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LATTICE IDEALS, THEIR INITIAL IDEALS AND (CO-)GENERIC MONOMIAL IDEALS

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This article is based on a joint work [12] with E. Miller and B. Sturmfels, which has been submitted to a journal. But §3 contains extra topics and results.

ABSTRACT. Monomial ideals which are generic with respect to either their generators or irreducible components have minimal free resolutions derived from simplicial complexes. For a generic monomial ideal, the associated primes satisfy a saturated chain condition, and the Cohen-Macaulay property implies shellability for both the Scarf complex and the Stanley-Reisner complex. Reverse lexicographic initial ideals of generic lattice ideals are generic. Cohen-Macaulayness for cogeneric ideals is characterized combinatorially; in the cogeneric case the Cohen-Macaulay type is greater than or equal to the number of irreducible components. Methods of proof include Alexander duality and Stanley's theory of local h-vectors.

1. GENERICITY OF MONOMIAL IDEALS REVISITED

Let M be a monomial ideal minimally generated by monomials $m_1, \dots, m_r$ in a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. For a subset $\sigma \subseteq \{1, \dots, r\}$, we set $m_\sigma := \mathrm{lcm}(m_i \mid i \in \sigma)$, and $\mathbf{a}_\sigma := \deg m_\sigma \in \mathbb{N}^n$ the exponent vector of m_σ. Here $m_\emptyset = 1$. For a monomial $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$, we set $\deg_{x_i}(\mathbf{x}^\mathbf{a}) := a_i$, and we call $\mathrm{supp}(\mathbf{x}^\mathbf{a}) := \{i \mid a_i \neq 0\} \subseteq \{1, \dots, n\}$ the *support* of $\mathbf{x}^\mathbf{a}$.

Definition 1.1. A monomial ideal $M = \langle m_1, \dots, m_r \rangle$ is called *generic* if for any two distinct generators m_i, m_j of M which have the same positive degree in some variable x_s there exists a third monomial generator $m_l \in M$ which divides $m_{\{i,j\}} = \mathrm{lcm}(m_i, m_j)$ and satisfies $\mathrm{supp}(m_{\{i,j\}}/m_l) = \mathrm{supp}(m_{\{i,j\}})$.

The above definition of genericity is more inclusive than the one given by Bayer-Peeva-Sturmfels [1], but we will see that this definition permits the same algebraic conclusions as the one in [1]. There are important families of monomial ideals which are generic in the sense of Definition 1.1 but not in the sense of [1]. One such family is the initial ideals of generic lattice ideals as in Theorem 3.3. Here is another one:

Example 1.2. The *tree ideal* $M = \langle \left(\prod_{s \in I} x_s\right)^{n-|I|+1} \mid \emptyset \neq I \subseteq \{1, \dots, n\} \rangle$ is generic in the new sense but very far from generic in the old sense. This ideal is Artinian of colength $(n+1)^{n-1}$, the number of trees on $n+1$ labelled vertices.

Recall that a monomial ideal $M \subset S$ can be uniquely written as a finite irredundant intersection $M = \bigcap_{i=1}^r M_i$ of irreducible monomial ideals (i.e., ideals generated by powers of variables). We say M_i is an *irreducible component* of M.

Definition 1.3. A monomial ideal with irreducible decomposition $M = \bigcap_{i=1}^{r} M_i$ is called *cogeneric* if the following condition holds: if distinct irreducible components M_i and M_j have a minimal generator in common, there is an irreducible component $M_l \subset M_i + M_j$ such that M_l and $M_i + M_j$ do not have a minimal generator in common.

A monomial ideal M is cogeneric if and only if its *Alexander dual ideal* $M^{\mathbf{a}}$ is generic. See [11] or Section 4 for the relevant definitions. Cogeneric monomial ideals will be studied in detail in Section 4. The remainder of this section is devoted to basic properties of generic monomial ideals.

Let $M \subset S$ be a monomial ideal minimally generated by monomials $m_1, \ldots, m_r$ again. The following simplicial complex on r vertices, called the *Scarf complex* of M, was introduced by Bayer, Peeva and Sturmfels in [1]:

$$\Delta_M \quad := \quad \{\sigma \subseteq \{1, \ldots, r\} \mid m_\tau \neq m_\sigma \text{ for all } \tau \neq \sigma\}.$$

Let $S(-\mathbf{a}_\sigma)$ denote the free S-module with one generator e_σ in multidegree $\mathbf{a}_\sigma$. The *algebraic Scarf complex* F_{Δ_M} is the free S-module $\bigoplus_{\sigma \in \Delta_M} S(-\mathbf{a}_\sigma)$ with the differential

$$d(e_\sigma) \quad = \quad \sum_{i \in \sigma} \operatorname{sign}(i, \sigma) \cdot \frac{m_\sigma}{m_{\sigma \setminus \{i\}}} \cdot e_{\sigma \setminus \{i\}}$$

where $\operatorname{sign}(i, \sigma)$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of σ. It is known that F_{Δ_M} is always contained in the minimal free resolution of S/M as a subcomplex [1, §3], although F_{Δ_M} need not be acyclic in general. However we will see in Theorem 1.5 that it is acyclic if M is generic, as was the case under the old definition.

Lemma 1.4. *Let $M = \langle m_1, \ldots, m_r \rangle$ be a generic monomial ideal. If $\sigma \notin \Delta_M$, then there is a monomial $m \in M$ such that m divides m_σ and* $\operatorname{supp}(m_\sigma/m) = \operatorname{supp}(m_\sigma)$.

Proof. Choose $\sigma \notin \Delta_M$ maximal among subsets of $\{1, \ldots, r\}$ with label $\mathbf{a}_\sigma$. Then $m_\sigma = m_{\sigma \setminus \{i\}}$ for some $i \in \sigma$. If $\operatorname{supp}(m_\sigma/m_i) = \operatorname{supp}(m_\sigma)$, the proof is done. Otherwise, there is $\sigma \ni j \neq i$ with $\deg_{x_s} m_i = \deg_{x_s} m_j > 0$ for some x_s. Since M is generic, there is a monomial $m \in M$ which divides $m_{\{i,j\}}$ and satisfies $\operatorname{supp}(m_{\{i,j\}}/m) = \operatorname{supp}(m_{\{i,j\}})$. Since $m_{\{i,j\}}$ divides m_σ, the monomial m has the desired property. □

The following theorem extends results in [1] and is the main result in this section.

Theorem 1.5. *A monomial ideal M is generic if and only if the following two hold:*
(a) The algebraic Scarf complex F_{Δ_M} equals the minimal free resolution of S/M.
(b) No variable x_s appears with the same non-zero exponent in m_i and m_j for any edge $\{i, j\}$ of the Scarf complex Δ_M.

Proof. Suppose that M is generic. Then (b) is straightforward from the definition, and, using Lemma 1.4, (a) is proved by the same argument as in [1, Theorem 3.2].

Assuming (a) and (b), we show that M is generic. For any generator m_i let

$$A_i \;:=\; \{m_j \mid m_j \neq m_i \text{ and } \deg_{x_s} m_j = \deg_{x_s} m_i > 0 \text{ for some } s\}.$$

The set A_i can be partially ordered by letting $m_j \preceq m_{j'}$ if $m_{\{i,j\}}$ divides $m_{\{i,j'\}}$. It is enough to produce a monomial m_l as in Definition 1.1 whenever $m_j \in A_i$ is a minimal element for this partial order. Supposing, then, that m_j is minimal, use (a) to write

$$(1) \qquad \frac{m_{\{i,j\}}}{m_i} \cdot e_i - \frac{m_{\{i,j\}}}{m_j} \cdot e_j = \sum_{\{u,v\} \in \Delta_M} b_{u,v} \cdot d(e_{\{u,v\}})$$

where we may assume (by picking such an expression with a minimal number of nonzero terms) that the monomials $b_{u,v}$ are 0 unless $m_{\{u,v\}}$ divides $m_{\{i,j\}}$. There is at least one monomial m_l such that $b_{l,j} \neq 0$, and we claim $m_l \notin A_i$. Indeed, m_l divides $m_{\{i,j\}}$ because $m_{\{l,j\}}$ does, so if $\deg_{x_t} m_i < \deg_{x_t} m_j$ (which must occur for some t because m_j does not divide m_i), then $\deg_{x_t} m_l \leq \deg_{x_t} m_j$. Applying (b) to $m_{\{l,j\}}$ we get $\deg_{x_t} m_l < \deg_{x_t} m_j$, and furthermore $\deg_{x_t} m_{\{i,l\}} < \deg_{x_t} m_{\{i,j\}}$, whence $m_l \notin A_i$ by minimality of m_j. So if $\deg_{x_s} m_{\{i,j\}} > 0$ for some s, then either $\deg_{x_s} m_l < \deg_{x_s} m_j$ by (b), or $\deg_{x_s} m_l < \deg_{x_s} m_i$ because $m_l \notin A_i$. □

Remark 1.6. Condition (a) in Theorem 1.5 splits into two parts: minimality and acyclicity. For the Scarf complex of *any* monomial ideal, minimality is automatic since face labels $\mathbf{a}_\sigma$ of Δ_M are distinct. It is acyclicity which must be checked.

