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Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa–Intriligator formula (joint work with Anatol N. Kirillov)

京大·理学研究科 前野 俊昭 (Toshiaki Maeno)

Introduction.

The cohomology ring of flag variety Fl_n of type A_{n-1} is isomorphic to the quotient ring of the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ by the ideal generated by symmetric polynomials. It is well-known that the dual classes of the Schubert cycles are represented by so-called Schubert polynomials in the cohomology ring. On the other hand, physicists introduced another ring structure on the cohomology group, which is called quantum cohomology ring. The quantum cohomology ring of the flag variety also has a structure of the quotient ring of the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ ([C],[GK]). In the quantum cohomology ring of flag variety, the Schubert polynomials do not represent the Schubert classes any more. Fomin, Gelfand and Postnikov [FGP] first introduced the quantum Schubert polynomials, which correspond to the Schubert classes in the quantum cohomology ring. They gave a commuting family of operators acting on the polynomial ring and the quantization of polynomials. In this note, we define the quantum Schubert polynomials based on the orthogonality of the Schubert classes with respect to the intersection pairing. We also introduce the quantum double Schubert polynomials and see that they satisfy an analogue of the Cauchy formula. As applications of quantum Cauchy formula, we consider the Vafa-Intriligator type formula and the quantization map. For details, see Kirillov and Maeno [KM1].

1 Cohomology ring of flag variety and Schubert polynomials.

First of all, we make a brief review of the Schubert polynomials. More details are found in Macdonald [M]. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbf{C}^n \otimes \mathcal{O}_F$ be the universal flag of subbundles over $F = Fl_n$. Then the cohomology ring $H^*(F, \mathbf{Z})$ is isomorphic to the quotient ring P_n/I_n, where $P_n = \mathbf{Z}[x_1, \ldots, x_n]$ and I_n is the ideal generated by symmetric polynomials, and the natural identification is given by

$$x_i \mapsto c_1(E_i/E_{i-1}), \quad i = 1, \ldots, n.$$

Let us consider the natural sequence of quotient bundles

$$E_n = L_n \to \cdots \to L_1 \to 0,$$

where $L_i = E_n/E_{n-i}$ on F. Now we fix a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbf{C}^n.$$

Then we have induced morphisms $f_{pq} : V_p \otimes \mathcal{O}_F \to L_q$ for $1 \leq p, q \leq n$.

Definition 1.1. *For a permutation $w \in S_n$, the Schubert cycle Ω_w is the locus where* $\mathrm{rank} f_{pq} \leq r_w(q,p)$ *for any $1 \leq p, q \leq n-1$, where $r_w(q,p) = \sharp\{i \mid i \leq q,\ w(i) \leq p\}$.*

It is easy to see that the dual class $[\Omega_w] \in H^{2l(w)}(F, \mathbf{Z})$ where $l(w)$ is the length of w and that the Schubert cycles give an orthonormal basis of of the cohomology group with respect to the intersection pairing, namely

$$\langle \Omega_u, \Omega_v \rangle = \delta_{v, w_0 u},$$

where $w_0 \in S_n$ is the permutation of maximal length.

Next we consider the Schubert polynomials which represent the Schubert classes in the cohomology ring. We introduce the divided difference operators ∂_i $i = 1, \ldots, n-1$ acting on the polynomial ring P_n as follows:

$$(\partial_i f)(x) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots x_n)}{x_i - x_{i+1}}.$$

We choose a permutation $w \in S_n$ and its reduced decomposition

$$w = s_{a_1} \cdots s_{a_p},$$

where s_i is a simple transposition $(i, i+1)$ and $p = l(w)$. Then the operator

$$\partial_w = \partial_{a_1} \cdots \partial_{a_p}$$

does not depend on the choice of reduced decomposition of w.

Definition 1.2(Lascoux and Schützenberger [LS1]).
For each permutation $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w(x)$ is defined to be

$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0}(x^\delta),$$

where δ is a multi-index $(n-1, n-2, \ldots, 0)$.

