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Comparison Theorems for Neutral Differential Equations

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1. Introduction

We shall be concerned with the oscillatory behavior of solutions of the even order neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+h(t)x(\tau(t))]+f(t,x(g(t)))=0, \tag{1.1}$$

where $n \geq 2$ is even and the following conditions (H1)–(H4) are assumed to holds:

(H1) $h \in C[t_0,\infty)$ and $h(t) \geq 0$ for $t \geq t_0$;
(H2) $\tau \in C[t_0,\infty)$ is strictly increasing and satisfies $\tau(t) < t$ for $t \geq t_0$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\lim_{t\to\infty} \tau(t)/t = 1$;
(H3) $g \in C[t_0,\infty)$ and $\lim_{t\to\infty} g(t) = \infty$;
(H4) $f \in C([t_0,\infty)\times\mathbb{R})$, $f(t,u)$ is nondecreasing in $u \in \mathbb{R}$ for each fixed $t \geq t_0$ and satisfies $uf(t,u) \geq 0$ for $(t,u) \in [t_0,\infty)\times\mathbb{R}$.

Moreover, we assume that one of the following cases (R1)–(R3) holds:

(R1) $0 \leq \mu \leq h(t) \leq \lambda < 1$ on $[t_0,\infty)$ for some constants μ and λ;
(R2) $1 < \mu \leq h(t) \leq \lambda < \infty$ on $[t_0,\infty)$ for some constants μ and λ;
(R3) $\lim_{t\to\infty} h(t) = \infty$.

By a solution of (1.1), we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x,\infty)$ for some $t_x \geq t_0$. Therefore, if $x(t)$ is a solution of (1.1), then $x(t)+h(t)x(\tau(t))$ is n-times continuously differentiable on $[t_x,\infty)$. Note that, in general, $x(t)$ itself is not n-times continuously differentiable.

A solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. This means that a solution $x(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ as $i \to \infty$ and $x(t_i) = 0$ $(i = 1,2,\dots)$, and a solution $x(t)$ is nonoscillatory if and only if $x(t) \neq 0$ for all large t. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory, and nonoscillatory if at least one solution of (1.1) is nonoscillatory.

The purpose of this paper is to present sufficient conditions for (1.1) to be oscillatory or nonoscillatory.

In recent years there has been an increasing interest in oscillation theory for even order neutral differential equations, and a number of results have been obtained. For typical results we refer to the papers [1, 2, 4–9, 12, 13, 16, 18, 19, 21, 24] and the monographs [3] and [10]. Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See Hale [11].

Now consider the equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+p(t)|x(t-\rho)|^{\gamma}\operatorname{sgn} x(t-\rho)=0, \tag{1.2}$$

where $n \geq 2$ is even, $\lambda > 0$, $\tau > 0$, $\rho \in \mathbb{R}$, $\gamma > 0$, $p \in C[t_0, \infty)$, $p(t) \geq 0$. We note here that if $\gamma = 1$, then (1.2) becomes the linear equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + p(t)x(t-\rho) = 0.$$

The following result was obtained by Jaroš and Kusano [12, Theorems 3.1 and 4.1].

Theorem A. *Let* $\gamma = 1$ *and* $\lambda \in (0,1)$. *If*

$$\int^{\infty} t^{n-1-\varepsilon} p(t)dt = \infty \quad \textit{for some } \varepsilon > 0, \tag{1.3}$$

then (1.2) *is oscillatory. If*

$$\int^{\infty} t^{n-1} p(t)dt < \infty, \tag{1.4}$$

(1.2) *is nonoscillatory.*

However, very little is known about the oscillation of (1.2) with $\gamma = 1$ and $\lambda \in (0,1)$ in the case where both conditions (1.3) and (1.4) fail, such as $p(t) = ct^{-n}$ $(c > 0)$.

The following result, which is a characterization of the oscillation of (1.2) with $\gamma \neq 1$ and $\lambda \in (0,1)$, has been established by Yūki Naito [19, Theorems 5.3 and 5.4].

Theorem B. *Let* $\gamma \neq 1$ *and* $\lambda \in (0,1)$. *Then* (1.2) *is oscillatory if and only if*

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t)dt = \infty. \tag{1.5}$$

For the case $\lambda > 1$, the following sufficient condition for (1.2) to be nonoscillatory has been obtained in [22].