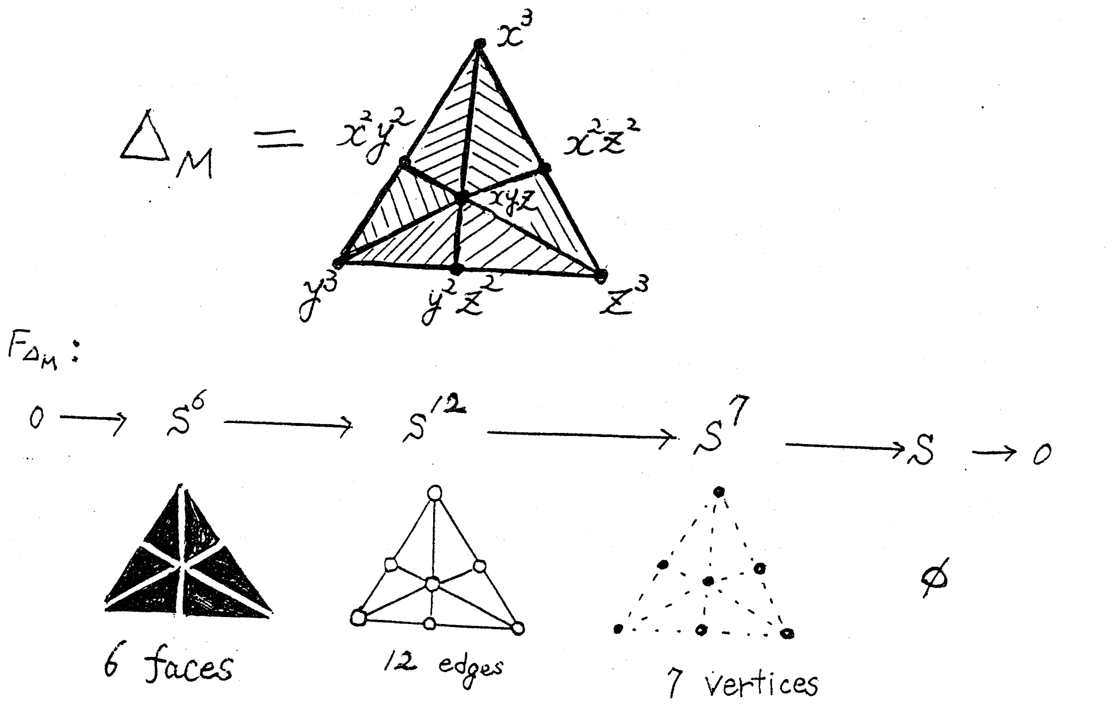
For an arbitrary monomial ideal M, Bayer and Sturmfels [2, §2] constructed a polyhedral complex hull(M) supporting a (not necessarily minimal) free resolution of M. Definition 1.1 suffices to imply that the hull complex equals the Scarf complex:

Proposition 1.7. *If M is a generic monomial ideal, then the hull complex* hull(M) *coincides with Δ_M, and in this case the hull resolution $F_{\mathrm{hull}(M)} = F_{\Delta_M}$ is minimal.*

Proof. Essentially unchanged from the proof of [2, Theorem 2.9]. □

Example 1.2 *(continued)* The Scarf complex Δ_M of M is the first barycentric subdivision of the $(n-1)$-simplex. By Theorem 1.5, F_{Δ_M} gives a minimal free resolution of S/M. Miller [11] also constructed a minimal free resolution of S/M as a *cohull resolution*, derived essentially from the coboundary complex of a permutahedron.

The following figure explains the (algebraic) Scarf complex of the tree ideal $M = \langle xyz, x^2y^2, y^2z^2, z^2x^2, x^3, y^3, z^3 \rangle$ of three variables.



2. Associated Primes and Irreducible Components

In this section we study the primary decomposition of a generic monomial ideal M. For a monomial prime P in S, we identify the homogeneous localization $(S/M)_{(P)}$ with the algebra $k[x_i \mid x_i \in P]/M_{(P)}$, where $M_{(P)}$ is the monomial ideal of $k[x_i \mid x_i \in P]$ gotten from M by setting equal to 1 all the variables not in P.

Remark 2.1. If M is a generic monomial ideal then so is $M_{(P)}$.

Let $M = \bigcap_{i=1}^{r} M_i$ be the irreducible decomposition of a monomial ideal M. Then we have $\{\text{rad}(M_i) \mid 1 \leq i \leq r\} = \text{Ass}(S/M)$. Note that distinct irreducible components may have the same radical. Bayer, Peeva and Sturmfels [1, §3] give a method for computing the irreducible decomposition of a generic monomial ideal (in the old definition). The generalization of this method by Miller [11, Theorem 5.12] shows that [1, Theorem 3.7] remains valid here, as we will show in Theorem 2.2 below.

Recall that $\text{codim}(I) \leq \text{codim}(P) \leq \text{proj-dim}_S(S/I) \leq n$ for any graded ideal $I \subset S$ and any associated prime $P \in \text{Ass}(S/I)$, and $\text{codim}(I) = \text{proj-dim}_S(S/I)$ if and only if S/I is Cohen-Macaulay. There always exists a minimal prime $P \in \text{Ass}(S/I)$ with $\text{codim}(P) = \text{codim}(I)$. But in general there is no $P \in \text{Ass}(S/I)$ with $\text{codim}(P) = \text{proj-dim}_S(S/I)$. For example, if $I = \langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle$, then $\text{proj-dim}_S(S/I) = 3$.

Theorem 2.2. *Let $M \subset S$ be a generic monomial ideal. Then*

(a) For each integer i with $\text{codim}(M) < i \leq \text{proj-dim}_S(S/M)$, there is an embedded associated prime $P \in \text{Ass}(S/M)$ with $\text{codim}(P) = i$.

(b) For all $P \in \text{Ass}(S/M)$ there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ with $\text{codim}(P_i) = \text{codim}(P_{i-1}) - 1$ for all i and P_t is a minimal prime of M.

Proof. (a) This was proved in [21] under the old definition of genericity. Using Theorem 1.5 and [11, Theorem 5.12], the argument in [21] also works here.

(b) It suffices to show that for any embedded prime P of M there is an associated prime $P' \in \text{Ass}(S/M)$ with $\text{codim}(P') = \text{codim}(P) - 1$ and $P' \subset P$. The localization $P_{(P)}$ of P is a maximal ideal of $S_{(P)}$, and an embedded prime of $M_{(P)}$, so there is a prime $P'_{(P)} \subset S_{(P)}$ such that $P'_{(P)} \in \text{Ass}(S/M)_{(P)}$, $\text{codim}(P'_{(P)}) = \text{codim}(P_{(P)}) - 1$ and $P'_{(P)} \subset P_{(P)}$ by (a) applied to the generic ideal $M_{(P)}$. The preimage $P' \subset S$ of $P'_{(P)} \subset S_{(P)}$ has the expected properties. □

Remark 2.3. Let $M \subset S$ be a generic monomial ideal, and $P, P' \in \text{Ass}(S/M)$ such that $P \supset P'$ and $\text{codim}\, P \geq \text{codim}\, P' + 2$. Theorem 2.2 does *not* state that there is an associated prime between P and P'. For example, set $M = \langle ac, bd, a^3b^2, a^2b^3\rangle$. Then $\langle a, b\rangle, \langle a, b, c, d\rangle \in \text{Ass}(S/M)$, but there is no associated prime between them.

Following [1, §3], we next define the *extended Scarf complex* Δ_{M^*} of M. Let

$$M^* := M + \langle x_1^D, \ldots, x_n^D\rangle \tag{2}$$

with D larger than any exponent on any minimal generator of M. We index the new monomials x_s^D just by their variables x_s; so the vertex set of Δ_{M^*} is a subset of $\{1, \ldots, r\} \cup \{x_1, \ldots, x_n\}$. This subset is proper if M contains a power of a variable.

Recall ([1, Corollary 5.5] for the old genericity or [11, Proposition 5.16] for the new) that Δ_{M^*} is a regular triangulation of an $(n-1)$-simplex Δ. The vertex set of Δ equals $\{x_1, \ldots, x_n\}$ unless M contains a power of a variable. The restriction of Δ_{M^*} to $\{1, \ldots, r\}$ equals the Scarf complex Δ_M of M. We next determine the restriction of Δ_{M^*} to $\{x_1, \ldots, x_n\}$.

The radical rad(M) of M is a square-free monomial ideal. Let $V(M)$ denote the corresponding *Stanley-Reisner complex*, which consists of all subsets of $\{x_1, \ldots, x_n\}$ which are not support sets of monomials in M. Then we have the following:

Lemma 2.4. *For a generic monomial ideal M, the restriction of the extended Scarf complex Δ_{M^*} to $\{x_1, \ldots, x_n\}$ coincides with the Stanley-Reisner complex $V(M)$.*

Proof. Every facet σ of Δ_{M^*} gives an irreducible component of M; see [1, Theorem 3.7] and [11, Theorem 5.12]. The radical of that component represents the face $\sigma \cap \{x_1, \ldots, x_n\}$ of $V(M)$. The facets of $V(M)$ arise in this way from the irreducible components whose associated primes are minimal. □

The following theorem generalizes [21, Corollary 2.4]. For the definition of shellability, see [16, §III.2] or [22, Lecture 8].

