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An application of de Jong's theorem to commutative ring theory –Positivity conjecture of Serre and symbolic powers–

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This is a joint work with Paul Roberts (see [8]).

Introduction

Around forty years ago, Serre [14] introduced a homological definition of intersection multiplicity for modules over a regular local ring and showed that it satisfied many of the properties which should hold for intersection multiplicities. We denote this multiplicity $\chi_R(M, N)$, where M and N are finitely generated modules over the regular local ring R such that $M \otimes_R N$ is a module of finite length (we recall the definition of $\chi_R(M, N)$ in section 1). Serre showed that the condition that $M \otimes_R N$ has finite length implies that $\dim(M) + \dim(N) \leq \dim(R)$, and he made the following conjectures:

1. (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, then $\chi_R(M, N) = 0$.

2. (Nonnegativity) It is always true that $\chi_R(M, N) \geq 0$.

3. (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, then $\chi_R(M, N) > 0$.

Serre proved these results for equicharacteristic local rings and unramified rings of mixed characteristic, leaving open the case in which R is a ramified regular local ring of mixed characteristic. The vanishing conjecture was proven about ten years ago by Roberts [11] and Gillet and Soulé [3] using K-theoretic methods. Recently, Gabber (see Berthelot [1], Hochster [4], or Roberts [13]) proved the nonnegativity conjecture using a recent result on resolution of singularities of de Jong [5]. In this paper we investigate the conditions which are necessary for the positivity conjecture to hold and show that the positivity conjecture implies a condition on the intersections of symbolic powers of prime ideals in regular local rings.

1 A criterion for positivity

In this section we set up notation, describe the construction of Gabber that we use, and give a criterion (Theorem 1.2) for the positivity conjecture to hold.

Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring with residue class field $k = R/\mathfrak{m}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy

1. $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$
2. $ht_R\,\mathfrak{p} + ht_R\,\mathfrak{q} = d.$

Serre defined the intersection multiplicity of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ to be

$$\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d}(-1)^i \text{length}(\text{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q})).$$

Serre's positivity conjecture states that in this situation, it follows that $\chi_R(R/\mathfrak{p}, R/\mathfrak{q})$ is positive.

In the next few paragraphs we describe the construction used in Gabber's proof of the nonnegativity conjecture, after which we state the consequence of his theorem that we will use. The whole proof is based on a theorem of de Jong [5] on the existence of regular alterations, which can be considered a slightly weaker version of resolution of singularities. It follows from de Jong's theorem that there exists a graded prime ideal I in the graded polynomial ring $A = R[X_0, X_1, \ldots, X_n]$ for some n such that the following three conditions hold:

1. $I \cap R = \mathfrak{q}$.
2. $Proj(A/I)$ is a regular scheme.
3. the morphism $Proj(A/I) \to Spec\,R/\mathfrak{q}$ is generically finite; that is, the extension of function fields defined by this morphism is finite.

We remark that the existence of the ideal I with the above properties requires that the regular local ring R is essentially of finite type over a field or a complete discrete valuation ring, but that the multiplicity conjectures can be reduced to this case. We refer to Berthelot [1] and Hochster [4] for details of this reduction. Throughout this section we assume that these properties hold and that this construction can be carried out. We let X denote $Proj(A/I)$.

The first step in Gabber's proof of nonnegativity reduces the computation of intersection multiplicities to a computation on associated graded rings. Let $gr_I(A)$ be the associated graded ring of I; since I is a graded ideal of the graded ring A, $gr_I(A)$ is a bigraded ring. One grading, which we sometimes refer to as the first grading, is induced by the grading on A, while the second grading is determined by the powers of I. To the ring $gr_I(A)$ we associate a scheme Y as follows. The component of $gr_I(A)$ of degree 0 in the second grading is A/I, which defines the projective scheme $X = Proj(A/I)$. For each element x of degree one in A/I there is an open affine subset of $Proj(A/I)$ with associated ring consisting of the elements of degree zero in the localization $(A/I)_x$. Each such x defines an element of $gr_I(A)$ of degree zero in the second grading. For each such x, we define an open affine subset of Y by taking $Spec(gr_I(A)_{(x)})$, where $gr_I(A)_{(x)}$ is the ring of elements of degree zero in the first grading in the localization $gr_I(A)_x$. We denote Y by $Proj(gr_I(A))$ and use similar notation for analogous schemes defined by other bigraded rings. Note that since I is locally generated by a regular sequence, the associated graded ring $gr_I(A)$ is locally isomorphic to the symmetric algebra of I/I^2 over A/I. Thus if we define a scheme $Proj(Sym_{A/I}(I/I^2))$ associated to the bigraded ring $Sym_{A/I}(I/I^2)$ using a similar definition, the map induced on schemes by the natural surjection from $Sym_{A/I}(I/I^2)$ to $gr_I(A)$ is an isomorphism.