Theorem C. *Let* $\lambda > 1$. *If*

$$\int^{\infty} t^{\min\{\gamma,1\}(n-1)} p(t)dt < \infty,$$

then (1.2) *is nonoscillatory.*

Sufficient conditions for (1.2) with $\lambda > 1$ to be oscillatory were obtained in [5], [6], [8] and [9]. All of them, however, assume that

$$\int^{\infty} p(t)dt = \infty.$$

In this paper we have the following results.

Theorem 1.1. *Let* $\gamma \neq 1$ *and* $\lambda \neq 1$. *Then* (1.2) *is oscillatory if and only if* (1.5) *holds.*

Theorem 1.2. *Let* $\gamma = 1$ *and* $\lambda \neq 1$.

(i) *Equation* (1.2) *is oscillatory if* (1.3) *holds.*

(ii) *Suppose that*

$$\int^{\infty} t^{n-2}p(t)dt < \infty. \tag{1.6}$$

Equation (1.2) *is oscillatory if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2}p(s)ds > (1+\lambda)(n-1)!,$$

or if

$$\liminf_{t\to\infty} t \int_t^{\infty} s^{n-2}p(s)ds > (1+\lambda)(n-1)!/4.$$

Equation (1.2) *is nonoscillatory if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2}p(s)ds < (1+\lambda)(n-2)!/4.$$

Remark 1.1. If (1.4) holds, then (1.6) is satisfied and

$$0 \le \lim_{t\to\infty} t \int_t^{\infty} s^{n-2}p(s)ds \le \lim_{t\to\infty} \int_t^{\infty} s^{n-1}p(s)ds = 0.$$

Hence, Theorem 1.2 implies that (1.2) with $\gamma = 1$ and $\lambda \neq 1$ is nonoscillatory if (1.4) holds.

We give an example illustrating Theorem 1.2.

Example 1.1. We consider the second order linear neutral differential equation

$$\frac{d^2}{dt^2}[x(t) + \lambda x(t-\tau)] + c t^{\alpha} x(t-\rho) = 0, \tag{1.7}$$

where $\lambda > 0$, $\lambda \neq 1$, $\tau > 0$, $\rho \in \mathbb{R}$, $c > 0$, $\alpha \in \mathbb{R}$. Applying Theorem 1.2 to (1.7), we conclude that: (1.7) is oscillatory if either $\alpha > -2$ or $\alpha = -2$ and $c > (1+\lambda)/4$; (1.7) is nonoscillatory if either $\alpha < -2$ or $\alpha = -2$ and $c < (1+\lambda)/4$.

Our approach is to compare the oscillation of neutral differential equations of the form (1.1) with that of non-neutral differential equations of the form

$$x^{(n)}(t) + f(t, x(g(t))) = 0. \tag{1.8}$$

Such an approach as this has been conducted by Tang and Shen [23], and Zhang and Yang [25]. The oscillatory behavior of solutions of (1.8) has been intensively studied in the last three decades. See, for example, Kitamura [15], Manabu Naito [17] and the references cited therein.

2. Comparison theorems

The main results of this paper are as follows.

Theorem 2.1. *Suppose that* (R1) *holds. If, for some* ε *with* $0 < \varepsilon < (1-\lambda)/(1-\mu^2)$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{1-\lambda}{1-\mu^2} - \varepsilon\right) f(t, x(g(t))) = 0$$

is oscillatory, then (1.1) *is oscillatory. If, for some* $\varepsilon > 0$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{1-\mu}{1-\lambda^2} + \varepsilon\right) f(t, x(g(t))) = 0$$

is nonoscillatory, then (1.1) *is nonoscillatory.*

Theorem 2.2. *Suppose that* (R2) *holds. If, for some* ε *with* $0 < \varepsilon < (\mu - 1)/(\lambda^2 - 1)$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{\mu-1}{\lambda^2-1} - \varepsilon\right) f(t, x(g(t))) = 0$$

is oscillatory, then (1.1) *is oscillatory. If, for some* $\varepsilon > 0$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{\lambda-1}{\mu^2-1} + \varepsilon\right) f(t, x(g(t))) = 0$$

is nonoscillatory, then (1.1) *is nonoscillatory.*

Theorem 2.3. *Suppose that* (R3) *holds. If, for some* $\varepsilon \in (0,1)$, *the differential equation*

$$x^{(n)}(t) + (1-\varepsilon) f(t, [h(\tau^{-1}(g(t)))]^{-1} x(g(t))) = 0$$

is oscillatory, then (1.1) *is oscillatory. If, for some* $\varepsilon > 0$, *the differential equation*

$$x^{(n)}(t) + (1+\varepsilon) f(t, [h(\tau^{-1}(g(t)))]^{-1} x(g(t))) = 0$$

is nonoscillatory, then (1.1) *is nonoscillatory. Here,* $\tau^{-1}(t)$ *is the inverse function of* $\tau(t)$.