Theorem 2.5. *Let M be a generic monomial ideal. If M has no embedded associated primes, then M is Cohen-Macaulay. In this case, both Δ_M and $V(M)$ are shellable.*

Proof. The first statement immediately follows from Theorem 2.2. For the second statement we note that all facets σ of Δ_{M^*} have the following property:

$$|\sigma \cap \{1, \ldots, r\}| = \operatorname{codim} M \quad \text{and} \quad |\sigma \cap \{x_1, \ldots, x_n\}| = \dim S/M. \tag{3}$$

In particular, both cardinalities in (3) are independent of the facet σ. On the other hand, Δ_{M^*} is shellable since it is a regular triangulation of a simplex. A theorem of Björner [3, Theorem 11.13] implies that the restrictions of Δ_{M^*} to $\{1, 2, \ldots, r\}$ and to $\{x_1, \ldots, x_n\}$ are both shellable. We are done in view of Lemma 2.4. □

Remark 2.6. The shellability of Δ_{M^*} also implies the following result. If M is generic and $P, P' \in \operatorname{Ass}(S/M)$, then there is a sequence of associated primes $P = P_0, P_1, \ldots,$ $P_t = P'$ with $\operatorname{codim}(P_i + P_{i-1}) = \min\{\operatorname{codim}(P_i), \operatorname{codim}(P_{i-1})\} + 1$ for all $1 \le i \le t$. If M is pure dimensional, this simply says that S/M is connected in codimension 1.

Theorem 2.5 and Remark 2.6 suggest the following combinatorial problems:

Problem 2.7. (a) Characterize all collections $\mathcal{A}$ of monomial primes for which there exists a generic monomial ideal M with $\mathcal{A} = \operatorname{Ass}(S/M)$.

(b) Characterize the Stanley-Reisner complexes $V(M)$ of Cohen-Macaulay generic monomial ideals M.

If we put further restrictions on the generators of a generic monomial ideal M, then, since the extended Scarf complex Δ_{M^*} is a triangulation of a simplex, we can apply Stanley's theory of local h-vectors [16]. The next two results will be reinterpreted in Section 4 in terms of cogeneric ideals using Alexander duality [11].

Again let M^* be as in (2), and define the *excess* of a face $\sigma \in \Delta_{M^*}$ to be $e(\sigma) :=$ $\#\operatorname{supp}(m_\sigma) - \#\sigma$. This agrees, in our situation, with the definition of excess in [16].

Theorem 2.8. *If M is generic and all r generators $m_1, \dots, m_r$ have support of size c, i.e. $\#\operatorname{supp}(m_i) = c$ for all i, then M has at least $(c-1)\cdot r + 1$ irreducible components.*

Example 2.9. This is false without the assumption that M is generic. For instance, the non-generic monomial ideal $M = \langle x_1, y_1\rangle \cap \dots \cap \langle x_n, y_n\rangle$ has $r = 2^n$ generators, and each generator has support of size $c = n$, but M has only n irreducible components.

Proof. If $c = 1$, there is nothing to prove, so we may assume that $c \geq 2$. Set $\Gamma = \Delta_{M^*}$. The hypothesis on the generators of M means that Γ has n vertices of excess 0 and r vertices of excess $c-1$. To prove the assertion, we use the decomposition

$$h(\Gamma, x) \;=\; \sum_{W \in \Delta} \ell_W(\Gamma_W, x) \tag{4}$$

of the h-polynomial of Γ into local h-polynomials [16, eqn. (3)]. Here Δ denotes the simplex on $\{x_1, \dots, x_n\}$ and Γ_W the restriction of Γ to a face W of Δ. We have

$$\ell_W(\Gamma_W, x) = 1 \qquad \text{if } W = \emptyset. \tag{5}$$

Next, we consider the case $\#W = c$. In the Γ_W, the vertices corresponding to generators of M have excess $c-1$, and all other faces have excess less than $c-1$. So we have

$$\ell_W(\Gamma_W, x) = \ell_1(\Gamma_W)x + \ell_2(\Gamma_W)x^2 + \cdots + \ell_{c-1}(\Gamma_W)x^{c-1} \qquad \text{if } \#W = c, \tag{6}$$

where $\ell_1(\Gamma_W)$ is the number of generators of M whose support corresponds to the face W of Δ by [16, Example 2.3(f)]. Moreover $\ell_i(\Gamma_W) \geq \ell_1(\Gamma_W)$ for all $1 \leq i \leq c-1$ by [16, Theorem 5.2 and Theorem 3.3].

The coefficients of $\ell_W(\Gamma_W, x)$ are non-negative for all $W \in \Delta$ by [16, Corollary 4.7]. We now substitute the expressions in (5) and (6) into the sum on the right hand side of (4), and then we evaluate at $x = 1$. The number of irreducible components of M equals the number $f_{n-1}(\Gamma) = h(\Gamma, 1)$ of facets of Γ by [11, Theorem 5.12], hence

$$h(\Gamma,1) \;\geq\; 1 + \sum_{\#W=c} \Big(\sum_{i=1}^{c-1} \ell_i(\Gamma_W)\Big) \;\geq\; 1 + \sum_{\#W=c} (c-1)\cdot \ell_1(\Gamma_W) \;=\; (c-1)\cdot r + 1.$$

This yields the desired inequality. ∎

The inequality in Theorem 2.8 is sharp for all c and r; see Example 4.17 below.

3. Initial Ideals of Lattice Ideals

One motivation for our new definition of genericity for monomial ideals is consistency with the notion of genericity for lattice ideals introduced in [15]. It is the purpose of this section to establish this connection. We fix a sublattice $\mathcal{L}$ of $\mathbb{Z}^n$ which contains no nonnegative vectors. The *lattice ideal* $I_{\mathcal{L}}$ associated to $\mathcal{L}$ is defined by

$$I_{\mathcal{L}} \;:=\; \langle\, \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \text{ and } \mathbf{a} - \mathbf{b} \in \mathcal{L} \,\rangle \quad \subset \quad S,$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. The ideal $I_{\mathcal{L}}$ is homogeneous with respect to some grading where $\deg(x_s)$ is a positive integer for each s. We have $\operatorname{codim}(I_{\mathcal{L}}) = \operatorname{rank}(\mathcal{L})$. Moreover, the ring $S/I_{\mathcal{L}}$ has a fine grading by $\mathbb{Z}^n/\mathcal{L}$ (cf. [14]).

The following three conditions are equivalent: (a) The abelian group $\mathbb{Z}^n/\mathcal{L}$ is torsion free, (b) $I_\mathcal{L}$ is a prime ideal, and (c) $I_\mathcal{L}$ is a toric ideal (i.e., $S/I_\mathcal{L}$ is an affine semigroup ring). Even if $I_\mathcal{L}$ is not prime, all monomials are non-zero divisors of $S/I_\mathcal{L}$, and all associated primes of $I_\mathcal{L}$ have the same codimension. If I_A is the toric ideal of an integer matrix A, as defined in [19], then I_A coincides with the lattice ideal $I_\mathcal{L}$ where $\mathcal{L} \subset \mathbb{Z}^n$ is the kernel of A.

Following Peeva and Sturmfels [15], we call a lattice ideal $I_\mathcal{L}$ *generic* if it is generated by binomials with full support, i.e.,

$$I_\mathcal{L} \;=\; \langle \mathbf{x}^{\mathbf{a}_1} - \mathbf{x}^{\mathbf{b}_1}, \mathbf{x}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_2}, \dots , \mathbf{x}^{\mathbf{a}_r} - \mathbf{x}^{\mathbf{b}_r} \rangle$$

where none of the r vectors $\mathbf{a}_i - \mathbf{b}_i \in \mathbb{Z}^n$ has a zero coordinate.

The minimal free resolution of a generic lattice ideal $I_\mathcal{L}$ is constructed in Peeva and Sturmfels [15], which is also called the *algebraic Scarf complex* and denoted by $F_\mathcal{L}$. If a (not necessarily generic) codimension 2 lattice ideal is not a complete intersection, then the minimal free resolution is also attained by the algebraic Scarf complex (see [14], also [7]). If the minimal free resolution of a lattice ideal $I_\mathcal{L}$ has the structure of the algebraic Scarf complex, then we can easily compute the extension module $\operatorname{Ext}^i_S(S/I_\mathcal{L}, S)$ for $i = \text{proj-dim}_S\, S/I_\mathcal{L}$ by an argument similar to the proof of [21, Theorem 2.6], so we can get the following.

Proposition 3.1. *Suppose that the minimal free resolution of a lattice ideal $I_\mathcal{L}$ is given by the algebraic Scarf complex (e.g. $I_\mathcal{L}$ is generic). If $S/I_\mathcal{L}$ satisfies Serre's condition (S_2), then $S/I_\mathcal{L}$ is Cohen-Macaulay.*

The above statement is not true, if $I_\mathcal{L}$ is not generic. (c.f. [4, § 6.2]).

