemma, we show that these cones are invariant, moreover expanded, under $(DF)_{\mathbf{x}}$ and $(DF^{-1})_{\mathbf{x}}$.

Lemma 4 *There is a $\lambda_1 > 1$ such that for any $1 < \lambda \leq \lambda_1$ the unstable and stable cones satisfy the following conditions for all $\mathbf{x} \in \Lambda$:*

(i) $(DF)_{\mathbf{x}}C^u(\mathbf{x}) \subset C^u(F(\mathbf{x}))$ and $\|(DF)_{\mathbf{x}}\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ for all $\mathbf{v} \in C^u(\mathbf{x})$;

(ii) $(DF^{-1})_{\mathbf{x}}C^s(\mathbf{x}) \subset C^s(F^{-1}(\mathbf{x}))$ and $\|(DF^{-2})_{\mathbf{x}}\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ for all $\mathbf{v} \in C^s(\mathbf{x})$,

where $\|\mathbf{v}\| = \max\{|\xi|,\ |\eta|,\ |\zeta|\}$ for $\mathbf{v} = (\xi,\ \eta,\ \zeta) \in C^u(\mathbf{x}),\ C^s(\mathbf{x})$.

Proof. Let $\mathbf{x} = (x,\ y,\ z)$ be a point in Λ. For any $\mathbf{v} = (\xi,\ \eta,\ \zeta) \in C^u(\mathbf{x})$, we denote $(DF)_{\mathbf{x}}\mathbf{v}$ as $(\xi_1,\ \eta_1,\ \zeta_1)$. The image of the vector by the derivative is given by

$$\begin{pmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{pmatrix} = \begin{pmatrix} -2x & 0 & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} -2x\xi + b\zeta \\ \xi \\ \eta \end{pmatrix}.$$

We estimate

$$\begin{aligned} |\xi_1| = |-2x\xi + b\zeta| &\geq 2|x||\xi| - |b||\zeta| \\ &\geq 2|x||\xi| - |b|\sqrt{|\xi/\lambda|^2 - \eta^2} \\ &\geq \left(2|x| - |b||\lambda^{-1}|\right)|\xi|. \end{aligned}$$

As $\mathbf{x} = (x,\ y,\ z) \in \widetilde{\mathcal{D}} \cap F(\widetilde{\mathcal{D}})$, $a = 5$, $b = 0.1$ and $r = 3$, we have

$$a + bz - x^2 \leq r, \ \textit{i.e.}\ 5 + 0.1z - x^2 \leq 3.$$

Then we have $x^2 \geq 2 + 0.1z \geq 2 - 0.3$. Thus $|x| \geq \sqrt{1.7} > 1.3$. We assume that $\lambda = 1.5$. As $|x| > 1.3$, we have

$$|\xi_1| > \left(2 \cdot 1.3 - 0.1 \cdot \frac{2}{3}\right)|\xi| > 2.533|\xi|.$$

We next estimate

$$\begin{aligned}
\lambda\sqrt{{\eta_1}^2+{\zeta_1}^2} &< \lambda\sqrt{\xi^2+|\xi/\lambda|^2-\zeta^2} \\
&< \lambda\sqrt{\xi^2+\xi^2\lambda^{-2}} \\
&< \left(\lambda\sqrt{1+\lambda^{-2}}\right)|\xi| < 2.167|\xi|.
\end{aligned}$$

Therefore,

$$|\xi_1| > 2.533|\xi| > 2.167|\xi| > \lambda\sqrt{{\eta_1}^2+{\zeta_1}^2},$$

Then, we obtain $(DF)_{\mathbf{x}}\mathbf{v} = (\xi_1,\ \eta_1,\ \zeta_1) \in C^u(F(\mathbf{x}))$, and

$$\|(\xi_1,\ \eta_1,\ \zeta_1)\| = |\xi_1| > 2.533|\xi| > 2.167|\xi| > 1.5|\xi| = \lambda\|(\xi,\ \eta,\ \zeta)\|.$$

This completes the proof of (1).

For the second claim of this lemma, for any $\mathbf{v} = (\xi,\ \eta,\ \zeta) \in C^s(\mathbf{x})$, we denote $(DF^{-1})_{\mathbf{x}}\mathbf{v}$ by $(\xi_{-1},\ \eta_{-1},\ \zeta_{-1})$ *i.e.*,

$$\begin{pmatrix} \xi_{-1} \\ \eta_{-1} \\ \zeta_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ b^{-1} & 2b^{-1}y & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \eta \\ \zeta \\ b^{-1}(\xi+2y\eta) \end{pmatrix}.$$