From Theorems 2.1 and 2.2, we have the following results.

Corollary 2.1. *Suppose that* $\lim_{t\to\infty} h(t) = \lambda$ *for some* $\lambda > 0$ *with* $\lambda \neq 1$. *If, for some* ε *with* $0 < \varepsilon < 1/(1+\lambda)$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{1}{1+\lambda} - \varepsilon\right) f(t, x(g(t))) = 0 \tag{2.1}$$

is oscillatory, then (1.1) *is oscillatory. If, for some* $\varepsilon > 0$, *the differential equation*

$$x^{(n)}(t) + \left(\frac{1}{1+\lambda} + \varepsilon\right) f(t, x(g(t))) = 0 \tag{2.2}$$

is nonoscillatory, then (1.1) *is nonoscillatory.*

Corollary 2.2. *Suppose that* $\lim_{t\to\infty} h(t) = 0$. *If, for some* $\varepsilon \in (0,1)$, *the differential equation*

$$x^{(n)}(t) + (1-\varepsilon) f(t, x(g(t))) = 0$$

is oscillatory, then (1.1) *is oscillatory. If, for some* $\varepsilon > 0$, *the differential equation*

$$x^{(n)}(t) + (1+\varepsilon) f(t, x(g(t))) = 0$$

is nonoscillatory, then (1.1) *is nonoscillatory.*

Proof of Corollaries 2.1 and 2.2. We give only the proof of Corollary 2.1 for the case $0 < \lambda < 1$. Likewise, we can be prove Corollary 2.1 for the case $\lambda > 1$ and Corollary 2.2. First suppose that (2.1) is oscillatory for some ε with $0 < \varepsilon < 1/(1+\lambda)$. There exists a number $\delta > 0$ such that

$$0 < \lambda - \delta < \lambda + \delta < 1 \quad \text{and} \quad \left| \frac{1-(\lambda+\delta)}{1+(\lambda-\delta)^2} - \frac{1}{1+\lambda} \right| < \frac{\varepsilon}{2}.$$

Put

$$\widetilde{\mu} = \lambda - \delta, \quad \widetilde{\lambda} = \lambda + \delta \quad \text{and} \quad \widetilde{\varepsilon} = \frac{1-\widetilde{\lambda}}{1-\widetilde{\mu}^2} - \frac{1}{1+\lambda} + \varepsilon.$$

Then

$$0 < \widetilde{\mu} < \lambda < \widetilde{\lambda} < 1, \quad \frac{1}{1+\lambda} - \varepsilon = \frac{1-\widetilde{\lambda}}{1-\widetilde{\mu}^2} - \widetilde{\varepsilon} \quad \text{and} \quad \widetilde{\varepsilon} > \frac{\varepsilon}{2} > 0.$$

Since $\lim_{t\to\infty} h(t) = \lambda$, we see that $\widetilde{\mu} \le h(t) \le \widetilde{\lambda}$ on $[t_1, \infty)$ for some $t_1 \ge t_0$. Hence, (R1) with μ, λ and t_0 replaced by $\widetilde{\mu}$, $\widetilde{\lambda}$ and t_1 holds. Theorem 2.1 implies that (1.1) is oscillatory. In the same way, we conclude that if (2.2) is nonoscillatory for some $\varepsilon > 0$, then (1.1) is nonoscillatory.

Now we are concerned with the oscillatory behavior of solutions of (1.2). It is possible to discuss more general neutral differential equations of the form (1.1). But, for simplicity, we restrict our attention to neutral differential equations of the form (1.2).

Consider the equation

$$x^{(n)}(t) + p(t)|x(t-\rho)|^{\gamma} \operatorname{sgn} x(t-\rho) = 0, \tag{2.3}$$

where $n \ge 2$ is even, $\rho \in \mathbb{R}$, $\gamma > 0$, $p \in C[t_0, \infty)$, $p(t) \ge 0$.

Lemma 2.1. *Let* $\gamma = 1$. *Then* (2.3) *is oscillatory if* (1.3) *holds.*

Lemma 2.2. *Let* $\gamma \ne 1$. *Then* (2.3) *is oscillatory if and only if* (1.5) *holds.*

For the proof of Lemmas 2.1 and 2.2, see Kitamura [15, Corollaries 3.1 and 5.1]. The following results was obtained by Manabu Naito [17, Theorems 2 and 4].