Proposition 3.2. (c.f. [8, Theorem 1.1 (2)]) *Suppose that the minimal free resolution of a lattice ideal $I_\mathcal{L}$ is given by the algebraic Scarf complex $F_\mathcal{L}$ (e.g. $I_\mathcal{L}$ is generic). If $S/I_\mathcal{L}$ is Gorenstein, then $I_\mathcal{L}$ is a principal ideal.*

Proof. If $S/I_\mathcal{L}$ is Gorenstein, we have $\operatorname{Hom}(F_\mathcal{L}, S) \simeq F_\mathcal{L}$ up to the degree shifting. So $\operatorname{Hom}(F_\mathcal{L}, S)$ gives the minimal free resolution of $S/I_\mathcal{L}$ again. If $\operatorname{codim} I_\mathcal{L} \geq 2$, then $I_\mathcal{L}$ must be a monomial ideal by the structure of $\operatorname{Hom}(F_\mathcal{L}, S)$. This is a contradiction. □

The following is a main result of this section.

Theorem 3.3. *Let $I_\mathcal{L}$ be a generic lattice ideal, and M the initial ideal of $I_\mathcal{L}$ with respect to a reverse lexicographic term order. Then M is a generic monomial ideal.*

Proof. Set $M = \operatorname{in}_{revlex}(I_\mathcal{L}) = \langle m_1, \dots , m_r \rangle$. Gasharov, Peeva and Welker [8] proved that the algebraic Scarf complex F_{Δ_M} is a minimal free resolution of S/M. Using Theorem 1.5, it suffices to prove that no variable x_s appears with the same non-zero exponent in m_i and m_j for any $i \neq j$ with $\{i, j\} \in \Delta_M$. Assume the contrary, that is, $\deg_{x_s} m_i = \deg_{x_s} m_j > 0$ for some $\{i, j\} \in \Delta_M$. By [15, Theorem 5.2], there are three monomials $m'_i, m'_j, m'_l \in S$ satisfying the following conditions.

(a) $\{m'_i, m'_j, m'_l\}$ is a *basic fiber* (see [15, §2]), in particular, $\gcd(m'_i, m'_j, m'_l) = 1$.

(b) $m_i = \frac{m'_i}{\gcd(m'_i, m'_l)}$ and $m_j = \frac{m'_j}{\gcd(m'_j, m'_l)}$.

By (b), we have $\deg_{x_s}(m_i') \geq \deg_{x_s}(m_i) > 0$ and $\deg_{x_s}(m_j') \geq \deg_{x_s}(m_j) > 0$. Since $\gcd(m_i', m_j', m_l') = 1$, we have $\deg_{x_s} m_l' = 0$. So $\deg_{x_s} m_i' = \deg_{x_s} m_i = \deg_{x_s} m_j = \deg_{x_s} m_j'$. Combining property (a) with [15, Theorem 3.2], we see that the binomial

$$\frac{m_i'}{\gcd(m_i', m_j')} - \frac{m_j'}{\gcd(m_i', m_j')}$$

is a minimal generator of $I_{\mathcal{L}}$. Since $\deg_{x_s} m_i' = \deg_{x_s} m_j'$, the variable x_s does not appear in the above binomial. This contradicts the genericity of $I_{\mathcal{L}}$. □

Example 3.4. Theorem 3.3 is false for the old definition of "generic monomial ideal" given in [1]. For example, consider the following generic lattice ideal in $k[a,b,c,d]$:

$$I_{\mathcal{L}} \;=\; \langle a^4 - bcd, a^3c^2 - b^2d^2, a^2b^3 - c^2d^2, ab^2c - d^3, b^4 - a^2cd, b^3c^2 - a^3d^2, c^3 - abd \rangle$$

This ideal was featured in [15, Example 4.5]; it defines the toric curve $(t^{20}, t^{24}, t^{25}, t^{31})$. Consider a reverse lexicographic term order with $a > b > c > d$. Then $M = \langle a^4, a^3c^2, a^2b^3, ab^2c, b^4, b^3c^2, c^3 \rangle$. Since a^3c^2 and b^3c^2 are minimal generators of M, it is not generic in the sense of [1]. But M satisfies Definition 1.1 since $ab^2c \in M$. □

An important problem in combinatorial commutative algebra is to characterize those monomial ideals which are initial ideals of lattice ideals. The recent "Chain Theorem" of Hoşten and Thomas [10] provides a remarkable necessary condition.

Theorem 3.5 (Hoşten–Thomas [10]). *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. For each $P \in \mathrm{Ass}(S/M)$, there is a chain of associated primes $P = P_0 \supset P_1 \supset \cdots \supset P_t$ of M such that P_t is a minimal prime and* $\mathrm{codim}(P_i) = \mathrm{codim}(P_{i-1}) - 1$ *for all i.*

In other words, initial ideals of lattice ideals satisfy conclusion (b) of Theorem 2.2, even if they are not generic. We do not know whether part (a) holds as well.

Conjecture 3.6. *Let M be the initial ideal of $I_{\mathcal{L}}$ with respect to some term order. Then there is an associated prime $P \in \mathrm{Ass}(S/M)$ with* $\mathrm{codim}(P) = \text{proj-dim}_S(S/M)$.

Note that all *minimal* primes of an initial ideal M have the same codimension.

Corollary 3.7. *Conjecture 3.6 holds for the reverse lexicographic term order if the lattice ideal $I_{\mathcal{L}}$ is generic.*

Proof. Immediate from Theorem 2.2 and Theorem 3.3. □

The following result appears implicitly in the work of Hoşten-Thomas [10] and Peeva-Sturmfels [14].

Lemma 3.8. *Let M be the initial ideal of a lattice ideal $I_{\mathcal{L}}$ with respect to any term order. Then we have* $\text{proj-dim}_S(S/M) \leq 2^c - 1$ *where* $c := \mathrm{codim}\, I_{\mathcal{L}} = \mathrm{codim}\, M$.

Proof. Following [14, Algorithm 8.2], we construct a lattice ideal $I_{\mathcal{L}'}$ in $S[t] = k[x_1, \ldots, x_n, t]$ whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and M respectively. Moreover t is a non-zero divisor of $S[t]/I_{\mathcal{L}'}$, and the codimension of $I_{\mathcal{L}'}$ in $S[t]$ is equal to $\mathrm{codim}(I_{\mathcal{L}})$. Since $S/M = S[t]/(I_{\mathcal{L}'} + \langle t \rangle)$, we have proj-dim$_S(S/M)$ = proj-dim$_{S[t]}(S[t]/I_{\mathcal{L}'}) \leq 2^c - 1$. The last inequality follows from [14, Theorem 2.3]. □

We note that Conjecture 3.6 is also true in codimension 2. In fact, we can prove more.

Theorem 3.9. *Let M be an initial ideal of a codimension 2 lattice ideal $I_{\mathcal{L}} \subset S$. Then the minimal free resolution of M is given by the algebraic Scarf complex F_{Δ_M}.*

Proof. If M is a complete intersection, then the assertion is obvious. So we may assume that M is not a complete intersection. Let $I_{\mathcal{L}'} \subset S[t] = k[x_1, \dots, x_n, t]$ be a lattice ideal whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and M respectively. Note that $M \subset S \subset S[t]$ is an initial ideal of $I_{\mathcal{L}'} \subset S[t]$ with respect to a reverse lexicographic order for t smallest, and the minimal generators of $I_{\mathcal{L}'}$ form a Gröbner bases with respect to this term order (see [14, Lemma 8.4]). Since $I_{\mathcal{L}'}$ is not a complete intersection and has codimension 2, the algebraic Scarf complex $F_{\mathcal{L}'}$ is the minimal free resolution. Hence the i-faces of Δ_M are in bijection to the $i+1$ faces of $\Delta_{\mathcal{L}'}/\mathcal{L}'$ (see [15] for the definition) for all i by the argument same to [15, Theorem 5.2]. Since t is a non-zero divisor on $S[t]/I_{\mathcal{L}'}$ and $S[t]/(I_{\mathcal{L}} + t) \simeq S/M$, the multi-graded Betti numbers of S/M (over S) coincide with those of $S[t]/I_{\mathcal{L}'}$ (over $S[t]$). By the construction of $F_{\mathcal{L}'}$ and the correspondence between the faces of $\Delta_{\mathcal{L}'}/\mathcal{L}'$ and Δ_M, the multi-graded Betti numbers of S/M are concentrated in Δ_M parts. Thus F_{Δ_M} is the minimal free resolution. □

An initial ideal of a codimension 2 lattice ideal may not be generic. Set $I_{\mathcal{L}} := \langle ac - b^2, ad - bc, bd - c^2 \rangle \subset S = k[a, b, c, d]$ be the defining ideal of the twisted cubic curve in $\mathbb{P}^3$. $S/I_{\mathcal{L}}$ is normal and Cohen-Macaulay. It is known that $I_{\mathcal{L}}$ has eight distinct initial ideals, when we consider all possible term orders (see §4 of [18]), but seven of them are *not* generic. We also remark that four of the eight initial ideals are not Cohen-Macaulay and have embedded associated primes of codimension 3.

Corollary 3.10. *Conjecture 3.6 holds for any term order if* $\mathrm{codim}(I_{\mathcal{L}}) = 2$.