Throughout this paper, we are concerned both with graded rings and with the schemes that they define. To maintain consistency, when we speak of the dimension of a graded ring or module, we mean the dimension of the associated scheme or sheaf, which is generally one less than the dimension as a ring or module.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R satisfying the conditions stated at the beginning of this section. Let $\tilde{I}$ be the image of the ideal I in $A/\mathfrak{p}A$. We have a surjective map from $Sym_{A/I}(I/I^2)$ to $gr_{\tilde{I}}(A/\mathfrak{p}A)$. Let K denote the kernel of this map, and let $\mathcal{K}$ denote the associated sheaf of ideals in $\mathcal{O}_Y$, where $\mathcal{O}_Y$ denotes the structure sheaf of Y. Then $\mathcal{K}$ defines a closed subscheme of Y which is the projective scheme defined by $gr_{\tilde{I}}(A/\mathfrak{p}A)$. The dimension of this subscheme is equal to the dimension of $Proj(A/\mathfrak{p}A)$, which is $n + \dim(R/\mathfrak{p})$. Let $\mathcal{I}$ denote the sheaf of ideals in $\mathcal{O}_Y$ defined by the ideal $I/I^2 \oplus I^2/I^3 \oplus \cdots$ of $gr_I(A)$.

We wish to define an Euler characteristic $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$. We first recall the construction of $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$. The sheaf $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ can be computed using a locally free resolution in either variable and tensoring with the other, or by taking the associated sheaves of the local computations of Tor. The locally free resolutions are finite since the graded ring $gr_I(A)$ is locally a polynomial ring over a regular ring, which follows from the fact that $Proj(A/I)$ is a regular scheme. Each $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ is a coherent sheaf over $Proj(A/I)$, so we can then take its sheaf cohomology and obtain a finitely generated R-module. Furthermore, since $\mathfrak{p} + \mathfrak{q}$ is primary to $\mathfrak{m}$, the support of $\mathcal{O}_Y/\mathcal{I} \otimes \mathcal{O}_Y/\mathcal{K}$ lies over the closed point of R. Hence the same is true for $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ for each i, and the sheaf cohomology modules they define are thus R-modules of finite length. For any coherent sheaf $\mathcal{F}$ of $Proj(A/I)$ with support lying over the maximal ideal of R, we define the Euler characteristic to be the alternating sum of lengths of sheaf cohomology:

$$\chi(\mathcal{F}) = \sum_i (-1)^i \text{length}(H^i(X, \mathcal{F})).$$

We then define

$$\begin{aligned}\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}) &= \sum_i (-1)^i \chi(\text{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})) \\ &= \sum_i \sum_j (-1)^{i+j} \text{length}(H^j(X, \text{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}))).\end{aligned}$$

We will use these definitions and this notation in similar situations below for other modules and for ideals and modules in other graded rings.

The first main step in this construction is to reduce the computation of $\chi_R(R/\mathfrak{p}, R/\mathfrak{q})$ to that of the Euler characteristic $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$ which we just defined (we refer to the references [1], [4], and [13] cited above for details of how this is done). The second step of the construction is to replace the graded ring $gr_I(A)$ by a polynomial ring over a quotient of A/I. Let $B = (A/I) \otimes_R k$, where $k = R/\mathfrak{m}$. We define a map $\alpha : I/I^2 \to \Omega_A \otimes_A B$ by sending x to $dx \otimes 1$, where Ω_A is the module of differentials on A. Since I annihilates B and d is a derivation, α is a graded A-module homomorphism and induces a ring homomorphism $\beta : Sym_{A/I}(I/I^2) \to Sym_B(\Omega_A \otimes_A B)$. The main point is that the assumption that $Proj(A/I)$ is regular implies that $\alpha \otimes_A 1 : I \otimes_A B \to \Omega_A \otimes_A B$ is locally a split injection so that the image in $Sym_B(\Omega_A \otimes_A B)$ can be used compute Euler characteristics. We give an algebraic version of this construction.