It is known that $\mathfrak{S}_w(x) = \Omega_w$ in $H^*(F, \mathbf{Z})$. Moreover, the intersection pairing on $H^*(F, \mathbf{Z})$ is naturally identified with the induced pairing from the residue pairing

$$\langle f, g\rangle_{I_n} = \mathrm{Res}_{I_n}(fg), \quad f, g \in P_n,$$

where Res_{I_n} is the Grothendieck residue ([GH]). Hence, the Schubert polynomials are orthonormal with respect to the pairing $\langle\ ,\ \rangle$. Conversely, the orthogonality characterizes the Schubert polynomials.

Proposition 1.3. *The Schubert polynomials are uniquely characterized by the following properties.*

1. $\langle \mathfrak{S}_u, \mathfrak{S}_v\rangle = \delta_{u, w_0 v}$.
2. $\mathfrak{S}_w(x) = x^{c(w)} + \sum \alpha_I x^I$,
where $c(w) = (c_1(w), \ldots, c_n(w))$ is the code of w and multi-indices $I = (i_1, \ldots, i_n)$ run over $I \subset \delta$ (i.e $0 \le i_k \le n-k$) and I degree-lexicographically smaller than $c(w)$.

Remarks. 1) (Definition of the code)([M], p.9).
For a permutation $w \in S_n$, we define

$$c_i(w) = \sharp\{j \mid i < j, w(i) > w(j)\}.$$

The sequence $c(w) = (c_1(w), \dots, c_n(w))$ is called the code of w.
2) The Schubert polynomials are obtained as Gram-Schmidt's orthogonalization of the monomials $(x^I)_{I\subset\delta}$ ordered degree-lexicographically.

2 Quantum Schubert polynomials and quantum double Schubert plynomials

Quantum cohomology ring is a deformed ring of the ordinary cohomology ring and its structure constants are given by the Gromov-Witten invariants. For general definition and properties of quantum cohomology ring, see [KM2], [MS] and [RT]. Let us remind the structure of quantum cohomology ring of flag variety.

Theorem 2.1(Givental and Kim [GK], Ciocan-Fontanine [C]).
The quantum cohomology ring of flag variety F is generated by $x_i = c_1(E_i/E_{i-1})$, $i = 1, \dots, n$, as a $\mathbf{Z}[q_1, \dots, q_{n-1}]$-algebra and

$$QH^*(F) \cong \mathbf{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]/\tilde{I}_n,$$

where the ideal $\tilde{I}_n$ is generated by the quantum elementary symmetric functions $\tilde{e}_i(x) := e_i(x|q)$, with generating function

$$\Delta_n(t|x) :=$$

$$\sum_{i=1}^{n} e_i(x|q)t^i = \det \begin{pmatrix} x_1+t & q_1 & 0 & \dots & \dots & 0 \\ -1 & x_2+t & q_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & x_{n-1}+t & q_{n-1} \\ 0 & \dots & \dots & 0 & -1 & x_n+t \end{pmatrix}.$$

We define a pairing on $\bar{P}_n = \mathbf{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]$ with values in $\mathbf{Z}[q_1, \dots, q_{n-1}]$ by

$$\langle f, g\rangle_Q = \mathrm{Res}_{\tilde{I}_n}(fg), \quad f, g \in \bar{P}_n.$$

This pairing induces a nondegenerate pairing on $QH^*(F)$.

Definition 2.2([KM1], Definition 5).

We define the quantum Schubert polynomials $\tilde{\mathfrak{S}}_w$ as Gram-Schmidt's orthogonalization of the classical Schubert polynomials $\mathfrak{S}_w$ with respect to the pairing $\langle\ ,\ \rangle_Q$:

1) $\langle \tilde{\mathfrak{S}}_u, \tilde{\mathfrak{S}}_v \rangle_Q = \delta_{v,w_0u}$,

2) $\tilde{\mathfrak{S}}_w(x) = x^{c(w)} + \Sigma_{I<c(w)}\, a_I(q)x^I$,

where $a_I(q) \in \mathbf{Z}[q_1, \ldots, q_{n-1}]$ and $I < c(w)$ means the degree-lexicographic order.

Remarks. 1)This definition is the analogue of the characterization of Schubert polynomials in Proposition 1.3.