Lemma 2.3. *Let* $\gamma = 1$. *Suppose that* (1.6) *holds. Then* (2.3) *is oscillatory if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s)ds > (n-1)!,$$

or if

$$\liminf_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s)ds > (n-1)!/4.$$

Lemma 2.4. *Let* $\gamma = 1$. *Suppose that* (1.6) *holds. Then* (2.3) *is nonoscillatory if*

$$\limsup_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s)ds < (n-2)!/4.$$

Combining Corollary 2.1 with Lemmas 2.1–2.4, we obtain Theorems 1.1 and 1.2.

3. Preparation for the proofs of Theorems 2.1–2.3

In this section we prepare for the proofs of Theorems 2.1–2.3.
We make use of the following well-known lemma of Kiguradze [14].

Lemma 3.1. *Let $n \geq 2$ be even and let $u \in C^n[t_0, \infty)$ satisfy*

$$u(t) \neq 0 \quad \text{and} \quad u(t)u^{(n)}(t) \leq 0 \quad \text{for } t \geq t_0.$$

Then there exist an integer $k \in \{1, 3, \dots, n-1\}$ and a number $T \geq t_0$ such that

$$\text{(3.1)} \qquad \begin{cases} u(t)u^{(i)}(t) > 0, & 0 \leq i \leq k-1, \quad t \geq T, \\ (-1)^{i-k}u(t)u^{(i)}(t) \geq 0, & k \leq i \leq n, \quad t \geq T. \end{cases}$$

A function $u(t)$ satisfying (3.1) is said to be a function of Kiguradze degree k.

Lemma 3.2. *Suppose that* (H2) *holds. Let $u(t)$ be a function of Kiguradze degree k with $k \geq 1$. Then*

$$\text{(3.2)} \qquad \lim_{t\to\infty} \frac{u(\tau(t))}{u(t)} = 1.$$

Proof. We may assume that $u(t) > 0$ for all large t, since $-u(t)$ is a function of Kiguradze degree k. It is easy to see that one of the following three cases holds:

$$\text{(3.3)} \qquad \lim_{t\to\infty} u(t)/t^k = \text{const} > 0;$$

$$\text{(3.4)} \qquad \lim_{t\to\infty} u(t)/t^k = 0 \quad \text{and} \quad \lim_{t\to\infty} u(t)/t^{k-1} = +\infty;$$

$$\text{(3.5)} \qquad \lim_{t\to\infty} u(t)/t^{k-1} = \text{const} > 0.$$

In the case (3.3), we find that

$$\lim_{t\to\infty} \frac{u(\tau(t))}{u(t)} = \lim_{t\to\infty} \frac{u(\tau(t))}{[\tau(t)]^k} \left[\frac{u(t)}{t^k}\right]^{-1} \left[\frac{\tau(t)}{t}\right]^k = 1.$$

In exactly the same way, (3.2) holds for the case (3.5). Assume that (3.4) holds. We can take a number $T > 0$ so large that $u(t)$ satisfies (3.1) and $\tau(t) > 0$ for $t \geq T$. There exists a function $\rho \in C^1[T, \infty)$ such that $0 < \rho(t) \leq \tau(t) < t$ for $t \geq T$ and $\lim_{t\to\infty} \rho(t)/t = \lim_{t\to\infty} \rho'(t) = 1$. In fact,

$$\rho(t) = \int_T^t \inf_{r \geq s} \frac{\tau(r)}{r} ds$$

is such a function, since

$$\rho(t) \leq \int_T^t \inf_{r \geq t} \frac{\tau(r)}{r} ds \leq \frac{\tau(t)}{t}(t - T) \leq \tau(t) < t, \quad t \geq T,$$

and $\lim_{t\to\infty} \rho'(t) = \lim_{t\to\infty} \inf_{r\geq t} \tau(r)/r = 1$. Now we claim that

$$\text{(3.6)} \qquad \lim_{t\to\infty} \frac{u^{(k-1)}(\rho(t))}{u^{(k-1)}(t)} = 1.$$

To see this, it is sufficient to show that

$$\lim_{t\to\infty} \frac{1}{u^{(k-1)}(t)} \int_{\rho(t)}^{t} u^{(k)}(s)ds = 0,$$

since

$$\frac{1}{u^{(k-1)}(t)} \int_{\rho(t)}^{t} u^{(k)}(s)ds = 1 - \frac{u^{(k-1)}(\rho(t))}{u^{(k-1)}(t)}.$$