Let $s_1, \ldots, s_d$ be a minimal set of generators for $\mathfrak{m}$. At this point we must assume that R is equicharacteristic or ramified, so that $\Omega_A \otimes_A B$ is a free B-module with basis $ds_1, \ldots, ds_d, dX_0, \ldots, dX_n$ (see [1], [4], or [13]). For simplicity of notation, we put $S_i = ds_i$, $T_j = dX_j$ for $i = 1, \ldots, d$ and $j = 0, \ldots, n$. Let $\{U_{jk} \mid j,\ k = 0, \ldots, n\}$ be an additional set of $(n+1)^2$ variables. Then we define a ring homomorphism

$$\varphi : B[S_1, \ldots, S_d, T_0, \ldots, T_n] \longrightarrow B[S_1, \ldots, S_d, U_{00}, U_{01}, \ldots, U_{nn}]$$

by letting $\varphi(T_j) = X_0 U_{j0} + X_1 U_{j1} + \cdots + X_n U_{jn}$ for each j. We then have maps

$$\begin{aligned}&Sym_{A/I}(I/I^2) \xrightarrow{\beta} Sym_B(\Omega_A \otimes_A B) \\ &= B[S_1, \ldots, S_d, T_0, \ldots, T_n] \xrightarrow{\varphi} B[S_1, \ldots, S_d, U_{00}, \ldots, U_{nn}].\end{aligned}$$

We denote $B[S_1, \ldots, S_d, T_0, \ldots, T_n]$ by F and $B[S_1, \ldots, S_d, U_{00}, \ldots, U_{nn}]$ by G. Both F and G have natural structures of bigraded rings. In this case the first grading is induced by that of A; in the first grading we let S_i and U_{jk} have degree zero, and we let T_j have degree one. All of the variables have degree 1 in the second grading. With these assumptions, the above maps are maps of bigraded rings. We thus have schemes $Proj(F)$ and $Proj(G)$ defined as above. Put $Z = Proj(F)$ and $W = Proj(G)$. We denote I_F and I_G the ideals generated by S_i, T_j and S_i, U_{jk} respectively in F and G, and $\mathcal{I}_F$ and $\mathcal{I}_G$ the associated ideal sheaves to I_F and I_G, respectively. We let K_F and K_G denote the ideals generated by the images of K in F and G respectively, and $\mathcal{K}_F$ and $\mathcal{K}_G$ the associated ideal sheaves to K_F and K_G, respectively. Since the maps from $Sym_{A/I}(I/I^2) \otimes_R k$ to F and G are locally inclusions of polynomial rings obtained by adjoining variables, the dimension of F/K_F is $n+d+1$ and the dimension of G/K_G is $d+(n+1)^2$. We have Euler characteristics $\chi_Z(\mathcal{O}_Z/\mathcal{I}_F, \mathcal{O}_Z/\mathcal{K}_F)$ and $\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$ defined by the same process which we outlined above. These Euler characteristics are simpler to compute than those on Y, since the ideals I_F and I_G are generated by variables in a polynomial ring, so that the Tors in the first step can be computed using Koszul complexes. We note that in the case of F the degrees of the T_i must be taken into account in computing Euler characteristics. We then have

Theorem 1.1 (Gabber) *With notation as above, we have*

1. $[R(X) : Q(R/\mathfrak{q})]\cdot\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) = \chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K})$, *where $R(X)$ is the function field of $X = Proj(A/I)$ and $Q(R/\mathfrak{q})$ is the field of fractions of $R/\mathfrak{q}$.*

2. $\chi_Y(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{K}) > 0$ *if and only if* $\chi_{Y'}((\mathcal{O}_Y/\mathcal{I}) \otimes_R k, (\mathcal{O}_Y/\mathcal{K}) \otimes_R k) > 0$, *where we put* $Y' = Proj(gr_I(A) \otimes_R k)$.

3. $\chi_{Y'}(\mathcal{O}_Y/\mathcal{I} \otimes_R k, \mathcal{O}_Y/\mathcal{K} \otimes_R k) = \chi_Z(\mathcal{O}_Z/\mathcal{I}_F, \mathcal{O}_Z/\mathcal{K}_F) = \chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$.

From the theorem above, it suffices to test the positivity of $\chi_W(\mathcal{O}_W/\mathcal{I}_G, \mathcal{O}_W/\mathcal{K}_G)$. We note that $k[S_i, U_{jk}]$ and $k[S_i]$ is a subring of G and F, respectively. In what follows, if J is an ideal of one of the graded rings we are considering, we denote $\overline{J}$ the ideal of elements a such that $(X_0, \dots, X_n)^k a \subseteq J$ for some integer k.