Proof. The assertion follows from Theorem 3.9 and [21, Corollary 2.7]. □

The above result also holds for the initial ideal $\mathrm{in}_\omega(I_{\mathcal{L}})$ with respect to a weight vector $\omega \in \mathbb{R}^n$ (c.f. [19]). Note that $\mathrm{in}_\omega(I_{\mathcal{L}})$ is not a monomial ideal in general. For any term order $\prec$, there is a weight vector $\omega \in \mathbb{R}^n$ such that $\mathrm{in}_\prec(I_{\mathcal{L}}) = \mathrm{in}_\omega(I_{\mathcal{L}})$ (c.f. [6]). As the usual term order case, we can construct a lattice ideal $I_{\mathcal{L}'}$ in $S[t] = k[x_1, \dots, x_n, t]$ whose images under the substitutions $t = 1$ and $t = 0$ are $I_{\mathcal{L}}$ and $\mathrm{in}_\omega(I_{\mathcal{L}})$ respectively. So Proposition 3.8 also holds for $\mathrm{in}_\omega(I_{\mathcal{L}})$.

Theorem 3.11. *Let $I_{\mathcal{L}} \subset S$ be a lattice ideal of codimension 2, and $\mathrm{in}_\omega(I_{\mathcal{L}})$ the initial ideal with respect to a weight vector $\omega \in \mathbb{R}^n$. If* proj-dim$_S(S/\mathrm{in}_\omega(I_{\mathcal{L}})) = 3$ *(equivalently, $S/\mathrm{in}_\omega(I_{\mathcal{L}})$ is not Cohen-Macaulay) and $\mathrm{in}_\omega(I_{\mathcal{L}}) \neq I_{\mathcal{L}}$, then there is a codimension 3 embedded associated prime of* $\mathrm{in}_\omega(I_{\mathcal{L}})$.

Proof. Let $I_{\mathcal{L}'} \subset S[t]$ be a codimension 2 lattice ideal representing the deformation from $I_{\mathcal{L}}$ to $\mathrm{in}_\omega(I_{\mathcal{L}})$. Suppose that proj-dim$_{S[t]}(S[t]/I_{\mathcal{L}'})$ = proj-dim$_S(S/\mathrm{in}_\omega(I_{\mathcal{L}})) = 3$.

In this case $\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t]) \neq 0$. Since the lattice ideal $I_{\mathcal{L}'}$ has pure codimension 2, we have $\dim(\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])) \leq n-4$ by [4, Theorem 8.1.1]. Let $F_{\mathcal{L}'}$ be a minimal free resolution of $S[t]/I_{\mathcal{L}'}$ constructed in [14]. Then $\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])$ is the third cohomology group of the cochain complex $(F_{\mathcal{L}'})^* := \mathrm{Hom}_{S[t]}(F_{\mathcal{L}'}, S[t])$. Let $e \in (F_{\mathcal{L}'})_3$ be a generator of a free summand in homological degree 3 corresponding to a *syzygy quadrangles* (c.f. [14]), and $e^* \in (F_{\mathcal{L}(\omega)})^*$ its dual. Since e^* is a cocycle of $(F_{\mathcal{L}'})^*$, we have the corresponding element $\overline{e^*} \in \mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])$. Then $S[t]/J \simeq S[t]\cdot\overline{e^*}$ is a submodule of $\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])$, where $J = \mathrm{ann}(\overline{e^*})$. By the construction of $F_{\mathcal{L}'}$, we have $d(e) = \sum_{i=1}^4 m_i \cdot e_i$ where m_i is a non-constant monomial and $e_i \in (F_{\mathcal{L}(\omega)})_2$ is a free base of homological degree 2. Set $J' := \langle m_1, \cdots, m_4\rangle \subset S[t]$. Then $J' \supseteq J$ by the construction. We have

$$(n+1)-4 \leq \dim(S[t]/J') \leq \dim(S[t]/J) \leq \dim(\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])) \leq (n+1)-4$$

by Krull's theorem. So $\dim(S[t]/J) = \dim(S[t]/J') = (n+1)-4$. Hence J' is complete intersection, and all its associated primes have codimension 4. If $\mathrm{in}_\omega(I_{\mathcal{L}}) \neq I_{\mathcal{L}}$, then we may assume that t is a zero divisor of S/J', after suitable choice of $e \in (F_{\mathcal{L}'})_3$. Then there is a monomial prime ideal $P \subset S[t]$ such that $t \in P$, $P \in \mathrm{Ass}(S[t]/J')$ and $\mathrm{codim}(P) = 4$. Since $\mathrm{codim}\, J = 4$ and $P \supset J' \supset J$, we have

$$P \in \mathrm{Ass}(S[t]/J) \subseteq \mathrm{Ass}(\mathrm{Ext}^3_{S[t]}(S[t]/I_{\mathcal{L}'}, S[t])).$$

Since $t \in P$ and $S/\mathrm{in}_\omega(I_{\mathcal{L}}) = S[t]/(I_{\mathcal{L}'} + (t))$, the image $\bar{P} \subset S = S[t]/(t)$ is an associated prime of $\mathrm{in}_\omega(I_{\mathcal{L}})$ by [13, Corollary 3.2]. Note that $\mathrm{codim}(\bar{P}) = 3$. So $\bar{P}$ is an expected ideal. □

4. A Study of Cogeneric Monomial Ideals

Cogeneric monomial ideals were introduced in Definition 1.3. As with genericity, our definition of cogenericity is slightly different from the original one of [20]. In Theorem 4.6 we shall see that the result of [20], an explicit description of the minimal free resolution of a cogeneric monomial ideal, is still true here. In fact, Alexander duality for arbitrary monomial ideals [11] allows us to shorten the construction of this resolution and clarify its relation to Theorem 1.5. For the reader's convenience, we briefly recall the definitions pertaining to Alexander duality. For details see [11].

The maximal $\mathbb{N}^n$-graded ideal $\langle x_1, \ldots, x_n\rangle \subset S$ will be denoted by $\mathfrak{m}$. Monomials and irreducible monomial ideals may each be specified by a single vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, so we will write $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$ and $\mathfrak{m}^{\mathbf{b}} = \langle x_s^{b_s} \mid b_s \geq 1\rangle$. Given a vector $\mathbf{a} = (a_1, \ldots, a_n)$ such that $b_s \leq a_s$ for all s, we define the Alexander dual vector $\mathbf{b}^{\mathbf{a}}$ with respect to $\mathbf{a}$ by setting its s^{th} coordinate to be

$$(\mathbf{b}^{\mathbf{a}})_s = \begin{cases} a_s + 1 - b_s & \text{if } b_s \geq 1 \\ 0 & \text{if } b_s = 0. \end{cases}$$

Whenever we deal with Alexander duality, we assume that we are given a vector $\mathbf{a}$ such that for each s, the integer a_s is larger than or equal to the s^{th} coordinate of any minimal monomial generator of M. This implies that a_s is also larger than or equal to the s^{th} coordinate of any irreducible component of M, and vice versa. The *Alexander dual ideal* $M^{\mathbf{a}}$ of M with respect to $\mathbf{a}$ is defined by

$$\begin{aligned} M^{\mathbf{a}} &= \langle \mathbf{x}^{\mathbf{b}^{\mathbf{a}}} \mid \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } M \rangle \\ &= \bigcap \{ \mathfrak{m}^{\mathbf{c}^{\mathbf{a}}} \mid \mathbf{x}^{\mathbf{c}} \text{ is a minimal generator of } M \}. \end{aligned}$$

That these two formulas give the same ideal is not obvious; it is equivalent to $(M^{\mathbf{a}})^{\mathbf{a}} = M$. It follows from these statements that M is generic if and only if $M^{\mathbf{a}}$ is cogeneric.

Example 4.1. The following monomial ideal in $S = k[x, y, z]$ is cogeneric:

$$M \;=\; \langle yz^2, xz^2, y^2z, xy^2, x^2 \rangle \;=\; \langle x, y \rangle \cap \langle x^2, y^2, z^2 \rangle \cap \langle x, z \rangle.$$

Its Alexander dual with respect to $\mathbf{a} = (2, 2, 2)$ is generic:

$$M^{\mathbf{a}} \;=\; \langle x^2y^2, xyz, x^2z^2 \rangle \;=\; \langle y^2, z \rangle \cap \langle x^2, z \rangle \cap \langle y, z^2 \rangle \cap \langle x^2, y \rangle \cap \langle x \rangle.$$

Example 4.2 ([11, Examples 1.9, 5.22]). If M is the tree ideal of Example 1.2 and $\mathbf{a} = (n, \ldots, n)$, then its Alexander dual $M^{\mathbf{a}}$ is the *permutahedron ideal*:

$$M^{\mathbf{a}} \quad = \quad \langle\, x_1^{\pi(1)} x_2^{\pi(2)} \cdots x_n^{\pi(n)} \;:\; \pi \text{ is a permutation of } \{1, 2, \ldots, n\} \,\rangle.$$

Thus the permutahedron ideal is cogeneric. Its minimal free resolution is the *hull resolution*, which is cellular and supported on a permutahedron [2, Example 1.9]. The following discussion reinterprets this resolution as a co-Scarf complex. □

Definition 4.3. Let $M = \bigcap_{i=1}^r M_i$ be a cogeneric monomial ideal. Set $\mathbf{a} = (D - 1, \ldots, D - 1)$ with D larger than any exponent on any minimal generator of M. The Alexander dual ideal $M^{\mathbf{a}}$ is minimally generated by monomials $m_1, \ldots, m_r$, where $m_i = \mathbf{x}^{\mathbf{b}_i^{\mathbf{a}}}$ for $M_i = \mathfrak{m}^{\mathbf{b}_i}$. We define the *co-Scarf complex* $\Delta_M^{\mathbf{a}}$ to be the extended Scarf complex of $M^{\mathbf{a}}$. More precisely, we set $(M^{\mathbf{a}})^* := M^{\mathbf{a}} + \langle x_1^D, \ldots, x_n^D \rangle$ and $\Delta_M^{\mathbf{a}}$ the Scarf complex of $(M^{\mathbf{a}})^*$. Since we index a new monomial x_s^D just by x_s, we see that $\Delta_M^{\mathbf{a}}$ is a simplicial complex on (a subset of) $\{1, \ldots, r, x_1, \ldots, x_n\}$.