Notice that $u^{(k-1)}(t)$ is nondecreasing and positive, and $u^{(k)}(t)$ is nonincreasing and nonnegative on $[T, \infty)$. We conclude that

$$u^{(k-1)}(t) \geq u^{(k-1)}(\rho(t)) = \int_{T}^{\rho(t)} u^{(k)}(s)ds + u^{(k-1)}(T) \geq (\rho(t) - T)u^{(k)}(\rho(t))$$

for all large $t > T$, so that

$$0 \leq \frac{u^{(k)}(\rho(t))}{u^{(k-1)}(t)} \leq \frac{1}{\rho(t) - T}$$

for all large $t > T$. We obtain

$$\begin{aligned} 0 \leq \frac{1}{u^{(k-1)}(t)} \int_{\rho(t)}^{t} u^{(k)}(s)ds &\leq \frac{u^{(k)}(\rho(t))}{u^{(k-1)}(t)}(t - \rho(t)) \\ &\leq \frac{t - \rho(t)}{\rho(t) - T} = \frac{1 - \rho(t)/t}{\rho(t)/t - T/t} \to 0 \quad (t \to \infty). \end{aligned}$$

Consequently, (3.6) holds as claimed. From (3.6) and the fact that $\lim_{t\to\infty} \rho'(t) = 1$, it follows that

$$\begin{aligned} \lim_{t\to\infty} \frac{u(\rho(t))}{u(t)} &= \lim_{t\to\infty} \frac{u'(\rho(t))\rho'(t)}{u'(t)} = \lim_{t\to\infty} \frac{u'(\rho(t))}{u'(t)} \cdot \lim_{t\to\infty} \rho'(t) \\ &= \lim_{t\to\infty} \frac{u'(\rho(t))}{u'(t)} = \cdots = \lim_{t\to\infty} \frac{u^{(k-1)}(\rho(t))}{u^{(k-1)}(t)} = 1. \end{aligned}$$

Since $u(t)$ is nondecreasing on $[T, \infty)$, we have

$$\frac{u(\rho(t))}{u(t)} \leq \frac{u(\tau(t))}{u(t)} \leq 1$$

for all large $t > T$. This implies that (3.2) holds.

A close look at the proof of the result of Onose [20] enable us to obtain the next result.

Lemma 3.3. *Let $n \geq 2$ be even. Suppose that* (H3) *and* (H4) *holds. Then the differential equation*

$$x^{(n)}(t) + f(t, x(g(t))) = 0$$

is oscillatory if and only if the differential inequality

$$\{x^{(n)}(t) + f(t, x(g(t)))\} \operatorname{sgn} x(t) \leq 0$$

has no nonoscillatory solution.

Now consider the functional equation

$$\omega(t) + h(t)\omega(\tau(t)) = 1. \tag{3.7}$$

The following results have been established in [22].

Lemma 3.4. *Suppose that* (H1), (H2) *and* (R1) *holds. If* $\omega(t)$ *is a continuous function satisfying* (3.7) *for all large* t, *then*

$$0 < \frac{1-\lambda}{1-\mu^2} \le \liminf_{t\to\infty} \omega(t) \le \limsup_{t\to\infty} \omega(t) \le \frac{1-\mu}{1-\lambda^2}.$$

Lemma 3.5. *Suppose that* (H1), (H2) *and* (R2) *holds. If* $\omega(t)$ *is a bounded continuous function satisfying* (3.7) *for all large* t, *then*

$$0 < \frac{\mu-1}{\lambda^2-1} \le \liminf_{t\to\infty} \omega(t) \le \limsup_{t\to\infty} \omega(t) \le \frac{\lambda-1}{\mu^2-1}.$$

Lemma 3.6. *Suppose that* (H1), (H2) *and* (R3) *holds. If* $\omega(t)$ *is a bounded continuous function satisfying* (3.7) *for all large* t, *then* $\lim_{t\to\infty} \omega(t)h(\tau^{-1}(t)) = 1$.

We regard $C[T,\infty)$ as the Fréchet space of all continuous functions on $[T,\infty)$ with the topology of uniform convergence on every compact subinterval of $[T,\infty)$. We introduce the operator $\Phi : C[T,\infty) \longrightarrow C[T,\infty)$ such that

$$\Phi[u](t) + h(t)\Phi[u](\tau(t)) = u(t), \quad u \in C[T,\infty).$$

This operator Φ is useful to discuss the existence of solutions of the neutral differential equation (1.1). The following propositions concerning this operator have been obtained in [22].