Then we have the following theorem:

Theorem 1.2 *With notation as above, the following statements are equivalent:*

1. $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

2. $\overline{K_G} \cap k[S_1, \dots, S_d, U_{00}, \dots U_{nn}] = 0$.

3. $\overline{K_F} \cap k[S_1, \dots, S_d] = 0$.

2 Positivity and Symbolic Powers

For a prime ideal $\mathfrak{q}$ of a commutative ring R, $\mathfrak{q}^{(k)} = \mathfrak{q}^k R_{\mathfrak{q}} \cap R$ is called the k-th *symbolic power* of $\mathfrak{q}$. The symbolic power can also be defined as the set of $r \in R$ such that there exists $s \notin \mathfrak{p}$ with $sr \in \mathfrak{p}^n$.

In this section we discuss the following conjecture:

Conjecture 2.1 *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $ht_R\,\mathfrak{p} + ht_R\,\mathfrak{q} = d$. Then*

$$\mathfrak{p} \cap \mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$$

for any $k > 0$.

We show below that in the case where $R \supset \mathbb{Q}$, we can easily solve the conjecture affirmatively.

Then we have the following theorem:

Theorem 2.2 *Let (R, m) be a regular local ring that is equicharacteristic or ramified. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R that satisfy the following three conditions;*

1. $\sqrt{\mathfrak{p}+\mathfrak{q}} = m$,
2. $ht_R\,\mathfrak{p} + ht_R\,\mathfrak{q} = d$,
3. $\chi_R(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Then we have $\mathfrak{p} \cap \mathfrak{q}^{(k)} \subseteq m^{k+1}$ for any positive integer k.

If $(R, \mathfrak{m})$ is equicharacteristic in Theorem 2.2, then the third assumption for $\mathfrak{p}$ and $\mathfrak{q}$ follows from the first and second ones by the positivity theorem due to Serre [14]. Therefore, if $(R, \mathfrak{m})$ is equicharacteristic, Conjecture 2.1 is true.

By Theorem 2.2, Serre's positivity conjecture implies Conjecture 2.1 in the case where R is ramified.

Since the connection between the positivity conjecture and Conjecture 2.1 depends on the construction of section 1, which requires that R is equicharacteristic or ramified, we do not know whether Theorem 2.2 is true in the case of an unramified regular local ring of mixed characteristic. Conjecture 2.1 is open in that case.

Here we give a remark on symbolic powers.

Remark 2.3 Let $(R, \mathfrak{m})$ be a regular local ring and $\mathfrak{q}$ a prime ideal of R.

(a) For any $k > 0$, $\mathfrak{q}^{(k)} \subseteq m^k$ (see Theorem (38.3) in Nagata [10]).

(b) Assume that $\mathfrak{q} \subseteq \mathfrak{m}^2$. If R contains a field of characteristic 0 (resp. $p > 0$), then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$ for any $k > 0$ (resp. $0 < k < p$) by Proposition 2.4 below.

On the other hand, we now present an example in positive characteristic in which $\mathfrak{q} \subseteq \mathfrak{m}^2$ but $\mathfrak{q}^{(p)} \not\subseteq \mathfrak{m}^{p+1}$.

Let E be a 3-dimensional regular local ring with regular system of parameters x, y, z, and let

$$\Phi \ : \ R = E[[S,T,U,V]] \longrightarrow E[[W]]$$

be the ring homomorphism of formal power series rings over E defined by letting $\Phi(S) = x^3W$, $\Phi(T) = y^3W$, $\Phi(U) = z^3W$, and $\Phi(V) = (xyz)^2W$. Put $\mathfrak{q} = Ker(\Phi)$. Then, $\mathfrak{q}$ is generated by the following 7 elements:

$$\begin{array}{lll} y^3S - x^3T, & z^3T - y^3U, & x^3U - z^3S, \\ xV - y^2z^2S, & yV - z^2x^2T, & zV - x^2y^2U, \\ V^2 - xyz^4ST & & \end{array}$$

Therefore $\mathfrak{q}$ is contained in $\mathfrak{m}^2$, where $\mathfrak{m} = (x,y,z,S,T,U,V)R$.