Remark 4.4. (a) There is nothing special about our choice of $\mathbf{a}$, except that it makes for convenient notation. Everything we do with $\Delta_M^{\mathbf{a}}$ is independent of which sufficiently large $\mathbf{a}$ is chosen. In particular, the regular triangulation of the $(n-1)$-simplex is independent of $\mathbf{a}$, as is the algebraic co-Scarf complex (Definition 4.5) it determines. We therefore set $\mathbf{a} = (D - 1, \ldots, D - 1)$ for the remainder of this section.

(b) For $\sigma \subseteq \{1, \ldots, r\}$, let M_σ be the irreducible monomial ideal $\sum_{i \in \sigma} M_i$. Then $m_\sigma = \mathbf{x}^{\mathbf{b}^{\mathbf{a}}}$ if $M_\sigma = \mathfrak{m}^{\mathbf{b}}$, and $\Delta_M^{\mathbf{a}} \cap \{1, \ldots, r\} = \{\sigma \subset \{1, \ldots, r\} \mid M_\tau \neq M_\sigma \text{ for all } \tau \neq \sigma\}$ is just the Scarf complex of $M^{\mathbf{a}}$.

A face σ of the co-Scarf complex $\Delta_M^{\mathbf{a}}$ fails to be in the (topological) boundary $\partial \Delta_M^{\mathbf{a}}$ of $\Delta_M^{\mathbf{a}}$ if and only if the monomial m_σ has full support, where m_σ is $\operatorname{lcm}(m_i \mid i \in \sigma)$ under the notation of Definition 4.3. Such a face will be called an *interior face* of $\Delta_M^{\mathbf{a}}$. The set $\operatorname{int}(\Delta_M^{\mathbf{a}})$ of interior faces is closed under taking supersets; that is, $\operatorname{int}(\Delta_M^{\mathbf{a}})$ is a *simplicial cocomplex*. Just as the algebraic Scarf complex is constructed from Δ_M for generic M, we construct an algebraic free complex from $\operatorname{int}(\Delta_M^{\mathbf{a}})$, but this time we use the coboundary map instead of the boundary map. The following is a special kind of *relative cocellular resolution* (in fact a cohull resolution) [11, §5].

Definition 4.5. Let $\mathbf{D} = (D, \dots, D) \in \mathbb{N}^n$ and $S(\mathbf{a}_\sigma - \mathbf{D})$ be the free S-module with one generator e_σ^* in multidegree $\mathbf{D} - \mathbf{a}_\sigma$. The *algebraic co-Scarf complex* $F^{\Delta_M^{\mathbf{a}}}$ of M is the free S-module

$$\bigoplus_{\sigma \in \mathrm{int}(\Delta_M^{\mathbf{a}})} S(\mathbf{a}_\sigma - \mathbf{D}) \quad \text{with differential} \quad d^*(e_\sigma^*) = \sum_{\substack{i \notin \sigma \\ \sigma \cup \{i\} \in \mathrm{int}(\Delta_M^{\mathbf{a}})}} \mathrm{sign}(i, \sigma \cup \{i\}) \cdot \frac{m_{\sigma \cup \{i\}}}{m_\sigma} \cdot e_{\sigma \cup \{i\}}^*$$

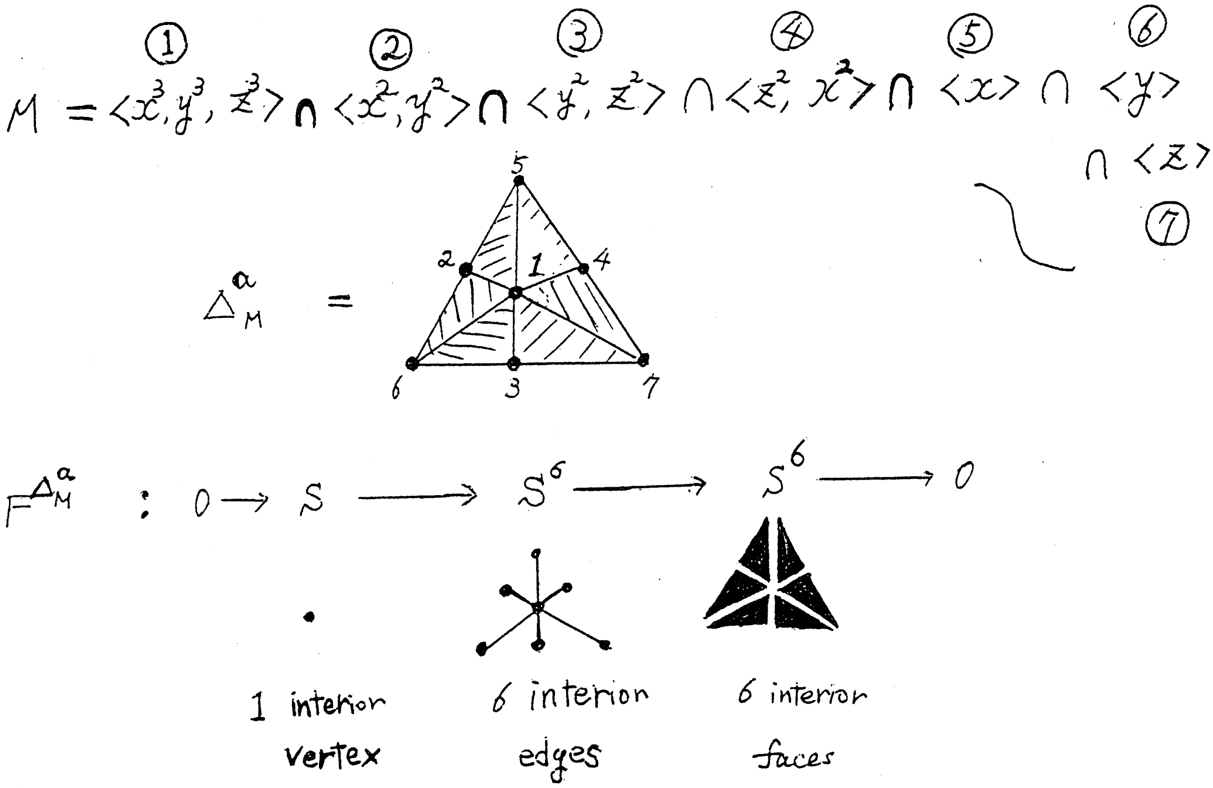
where $\mathrm{sign}(i, \sigma \cup \{i\})$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of $\sigma \cup \{i\}$. Put the summand $S(\mathbf{a}_\sigma - \mathbf{D})$ in homological degree $n - \#\sigma = n - \dim(\sigma) - 1$.

Theorem 4.6. *If M is a cogeneric monomial ideal, then the algebraic co-Scarf complex $F^{\Delta_M^{\mathbf{a}}}$ equals the minimal free resolution of M over S. In particular, M is minimally generated by the set of monomials $\{\mathbf{x}^{\mathbf{D}-\mathbf{a}_\sigma} \mid \sigma$ is a facet of $\Delta_M^{\mathbf{a}}\}$.*

Proof. This follows from Proposition 1.7 and [11, Theorem 5.8]. □

Example 4.1 *(continued)* For the cogeneric ideal $M = \langle x, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle$, the interior faces of $\Delta_M^{\mathbf{a}}$ are $\{2\}$, $\{1,2\}$, $\{2,3\}$, $\{2,x\}$, $\{2,y\}$, $\{2,z\}$, $\{1,2,x\}$, $\{1,2,y\}$, $\{2,3,x\}$, $\{2,3,z\}$ and $\{2,y,z\}$. The co-Scarf resolution is $0 \to S \to S^5 \to S^5 \to M \to 0$. The generators of M have exponent vectors $\mathbf{D} - \mathbf{a}_{\{1,2,x\}} = (0,1,2)$, $\mathbf{D} - \mathbf{a}_{\{1,2,y\}} = (1,0,2)$, $\mathbf{D} - \mathbf{a}_{\{2,3,x\}} = (0,2,1)$, $\mathbf{D} - \mathbf{a}_{\{2,3,z\}} = (1,2,0)$ and $\mathbf{D} - \mathbf{a}_{\{2,y,z\}} = (2,0,0)$.

Example 4.2 *(continued)* The following figure explains the (algebraic) co-Scarf complex of the permutahedron ideal of three variables.



We saw in Theorem 2.5 that for generic monomial ideals, the Cohen-Macaulay condition is equivalent to the much weaker condition of purity (all associated primes have the same dimension). For cogeneric monomial ideals, on the other hand, purity is obviously too easy to attain. Nonetheless, a cogeneric ideal is forced to be Cohen-Macaulay by *a priori* much weaker conditions. Before stating these in Theorem 4.8, we characterize depth for cogeneric ideals using a polyhedral criterion.