As before we take $b = 0.1$ and $\lambda = 1.5$. As $\mathbf{x} \in \widetilde{\mathcal{D}} \cap F^{-1}(\widetilde{\mathcal{D}})$, we have

$$\left|b^{-1}\left(-a+x+y^2\right)\right| \leq r,$$

and then

$$y^2 \geq a - x - br \geq 5 - 3 - 0.1\cdot 3 = 1.7.$$

Thus $|y| \geq \sqrt{1.7} > 1.3$. When the vector $\mathbf{v}$ satisfies $|\zeta| \geq \lambda|\eta|$, that is, $|\zeta| \geq 1.5|\eta|$, we estimate

$$\sqrt{{\eta_{-1}}^2+{\zeta_{-1}}^2} = \sqrt{\zeta^2+b^{-2}(\xi+2y\eta)^2} \geq |\zeta| \geq 1.5|\eta| = \lambda|\xi_{-1}|.$$

Similarly, when the vector $\mathbf{v}$ satisfies $\lambda|\eta| \geq |\zeta|$, we get

$$\begin{aligned}
\sqrt{{\eta_{-1}}^2+{\zeta_{-1}}^2} &= \sqrt{\zeta^2+b^{-2}\left(\xi+2y\eta\right)^2} \geq |b^{-1}|\,|\xi+2y\eta| \\
&\geq |b^{-1}|\left(2|y||\eta|-|\xi|\right) \\
&\geq |b^{-1}|\left(2|y||\eta|-\lambda^{-1}\sqrt{\eta^2+\lambda^2\eta^2}\right) \\
&\geq |b^{-1}|\left(2|y|-\sqrt{\lambda^{-2}+1}\right)|\eta| \geq 13.98|\eta| > 1.5|\eta| = \lambda|\xi_{-1}|.
\end{aligned}$$

Therefore, we obtain

$$(\xi_{-1},\ \eta_{-1},\ \zeta_{-1}) \in C^s(F^{-1}(\mathbf{x})).$$

We now estimate $\| (DF^{-n})_{\mathbf{x}}\, \mathbf{v}\|$ for every $\mathbf{v} \in C^s(\mathbf{x})$. We divide the proof which the vector $\mathbf{v}$ is expanded by the derivative of the inverse into three parts. When $\|\mathbf{v}\| = \max\{|\xi|,\ |\eta|,\ |\zeta|\} = |\xi|$, that is, $|\xi| \geq |\eta|$ and $|\xi| \geq |\zeta|$, we estimate

$$\left|b^{-1}(\xi + 2y\eta)\right| \geq |b^{-1}|\left(2|y||\eta| - |\xi|\right),$$

as $\sqrt{\eta^2 + \zeta^2} \geq \lambda|\xi|$ and $|\xi| \geq |\zeta|$, we have

$$|b^{-1}|\left(2|y||\eta| - |\xi|\right) \geq |b^{-1}|\left(2|y|\sqrt{\lambda^2 - 1} - 1\right)|\xi| > 19.07|\xi|.$$

Therefore we have $\|(DF^{-1})_{\mathbf{x}}\mathbf{v}\| = |b^{-1}(\xi + 2y\eta)| \geq \lambda\|\mathbf{v}\|$. When $\|\mathbf{v}\| = |\eta|$, we similarly estimate

$$\begin{aligned}\|(DF^{-1})_{\mathbf{x}}\mathbf{v}\| &= \left|b^{-1}(\xi + 2y\eta)\right| \geq |b^{-1}|\left(2|y||\eta| - |\xi|\right)\\ &\geq |b^{-1}|(2|y| - 1)|\eta| > 16|\eta| > \lambda\|\mathbf{v}\|.\end{aligned}$$

Let $\|\mathbf{v}\| = |\zeta|$. We denote $(DF^{-2})_{\mathbf{x}}\mathbf{v}$ as $(\xi_{-2},\ \eta_{-2},\ \zeta_{-2})$ which is given by,

$$\begin{pmatrix}\xi_{-2}\\ \eta_{-2}\\ \zeta_{-2}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ b^{-1} & 2b^{-1}y_{-1} & 0\end{pmatrix}\begin{pmatrix}\xi_{-1}\\ \eta_{-1}\\ \zeta_{-1}\end{pmatrix} = \begin{pmatrix}\zeta\\ b^{-1}(\xi + 2y\eta)\\ b^{-1}(\eta + 2y_{-1}\zeta)\end{pmatrix},$$

where $(x_{-1},\ y_{-1},\ z_{-1}) = F^{-1}(x,\ y,\ z)$. As similarly we have

$$\begin{aligned}\|(DF^{-2})_{\mathbf{x}}\mathbf{v}\| &= \left|b^{-1}(\eta + 2y_{-1}\zeta)\right| \geq |b^{-1}|\left(2|y_{-1}||\zeta| - |\eta|\right)\\ &\geq |b^{-1}|(2|y_{-1}| - 1)|\zeta| > 16|\zeta| > \lambda\|\mathbf{v}\|.\end{aligned}$$

Therefore we get $\|(DF^{-2})_{\mathbf{x}}\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ for all $\mathbf{v} \in C^s(\mathbf{x})$. This completes the proof of the lemma. ■

Acknowledgments

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