If E contains a field of characteristic 0, then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$, but it can be shown (see [12]) that $\mathfrak{q}^{(k)} \not\subseteq \mathfrak{m}^{k+2}$ for any $k > 0$, and it follows that the *symbolic Rees ring* $\oplus_{k\geq 0}\mathfrak{q}^{(k)}$ is not Noetherian in this case. It is shown in [12] that a certain subring of $\oplus_{k\geq 0}\mathfrak{q}^{(k)}$ gives a counterexample to the Hilbert's fourteenth problem which is in many ways simpler than that of Nagata [9].

If E contains a field of characteristic $p > 0$, then $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^{k+1}$ for $0 < k < p$, but $\mathfrak{q}^{(p)} \not\subseteq \mathfrak{m}^{p+1}$ for each prime integer p (see [7]; this example is similar to the one described below in mixed characteristic). In this case, an element in $\mathfrak{q}^{(p)} \setminus \mathfrak{m}^{p+1}$ makes the symbolic Rees ring $\oplus_{k\geq 0}\mathfrak{q}^{(k)}$ Noetherian for each p ([6], [7]).

We give an example in the case where E has mixed characteristic. Assume that E is a regular local ring of mixed characteristic such that its residue class field is of characteristic 2 and that x divides 2. Since $\mathfrak{q} \cap E = 0$, we have

$$\begin{aligned} & V^2 - 2x^{-1}y^2z^2SV + xy^4zSU + xyz^4ST - x^4yzTU \\ = \ & \frac{1}{x^2}\left\{(xV - y^2z^2S)^2 + yz(y^3S - x^3T)(x^3U - z^3S)\right\} \in \mathfrak{q}^{(2)}. \end{aligned}$$

Therefore $\mathfrak{q}^{(2)} \not\subseteq \mathfrak{m}^3$ in this case.

(c) Eisenbud-Mazur [2] studied conditions that imply $\mathfrak{m}\mathfrak{q}^{(k)} \supseteq \mathfrak{q}^{(k+1)}$.

The next proposition implies Conjecture 2.1 in the case where $R \supset \mathbb{Q}$.

Proposition 2.4 *Let R be a regular local ring containing a field and $\mathfrak{q}$ a prime ideal of R. Suppose that $\mathfrak{q}^{(k)} \subseteq \mathfrak{m}^l$ for some positive integers k and l.*

a) If the characteristic of R is 0, then $\mathfrak{q}^{(k+1)} \subseteq \mathfrak{m}^{l+1}$.

b) If the characteristic of R is $p > 0$ and if $l < p$, then $\mathfrak{q}^{(k+1)} \subseteq \mathfrak{m}^{l+1}$.

References

[1] P. Berthelot, *Altérations de variétés algébriques [d'aprés A. J. de Jong]*, Sém. Bourbaki **48** 1995/96 No. 815.

[2] D. Eisenbud and B. Mazur, *Evolutions, symbolic squares, and Fitting ideals*, J. reine angew. Math. **488** (1997), 189–201.

[3] H. Gillet and C. Soulé, *K-théorie et nullité des multiplicites d'intersection*, C. R. Acad. Sci. Paris Ser. I Math. **300** (1985), 71–74.

[4] M. Hochster, *Nonnegativity of intersection multiplicities in ramified regular local rings following Gabber/De Jong/Berthelot*, unpublished notes.

[5] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. IHES 83 (1996), 51–93.

[6] K. Kurano, *On finite generation of Rees rings defined by filtration*, J. Math. Kyoto Univ. **34** (1994), 73–86.

[7] K. Kurano, *Positive characteristic finite generation of symbolic Rees algebras and Roberts' counterexamples to the fourteenth problem of Hilbert*, Tokyo J. Math. **16** (1993), 473–496.

[8] K. Kurano and P. Roberts, *The positivity of intersection multiplicities and symbolic powers of prime ideals*, preprint.

[9] M. Nagata, *On the fourteenth problem of Hilbert*, Proc. Internat. Congress Math. (1958), Cambridge Univ. Press (1960), 452–462.

[10] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Appl. Math., Wiley, New York, 1962.

[11] P. Roberts, *The vanishing of intersection multiplicities and perfect complexes*, Bull. Amer. Math. Soc. **13** (1985), 127–130.

[12] P. ROBERTS, *An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem*, J. Alg. **132** (1990), 461–473.

[13] P. ROBERTS, *Recent developments on Serre's multiplicity conjectures: Gabber's proof of the nonnegativity conjecture*, to appear in L'Enseignement Mathématique.

[14] J-P. SERRE, *Algèbre locale. Multiplicités*, Lect. Note in Math., vol. 11, Springer-Verlag, Berlin, New York, 1965.

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