Proposition 3.1. *Suppose that* (H1) *and* (H2) *holds. Let* T_* *and* T *be numbers such that* $t_0 \le T_* \le \tau(T)$ *and let* $r \in C[T_*,\infty)$ *with* $r(t) > 0$ *for* $t \ge T_*$. *Assume moreover that* $0 \le h(t)[r(\tau(t))/r(t)] \le \lambda < 1$ *on* $[T,\infty)$ *for some* λ. *Then there exists a mapping* $\Phi : C[T_*,\infty) \longrightarrow C[T_*,\infty)$ *which satisfies the following properties*:

(i) *the mapping* Φ *is continuous in the* $C[T_*,\infty)$*-topology*;
(ii) *for each* $u \in C[T_*,\infty)$, Φ *satisfies* $\Phi[u](t) + h(t)\Phi[u](\tau(t)) = u(t)$ *for* $t \ge T$.

Proposition 3.2. *Suppose that* (H1) *and* (H2) *holds. Let* T_* *and* T *be numbers such that* $t_0 \le T_* \le \tau(T)$ *and let* $r \in C[T_*,\infty)$ *with* $r(t) > 0$ *for* $t \ge T_*$. *Assume moreover that* $h(t)[r(\tau(t))/r(t)] \ge \mu > 1$ *on* $[T,\infty)$ *for some* μ. *Define*

$$U = \{u \in C[T_*,\infty) : |u(t)| \le r(t),\ t \ge T\}.$$

Then there exists a mapping $\Psi : U \longrightarrow C[T_*,\infty)$ *which satisfies the following properties*:

(i) *the mapping* Ψ *is continuous in the* $C[T_*,\infty)$*-topology*;
(ii) *for each* $u \in U$, Ψ *satisfies* $\Psi[u](t) + h(t)\Psi[u](\tau(t)) = u(t)$ *for* $t \ge T$;
(iii) *if* $u \in U$, $u(t) > 0$ *for* $t \ge T_*$ *and*

$$\limsup_{t\to\infty} \frac{u(t)r(\tau(t))}{u(\tau(t))r(t)} \le 1,$$

then $\Psi[u](t)/u(t)$ *is bounded on* $[T_*,\infty)$.

4. Proofs of Theorems 2.1–2.3

In this section we give the proofs of Theorems 2.1–2.3.

Proofs of Theorems 2.1–2.3. Suppose that one of conditions (R1)–(R3) holds. Define the constants c_* and c^*, and the function $H(t)$ by

$$c_* = (1-\lambda)/(1-\mu^2), \quad c^* = (1-\mu)/(1-\lambda^2), \quad H(t) = 1 \quad \text{if (R1) holds,}$$
$$c_* = (\mu-1)/(\lambda^2-1), \quad c^* = (\lambda-1)/(\mu^2-1), \quad H(t) = 1 \quad \text{if (R2) holds,}$$
$$c_* = 1, \quad c^* = 1, \quad H(t) = [h(\tau^{-1}(t))]^{-1} \quad \text{if (R3) holds,}$$

First we show the first halves of Theorems 2.1–2.3. Suppose that

$$v^{(n)}(t) + (c_* - \varepsilon) f(t, H(g(t))v(g(t))) = 0 \tag{4.1}$$

is oscillatory for some $\varepsilon \in (0, c_*)$. Assume to the contrary that (1.1) has a nonoscillatory solution $x(t)$. We may suppose without loss of generality that $x(t) > 0$ for all large t, since the case $x(t) < 0$ can be treated similarly. Then we easily see that $y(t) \equiv x(t) + h(t)x(\tau(t))$ is a function of Kiguradze degree k for some $k \in \{1, 3, \dots, n-1\}$ and $y(t) > 0$ for all large $t \geq t_0$. Observe that

$$\frac{x(t)}{y(t)} + h(t)\frac{y(\tau(t))}{y(t)}\frac{x(\tau(t))}{y(\tau(t))} = 1$$

for all large $t \geq t_0$. Put $\omega(t) = x(t)/y(t)$ and $\widetilde{h}(t) = h(t)y(\tau(t))/y(t)$. Then

$$\omega(t) + \widetilde{h}(t)\omega(\tau(t)) = 1.$$

Since $0 < \omega(t) = 1 - \widetilde{h}(t)\omega(\tau(t)) \leq 1$ for all large t, we find that $\omega(t)$ is bounded.