2) The polynomials obtained by the Gram-Schmidt type orthogonalization are, a priori, defined over the field of fractions of $\mathbf{Z}[q_1, \ldots, q_{n-1}]$. However, it turns out that they are defined over $\mathbf{Z}[q_1, \ldots, q_{n-1}]$ from Theorem 3.1.

Next we introduce quantum double Schubert polynomials.

Definition 2.3([KM1], Definition 4).

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be two sets of variables, put

$$\tilde{\mathfrak{S}}_{w_0}(x,y) := \tilde{\mathfrak{S}}_{w_0}^{(q)}(x,y) = \prod_{i=1}^{n-1} \Delta_i(y_{n-i} \mid x_1, \ldots, x_i).$$

For each permutation $w \in S_n$, the quantum double Schubert polynomial is defined to be

$$\tilde{\mathfrak{S}}_w(x,y) = \partial^{(y)}_{ww_0} \tilde{\mathfrak{S}}_{w_0}(x,y),$$

where divided difference operator $\partial^{(y)}_{ww_0}$ acts on the y variables.

Remarks. 1) If we put $q_1 = \cdots = q_{n-1} = 0$ in the quantum double Schubert polynomial, we get the classical one, namely

$$\tilde{\mathfrak{S}}_w(x,y)\ |_{q=0} = \mathfrak{S}_w(x,y),$$

where the classical double Schubert polynomial $\mathfrak{S}_w(x,y)$ defined as follows ([LS2]):

$$\mathfrak{S}_{w_0}(x,y) = \prod_{i+j \le n} (x_i + y_j),$$

$$\mathfrak{S}_w(x,y) = \partial^{(y)}_{ww_0}\mathfrak{S}_{w_0}(x,y).$$

2) The quantum double Schubert polynomials are stable under the natural embedding $i : S_n \to S_m$ for $m > n$:

$$\mathfrak{S}_w(x,y) = \mathfrak{S}_{i(w)}(x,y).$$

3 Quantum Cauchy formula and its applications.

Theorem 3.1([KM1], Theorem 3). *The following analogue of the Cauchy formula holds for the quantum double Schubert polynomial:*

$$\tilde{\mathfrak{S}}_{w_0}(x,y) = \sum_{w\in S_n} \tilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y).$$

For multi-index $I = (i_1,\dots,i_n) \subset \delta$, the elementary and the quantum elementary polynomials $e_I(x)$ and $\tilde{e}_I(x)$ are defined to be

$$e_I(x) = \prod_{k=1}^{n-1} e_{i_k}(x_1,\dots,x_{n-k}),$$

$$\tilde{e}_I(x) = \prod_{k=1}^{n-1} \tilde{e}_{i_k}(x_1,\dots,x_{n-k}).$$

The theorem above is equivalent to that

$$\langle \tilde{e}_I(x), \tilde{\mathfrak{S}}_w(x)\rangle_Q = \langle e_I(x), \mathfrak{S}_w(x)\rangle,$$

for any permutation $w \in S_n$. This means that $\tilde{e}_I(x)$ represents in the quantum cohomology ring the same class that $e_I(x)$ represents in the classical cohomology ring. We can prove this fact geometrically. The proof can be found in [KM1], Section 7.

Corollary 3.2. *The quantum Schubert polynomial $\tilde{\mathfrak{S}}_w(x)$ can be calculated by the formula*

$$\tilde{\mathfrak{S}}_w(x) = \left(\partial^{(y)}_{ww_0}\mathfrak{S}_{w_0}(x,y)\right)|_{y=0} .$$

In particular, the quantum Schubert polynomials are stable under the natural embedding $S_n \to S_m$, $m > n$.