Lemma 4.7. *Let M be a cogeneric monomial ideal. Then* $\operatorname{depth}(S/M) \leq d$ *if and only if the co-Scarf complex $\Delta_M^{\mathbf{a}}$ has an interior face of dimension d.*

Proof. By Theorem 4.6, the shifted augmentation $F^{\Delta_M^{\mathbf{a}}} \to S$ (obtained by including $\operatorname{coker}(F^{\Delta_M^{\mathbf{a}}}) = M$ into S and shifting homological degrees up one) is a minimal free resolution of S/M. The co-Scarf complex $\Delta_M^{\mathbf{a}}$ has an interior face of dimension d if and only if this shifted augmented complex is nonzero in homological dimension $n-d$. The lemma now follows from the Auslander-Buchsbaum formula. □

Recall that a module N satisfies Serre's condition (S_k) if for every prime $P \subset S$, $\operatorname{depth}(N_P) < k \Rightarrow \operatorname{depth}(N_P) = \dim(N_P)$. Using [4, Chapter 2.1] and homogeneous localization, it follows that if S/M satisfies (S_k) then

(7) $$\operatorname{depth}((S/M)_{(P)}) < k \quad \Longrightarrow \quad \dim((S/M)_{(P)}) = \operatorname{depth}((S/M)_{(P)}).$$

Observe that $M_{(P)}$ is cogeneric if M is, in analogy to Remark 2.1. For condition (d) below, recall the definition of *excess* from before Theorem 2.8.

Theorem 4.8. *Let $M \subset S$ be a cogeneric monomial ideal of codimension c with the irreducible decomposition $M = \bigcap_{i=1}^r M_i$. Then the following conditions are equivalent.*

(a) S/M is Cohen-Macaulay.
(b) S/M satisfies Serre's condition (S_2).
(c) $\operatorname{codim} M_i = c$ *for all i, and* $\operatorname{codim}(M_i + M_j) \leq c+1$ *for all edges $\{i,j\} \in \Delta_M^{\mathbf{a}}$.*
(d) Every face of $\Delta_M^{\mathbf{a}}$ has excess $< c$.
(e) $\Delta_M^{\mathbf{a}}$ has no interior faces of dimension $< n-c$.

Proof. (a) $\Rightarrow$ (b) : Cohen-Macaulay $\Leftrightarrow$ (S_k) for all k.

(b) $\Rightarrow$ (c) : The initial equality follows from [9, Remark 2.4.1], so it suffices to prove the inequality. Suppose $i \neq j$ with $\{i,j\} \in \Delta_M^{\mathbf{a}}$. Let $P = \operatorname{rad}(M_i + M_j)$, and denote by F the face of $\Delta = 2^{\{x_1,\ldots,x_n\}}$ whose vertices are the variables in P. By [11, Proposition 4.6], the co-Scarf complex of $M_{(P)}$ is, as a triangulation of the simplex 2^F, the restriction $(\Delta_M^{\mathbf{a}})_F$ of the triangulation $\Delta_M^{\mathbf{a}}$ to 2^F. By our choice of F, $\{i,j\}$ is an interior edge of $(\Delta_M^{\mathbf{a}})_F$, so Lemma 4.7 implies that $\operatorname{depth}((S/M)_{(P)}) \leq 1$, whence (7) implies that $\dim((S/M)_{(P)}) \leq 1$. Equivalently, $\operatorname{codim}(M_i + M_j) \leq c+1$.

(c) $\Rightarrow$ (d) : The purity of the irreducible components means that all vertices have excess $c-1$ or 0, while the condition on the edges implies that the excess of a nonempty face can only decrease or remain the same upon the addition of a vertex.

(d) $\Rightarrow$ (e) : In particular, the interior faces have excess less than c.

(e) $\Rightarrow$ (a) : Lemma 4.7. □

Remark 4.9. Hartshorne [9] proved that a catenary local ring satisfying Serre's condition (S_2) is pure and connected in codimension 1. The converse is not true even for cogeneric monomial ideals. If we take $M = \langle x, y^2\rangle \cap \langle y, z\rangle \cap \langle z^2, w\rangle$ then S/M is pure and connected in codimension 1, but does not satisfy the condition (S_2); in fact, $\operatorname{depth}(S/M) = 1$. On the other hand, $M' = \langle x, y\rangle \cap \langle y^2, z^2\rangle \cap \langle z, w\rangle$ is Cohen-Macaulay, although $\operatorname{Ass}(M) = \operatorname{Ass}(M')$.

The above theorem and remark leads to a natural question.

Problem 4.10. Which Cohen-Macaulay simplicial complexes have Stanley-Reisner ideal $\operatorname{rad}(M)$ for some Cohen-Macaulay cogeneric monomial ideal M?

Recall that the *type* of a Cohen-Macaulay quotient S/M is the nonzero total Betti number of highest homological degree; if M is cogeneric then this Betti number equals the number of interior faces of minimal dimension in $\Delta_M^{\mathbf{a}}$ by Theorem 4.6.

Theorem 4.11. *Let M be a Cohen-Macaulay cogeneric monomial ideal of codimension ≥ 2. The type of S/M is at least the number of irreducible components of M.*

Recall that S/M is *Gorenstein* if its Cohen-Macaulay type equals 1. This implies:

Corollary 4.12. *Let M be a cogeneric monomial ideal. Then S/M is Gorenstein if and only if M is either a principal ideal or an irreducible ideal.*

Remark 4.13. In the generic monomial ideal case, we have the opposite inequality to the one in Theorem 4.11. More precisely, if M is Cohen-Macaulay and generic then

$$\begin{aligned} \text{Cohen-Macaulay type of } S/M \quad &= \quad \#\{\text{facets of the Scarf complex } \Delta_M\} \\ \leq \quad \#\{\text{facets of } \Delta_{M^*}\} \quad &= \quad \#\{\text{irreducible components of } M\}, \end{aligned}$$

because the map $\Delta_{M^*} \to \Delta_M$, $\sigma \mapsto \sigma \cap \{1, \ldots, r\}$ is surjective on facets. Also here, S/M is Gorenstein if and only if it is complete intersection [21, Corollary 2.11].

We present two proofs of Theorem 4.11. The first is algebraic and uses Alexander duality, in particular the following result. For notation, define $\mathbf{b} \cdot F \in \mathbb{N}^n$, for $F \subseteq \{1, \ldots, n\}$ and $\mathbf{b} \in \mathbb{N}^n$, to have s^{th} coordinate b_s if $s \in F$ and 0 otherwise. Also, set $\beta_{i,\mathbf{b}}(M) = \dim_k(\operatorname{Tor}_i^S(M, k))_{\mathbf{b}}$, the i^{th} Betti number of M in $\mathbb{Z}^n$-degree $\mathbf{b}$.

Theorem 4.14 (E. Miller [11, Theorem 4.13]). *Let $M \subset S$ be any monomial ideal and let $F \subseteq \{1, \ldots, n\}$. If $\operatorname{supp}(\mathbf{b}) = F$ and $b_s \leq a_s$ for all s, then*

$$\beta_{i,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) \;\leq\; \sum_{\substack{\mathbf{c} \in \mathbb{N}^n \\ \mathbf{c} \cdot F = \mathbf{b}}} \beta_{\#F - i - 1, \mathbf{c}}(M).$$

Proof of Theorem 4.11. Let $\operatorname{Irr}(S/M)$ denote the set of vectors $\mathbf{b} \in \mathbb{N}^n$ for which $\mathbf{m}^{\mathbf{b}}$ is an irreducible component of M. For any $\mathbf{c} \in \mathbb{N}^n$, we define

$$\gamma_{\mathbf{c}} \;:=\; \#\{F \subseteq \{1, \ldots, n\} \mid \mathbf{c} \cdot F \in \operatorname{Irr}(S/M)\}.$$

Set $d = \operatorname{codim}(M)$. The first aim is to show that

$$\#\operatorname{Irr}(S/M) \;\leq\; \sum_{\mathbf{c} \in \mathbb{N}^n} \gamma_{\mathbf{c}} \cdot \beta_{d-1, \mathbf{c}}(M). \tag{8}$$

In fact, this inequality holds even if M is not cogeneric: by the construction of $M^{\mathbf{a}}$,

$$\# \operatorname{Irr}(S/M) \;=\; \sum_{\mathbf{b}\in\operatorname{Irr}(S/M)} \beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) \;=\; \sum_{\mathbf{b}\in\mathbb{N}^n} \beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}).$$

Since S/M is Cohen-Macaulay of codimension d, each $\mathbf{b} \in \operatorname{Irr}(S/M)$ has precisely d non-zero coordinates, and $\beta_{i,\mathbf{c}}(M) = 0$ for $i \geq d$. Thus Theorem 4.14 specializes to

$$\beta_{0,\mathbf{b}^{\mathbf{a}}}(M^{\mathbf{a}}) \;\leq\; \sum_{\mathbf{c}\cdot F = \mathbf{b}} \beta_{d-1,\mathbf{c}}(M)$$

for fixed $\mathbf{b} = (b_1, \ldots, b_n)$ and $F = \operatorname{supp}(\mathbf{b})$. Summing over all $\mathbf{b}$ proves (8).