Now we assume that (R1) holds. By Lemma 3.2, there are numbers t_1 and $\delta \in (0,1)$ such that $(1+\delta)\lambda < 1$,

$$\frac{1-\lambda(1+\delta)}{1-[\mu(1-\delta)]^2} \geq \frac{1-\lambda}{1-\mu^2} - \frac{\varepsilon}{2} \quad \text{and} \quad 1-\delta < \frac{y(\tau(t))}{y(t)} < 1+\delta, \quad t \geq t_1.$$

Put $\widetilde{\mu} = \mu(1-\delta)$ and $\widetilde{\lambda} = \lambda(1+\delta)$. Then $0 \leq \widetilde{\mu} \leq \widetilde{h}(t) \leq \widetilde{\lambda} < 1$ for $t \geq t_1$. Consequently, form Lemma 3.4 it follows that

$$\frac{x(t)}{y(t)} = \omega(t) \geq \frac{1-\widetilde{\lambda}}{1-\widetilde{\mu}^2} - \frac{\varepsilon}{2} \geq \frac{1-\lambda}{1-\mu^2} - \varepsilon = (c_* - \varepsilon)H(t), \quad t \geq t_2$$

for some $t_2 \geq t_1$. Likewise, using Lemmas 3.5 and 3.6, we can prove that $x(t) \geq (c_* - \varepsilon)H(t)y(t)$ on $[t_2, \infty)$ for some $t_2 \geq t_1$ in the case where (R2) or (R3) holds.

By virtue of (1.1) and the monotonicity of f, we see that

$$y^{(n)}(t) + f(t, (c_* - \varepsilon)H(g(t))y(g(t))) \leq 0$$

for all large $t \geq t_0$. Put $z(t) = (c_* - \varepsilon)y(t)$ and $F(t,u) = (c_* - \varepsilon)f(t, H(g(t))u)$. Then $z(t)$ is a nonoscillatory solution of the differential inequality

$$z^{(n)}(t) + F(t, z(g(t))) \leq 0.$$

From Lemma 3.3 it follows that

$$z^{(n)}(t) + F(t, z(g(t))) = 0$$

has a nonoscillatory solution, which implies that (4.1) is nonoscillatory. This is a contradiction. The proof of the first halves of Theorem 2.1–2.3 is complete.

Now let us show the second halves of Theorems 2.1–2.3.

Assume that, for some $\varepsilon > 0$, the differential equation

$$v^{(n)}(t) + (c^* + \varepsilon)f(t, H(g(t))v(g(t))) = 0$$

has a nonoscillatory solution $v(t)$. Set $w(t) = (c^* + \varepsilon)^{-1}v(t)$. Then $w(t)$ is a nonoscillatory solution of

$$w^{(n)}(t) + f(t, (c^* + \varepsilon)H(g(t))w(g(t))) = 0. \tag{4.2}$$

We may assume that $w(t) > 0$ and $w(g(t)) > 0$ for all large t, say $t \geq T_1$. It is easy to verify that $w(t)$ is a function of Kiguradze degree k for some $k \in \{1, 3, \dots, n-1\}$, $\lim_{t\to\infty} w^{(i)}(t) = 0$ $(i = k+1, \dots, n-1)$ and $\lim_{t\to\infty} w^{(k)}(t)$ exists in $[0, \infty)$. By Lemma 3.2, we have $\lim_{t\to\infty} w(\tau(t))/w(t) = 1$. Thus we can take a number $T \geq T_1$ so large that $T_* \equiv \min\{\tau(T), \inf\{g(t) : t \geq T\}\} \geq T_1$, $w^{(i)}(T) > 0$ for $i = 0, 1, 2, \dots, k-1$, and if (R1) holds, then

$$0 \leq h(t)[w(\tau(t))/w(t)] \leq \overline{\lambda} < 1, \quad t \geq T \quad \text{for some } \overline{\lambda},$$

and if (R2) or (R3) holds, then

$$h(t)[w(\tau(t))/w(t)] \geq \overline{\mu} > 1, \quad t \geq T \quad \text{for some } \overline{\mu}.$$

Integration of (4.2) yields

(4.3)

$$w(t) = P(t) + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, (c^* + \varepsilon)H(g(r))w(g(r)))drds$$

for $t \geq T$, where

$$P(t) = \sum_{i=0}^{k-1} \frac{(t-T)^i}{i!} w^{(i)}(T) + \frac{(t-T)^k}{k!} w^{(k)}(\infty).$$

Notice that $P(t) \geq P(T) = w(T) > 0$ for $t \geq T$. We define the set Y of functions $y \in C[T_*, \infty)$ satisfying

$$P(t) \leq y(t) \leq w(t), \quad t \geq T \quad \text{and} \quad y(t) = y(T), \quad t \in [T_*, T].$$