Corollary 3.3. The quantum Schubert polynomial $\tilde{\mathfrak{S}}_{w_0}(x)$ is obtained as a product of quantum elementary symmetric polynomials

$$\tilde{\mathfrak{S}}_{w_0}(x) = \tilde{e}_1(x_1)\tilde{e}_2(x_1, x_2) \cdots \tilde{e}_{n-1}(x_1, \ldots, x_{n-1}).$$

For a polynomial $f(x) \in P_n$, we denote by $[f(x)]$ the corresponding class in $QH^*(F)$. The correlation function (of genus zero) is given by

$$\langle f(x) \rangle = \int_F [f(x)].$$

This correlation function is a generating function of Gromov-Witten invariants of genus zero. By using genus g invariants instead of genus zero invariants, we can define the correlation function of genus g, which is denoted by $\langle f(x) \rangle_g$. For Gromov-Witten invariants of higher genus, see [KM2] and [RT]. In order to calculate the higher genus correlation function, we introduce a polynomial

$$C^{(q,q')}(x, y) = \sum_{w \in S_n} \tilde{\mathfrak{S}}_w^{(q)}(x) \tilde{\mathfrak{S}}_{w_0 w}^{(q')}(y).$$

From Theorem 3.1, we have

$$C^{(q,q')}(x, y) = \langle \tilde{\mathfrak{S}}_{w_0}^{(q)}(x, z), \tilde{\mathfrak{S}}_{w_0}^{(q')}(y, z) \rangle^{(z)},$$

where $\langle \ , \ \rangle^{(z)}$ is the classical pairing with respect to the variables z. The higher genus correlation functiton can be calculated by the following Vafa-Intriligator type formula.

Theorem 3.4.([KM1], Theorem 10).
Let $\Phi(x) = C^{(q,q)}(x, x)$. Then we have

$$\langle f(x) \rangle_g = \mathrm{Res}_{\tilde{I}_n}(f(x)\Phi(x)^g).$$

As another application of the quantum Cauchy formula, we consider the quantization map on the polynomial ring P_∞. Let $f(x) \in P_\infty$, then we have the interpolation formula

$$f(x) = \sum_{w \in S^{(n)}} \mathfrak{S}_w(x, y) \partial_w^{(y)} f(y),$$

where $S^{(n)}$ is the set of permutations whose codes have length less than or equal to n. We define the quantization of a polynomial $f(x) \in P_\infty$ by

$$\tilde{f}(x) = \sum_{w \in S^{(n)}} \tilde{\mathfrak{S}}_w(x,y)\partial_w^{(y)} f(y),$$

and obtain a linear map $P_\infty \to \bar{P}_\infty$. If we put $x_{n+1} = x_{n+2} = \cdots = 0$, and $q_n = q_{n+1} = \cdots = 0$, we have a quantization map $P_n \to \bar{P}_n$. This quantization map preserves the pairings:

$$\langle \tilde{f}, \tilde{g} \rangle_Q = \langle f, g \rangle_{I_n}.$$

for $f, g \in P_n$. Moreover, the quantization map maps the ideal $I_n \subset P_n$ into the ideal $\tilde{I}_n \subset \bar{P}_n$.

Quantization map does not commute with the product, namely $\widetilde{fg} \neq \tilde{f} \cdot \tilde{g}$ in general. The difference $\widetilde{fg} - \tilde{f} \cdot \tilde{g}$ reflects the structure constants of the quantum cohomology ring. If f is a linear form, the following quantum Pieri formula gives an explicit result.

Theorem 3.5. *Let t_{ij} be a transposition of the integers i and j. In the quantum cohomology ring $QH^*(F)$,*

$$(x_1 + \cdots x_k) \cdot \tilde{\mathfrak{S}}_w(x) = \sum_{(*)} \tilde{\mathfrak{S}}_{wt_{ij}}(x) + \sum_{(**)} q_i q_{i+1} \cdots q_{i+s-1} \tilde{\mathfrak{S}}_{wt_{ij}}(x),$$

where $(*)$ ranges the integers i, j such that $1 \leq i \leq k < j \leq n$ and $l(wt_{ij}) = l(w) + 1$, and $(**)$ ranges the integers i, j such that $1 \leq i \leq k < j \leq n$ and $2s = l(w) - l(wt_{ij}) + 1 \geq l(t_{ij})$.

Remark. The quantum Pieri formula was first proved in [FGP]. For the quantum Pieri formula for $\tilde{\mathfrak{S}}_{w_0}(x,y)$ and equivariant quantum Pieri formula, see also [KM1], Section 9.

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