The Cohen-Macaulay type of S/M is $\sum_{\mathbf{c}\in\mathbb{N}^n} \beta_{d-1,\mathbf{c}}(M)$, so it suffices to prove that if $\beta_{d-1,\mathbf{c}}(M) \neq 0$ then $\gamma_{\mathbf{c}} \leq 1$. Suppose the opposite, that is, $\gamma_{\mathbf{c}} \geq 2$ and $\beta_{d-1,\mathbf{c}}(M) \neq 0$. Then there are sets $F, F' \subseteq \{1, \ldots, n\}$ such that $\mathbf{c}\cdot F, \mathbf{c}\cdot F' \in \operatorname{Irr}(S/M)$ are distinct. Let $M_i = \mathfrak{m}^{\mathbf{c}\cdot F}$ and $M_j = \mathfrak{m}^{\mathbf{c}\cdot F'}$ be the irreducible components M corresponding to $\mathbf{c}\cdot F$ and $\mathbf{c}\cdot F'$. Since the algebraic co-Scarf complex of M is the minimal free resolution of M and $\beta_{d-1,\mathbf{c}}(M) \neq 0$, there is an interior face σ of the co-Scarf complex $\Delta_M^{\mathbf{a}}$ with $\mathbf{a}_\sigma = \mathbf{D} - \mathbf{c}$. Since $m_i = \mathbf{x}^{(\mathbf{c}\cdot F)^{\mathbf{a}}}$ and $m_j = \mathbf{x}^{(\mathbf{c}\cdot F')^{\mathbf{a}}}$ divide m_σ by construction, σ contains both i and j. In particular, $\{i, j\}$ is an edge of $\Delta_M^{\mathbf{a}}$. Now S/M is Cohen-Macaulay of codimension ≥ 2, so $\operatorname{supp}(m_i) \cap \operatorname{supp}(m_j) \neq \emptyset$ by Theorem 4.8. But $\deg_{x_s} m_i = \deg_{x_s} m_j = D - c_s > 0$ for any $s \in \operatorname{supp}(m_i) \cap \operatorname{supp}(m_j)$, contradicting the genericity of $M^{\mathbf{a}}$. □

After we had gotten the above proof, we conjectured the following more general result about arbitrary triangulations of a simplex. Margaret Bayer proved our conjecture for *quasigeometric triangulations*, using local h-vectors [16]. Since the co-Scarf complex is a quasigeometric triangulation, Theorem 4.15 provides a second proof of Theorem 4.11.

Theorem 4.15 (M. Bayer, personal communication). *Let $p_1, p_2, \ldots, p_r$ be points which lie in the relative interior of $(c-1)$-faces of a $(n-1)$-simplex Δ. Let Γ be a quasigeometric triangulation of Δ having the p_i among its vertices and having no interior $(n-c-1)$-face. Then the number of interior $(n-c)$-faces is at least r.*

Proof. According to the hypothesis, we have $\sum_{\substack{F\in\Delta\\ \#F=c}} f_0(\operatorname{int}(\Gamma_F)) \geq r$, and $f_i(\operatorname{int}(\Gamma)) = 0$ for all $-1 \leq i \leq n-c-1$. By the decomposition of the h-polynomial of Γ into local h-polynomials and the positivity of local h-vectors [16, Theorem 4.6], we have

$$h_{c-1}(\Gamma) \;=\; \sum_{F\in\Delta} \ell_{c-1}(\Gamma_F) \;\geq\; \sum_{\substack{F\in\Delta\\ \#F=c}} \ell_{c-1}(\Gamma_F).$$

On the other hand, we have seen that $\ell_1(\Gamma_F) = f_0(\operatorname{int}(\Gamma_F))$ in the proof of Theorem 2.8. Since a local h-vector is symmetric [16, Theorem 3.3], we have $\ell_{c-1}(\Gamma_F) = \ell_1(\Gamma_F) = f_0(\operatorname{int}(\Gamma_F))$. So

$$h_{c-1}(\Gamma) \;\geq\; \sum_{\substack{F\in\Delta\\ \#F=c}} \ell_{c-1}(\Gamma_F) \;=\; \sum_{\substack{F\in\Delta\\ \#F=c}} f_0(\operatorname{int}(\Gamma_F)) \geq r.$$

Since the h-vector of $\mathrm{int}(\Gamma)$ is the reverse of the h-vector of Γ (see the comment preceding [17, Theorem 10.5]), we have

$$\begin{aligned} h_{c-1}(\Gamma) &= h_{n+1-c}(\mathrm{int}(\Gamma)) \\ &= \sum_{i=0}^{n-c+1} (-1)^{n+1-c-i}\binom{n-i}{c-1}(f_{i-1}(\mathrm{int}(\Gamma))) \\ &= f_{n-c}(\mathrm{int}(\Gamma)). \end{aligned}$$

Thus, the number of interior $(n-c)$-faces of Γ is at least r. □

Our final results demonstrate the effective translation between generic and cogeneric monomial ideals via Alexander duality.

Theorem 4.16. *Let M be a cogeneric monomial ideal with r irreducible components, each having the same codimension c. Then M has at least $(c-1)\cdot r+1$ minimal generators. If M has exactly $(c-1)\cdot r+1$ generators then S/M is Cohen-Macaulay.*

Proof. The former statement is Alexander dual to Theorem 2.8. To prove the latter statement, we recall the proof of Theorem 2.8. Assume that S/M is not Cohen-Macaulay. Then $\Gamma := \Delta_M^{\mathbf{a}}$ has an edge $\{i,j\}$ whose excess e satisfies $e \geq c$, by Theorem 4.8. Let $W \in \Delta$ be the support of $m_{\{i,j\}}$. Then $\#W = e+2$. By [16, Proposition 2.2],

$$\ell_W(\Gamma_W, x) = \ell_2(\Gamma_W)x^2 + \ell_3(\Gamma_W)x^3 + \cdots,$$

where $\ell_2(\Gamma_W)$ is the number of edges of Γ whose supports are W. So we have $f_{n-1}(\Gamma) = h(\Gamma,1) \geq (c-1)\cdot r + 1 + \ell_2(\Gamma_W) > (c-1)\cdot r+1$ by an argument similar to the proof of Theorem 2.8. Since $f_{n-1}(\Gamma)$ is equal to the number of generators of M, the proof is done. □

Let $M = \bigcap_{i=1}^r M_i$ be a cogeneric monomial ideal without codimension 1 component, and $\Gamma := \Delta_M^{\mathbf{a}}$ its co-Scarf complex. Since Γ is shellable, the Stanley-Reisner ring $k[\Gamma]$ is always Cohen-Macaulay. Let $(h_0, h_1, \ldots, h_n)$ be the h-vector of Γ. Since Γ is Cohen-Macaulay, $h_i \geq 0$ for all i. Moreover we have $h_0 = 1$, $h_1 = r$, $\text{proj-dim}_S\, M = \min\{i \geq 0 \mid h_i = 0\}$, and the number of minimal generators of M is equal to $f_{n-1}(\Gamma) = \sum_{i=0}^n h_i$. In particular, when M has pure codimension c, then M is Cohen-Macaulay if and only if $h_c = h_{c+1} = \cdots = h_n = 0$. In this case, $k[\Gamma]$ is a level ring (see [4] for the definition), and the Cohen-Macaulay types of both S/M and $k[\Gamma]$ are equal to h_{c-1}. Note that $k[\Gamma]$ can be level, even if M is not Cohen-Macaulay. The essential part of the proof of Theorem 2.8 is to show $h_i \geq h_1$ for all $1 \leq i \leq c-1$, when M has pure codimension c. We can understand Theorem 4.16 more clearly from this point of view.

Example 4.17. (a) The ideal $M = \bigcap_{i=1}^r \langle x_1^i, x_2^i, \cdots, x_{c-1}^i, x_{c-1+i}\rangle$ is cogeneric and has $(c-1)\cdot r+1$ minimal generators. Thus the inequality in Theorem 4.16 is tight.

(b) The converse of the latter statement of Theorem 4.16 is false. For instance, $M = \langle a^4, b, c\rangle \cap \langle a^2, b^4, d\rangle \cap \langle a, b^3, e\rangle \cap \langle a^3, b^2, e^2\rangle \subset k[a, \ldots, e]$ is a Cohen-Macaulay cogeneric monomial ideal with 4 irreducible components, but M needs 12 generators.

We also note that the Cohen-Macaulay type of S/M is 7, this is larger than the number of irreducible components.

But in the codimension 2 case, we can prove the converse of Theorem 4.16.

Proposition 4.18. *Let M be a cogeneric monomial ideal with r irreducible components, all of codimension 2. Then S/M is Cohen-Macaulay if and only if M has exactly $r+1$ generators.*

Proof. Let $(h_0, \cdots, h_n)$ be the h-vector of $\Delta_M^{\mathbf{a}}$. We always have $h_0 = 1$, $h_1 = r$ and $h_i \geq 0$ for all $0 \leq i \leq n$. By the remark before Example 4.17, M needs $\sum_{i=0}^{n} h_i$ generators and M is Cohen-Macaulay if and only if $h_2 = h_3 = \cdots = 0$. □

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