We use Proposition 3.1 or 3.2 with $r(t) = w(t)$. Then there exists a continuous mapping $\Lambda : Y \longrightarrow C[T_*, \infty)$ such that

$$\Lambda[y](t) + h(t)\Lambda[y](\tau(t)) = y(t), \quad t \geq T, \quad y \in Y \tag{4.4}$$

and if (R2) or (R3) holds and $y \in Y$ satisfies $\lim_{t\to\infty} y(\tau(t))/y(t) = 1$, then $\Lambda[y](t)/y(t)$ is bounded on $[T_*, \infty)$. We define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} P(t) + \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \overline{f}(r, \Lambda[y](g(r)))drds, & t \geq T, \\ (\mathcal{F}y)(T), & t \in [T_*, T], \end{cases}$$

where

$$\overline{f}(t,u) = \begin{cases} f(t,(c^*+\varepsilon)H(g(t))w(g(t))), & u \geq (c^*+\varepsilon)H(g(t))w(g(t)), \\ f(t,u), & 0 \leq u \leq (c^*+\varepsilon)H(g(t))w(g(t)), \\ 0, & u \leq 0. \end{cases}$$

We note that

$$(4.5) \qquad 0 \leq \overline{f}(t,u) \leq f(t,(c^*+\varepsilon)H(g(t))w(g(t))) \quad \text{for all } (t,u) \in [T,\infty) \times \mathbb{R}.$$

Then it is easy to see that $\mathcal{F}$ is well defined on Y and maps Y into itself, by (4.3). Since Λ is continuous on Y, the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. In view of $\mathcal{F}(Y) \subset Y$, we find that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[T_*,\infty)$. By the Ascoli-Arzelà theorem, it suffices to verify that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. By (4.5), we easily see that there exists a function $G \in C[T,\infty)$ which is independent of $y \in Y$ and satisfies, for each $y \in Y$, $|(\mathcal{F}y)'(t)| \leq G(t)$ on $[T,\infty)$. Let I be an arbitrary compact subinterval of $[T,\infty)$. Then we see that $\{(\mathcal{F}y)'(t) : y \in Y\}$ is uniformly bounded on I. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for t_1, $t_2 \in [T_*,T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$.

By applying the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$, there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. It is easy to verify that $\widetilde{y}(t)$ satisfies $\widetilde{y}(t) > 0$ for $t \geq T_*$ and is a function of Kiguradze degree k. Lemma 3.2 implies that $\lim_{t\to\infty} \widetilde{y}(\tau(t))/\widetilde{y}(t) = 1$. From (4.4), we see that

$$\frac{\Lambda[\widetilde{y}](t)}{\widetilde{y}(t)} + h(t)\frac{\widetilde{y}(\tau(t))}{\widetilde{y}(t)}\frac{\Lambda[\widetilde{y}](\tau(t))}{\widetilde{y}(\tau(t))} = 1, \quad t \geq T,$$

and $\Lambda[\widetilde{y}](t)/\widetilde{y}(t)$ is bounded on $[T_*,\infty)$ if (R2) or (R3) holds. By the same arguments as in the proofs of the first halves of Theorems 2.1–2.3 and using Lemmas 3.4–3.6, we obtain

$$0 < \frac{\Lambda[\widetilde{y}](t)}{\widetilde{y}(t)} \leq (c^*+\varepsilon)H(t) \quad \text{for all large } t \geq T.$$

Since $0 < \widetilde{y}(t) \leq w(t)$ for $t \geq T$, we have $0 < \Lambda[\widetilde{y}](g(t)) \leq (c^*+\varepsilon)H(g(t))w(g(t))$ for all large t, say $t \geq \widetilde{T}$. Hence, $\overline{f}(t,\Lambda[\widetilde{y}](g(t))) = f(t,\Lambda[\widetilde{y}](g(t)))$ for $t \geq \widetilde{T}$. This and (4.4) imply that

$$\frac{d^n}{dt^n}[\Lambda[\widetilde{y}](t) + h(t)\Lambda[\widetilde{y}](\tau(t))] = \widetilde{y}^{(n)}(t) = -\overline{f}(t,\Lambda[\widetilde{y}](g(t))) = -f(t,\Lambda[\widetilde{y}](g(t)))$$

for $t \geq \widetilde{T}$, which means that $\Lambda[\widetilde{y}](t)$ is a nonoscillatory solution of (1.1). This completes the proof.

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