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MOTIVIC COHOMOLOGY

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ABSTRACT. We give an overview over the axioms of Beilinson and Lichtenbaum for motivic complexes, and explain which of these axioms have been proven for Bloch's higher Chow groups.

1. BASIC PROPERTIES

Let X be a smooth variety over a field k, and let

$$\Delta^i = \operatorname{Spec} k[t_0, \cdots, t_i]/(\sum t_s - 1)$$

be the algebraic i-simplex. Define $z^n(X,i)$ to be the free abelian group generated by algebraic cycles of codimension n on $\Delta^i \times X$ which intersect all faces $t_{j_1} = \ldots = t_{j_r} = 0$ in codimension n.

The inclusion of the face $t_j = 0$ induces on $z^n(X,*)$ the structure of a simplicial abelian group. Taking alternating sums one gets a chain complex. We normalize this chain complex in such a way that we get a cohomological complex with $z^n(X,2n)$ in degree 0 and $z^n(X,0)$ in degree $2n$. One defines $H^i(X,\mathbb{Z}(n))$ to be the jth cohomology of this complex. Similarly, $H^i(X,\mathbb{Z}/m(n))$ is the jth cohomology of $z^n(X,*)\otimes\mathbb{Z}/m$. Note that we don't have to take the derived tensor product since the $z^n(X,i)$ are free. There is an obvious long exact coefficient sequence.

Proposition 1.1. *a) $H^i(X,\mathbb{Z}(n)) = 0$ for $i > 2n$ or $i > n + \dim X$.*
b) $H^{2n}(X,\mathbb{Z}(n)) \cong \mathrm{CH}^n(X)$.

The first part follows because $z^n(X,*)$ is zero in the given range. The second part follows from the definition of rational equivalence. Because of this result, the motivic cohomology groups are also called "higher Chow groups".

It is easy to see that the motivic cohomolgy groups are contravariant for flat maps and covariant for proper maps.

The complex $z^n(-,*)$ forms a sheaf for the Zariski and the étale topology. We denote these complexes of sheaves by $\mathbb{Z}(n)$ and $\mathbb{Z}(n)_{\text{ét}}$, respectively. We define $\mathbb{H}^i(X_{\text{Zar}},\mathbb{Z}(n))$ and $\mathbb{H}^i(X_{\text{ét}},\mathbb{Z}(n))$ as the hypercohomology of these complexes of sheaves.

Let $Z \longrightarrow X$ be a closed immersion of pure codimension c with open complement U. There is an exact sequence of complexes

$$0 \longrightarrow z^{n-c}(Z,*) \longrightarrow z^n(X,*) \longrightarrow z^n(U,*) \longrightarrow C(*) \longrightarrow 0.$$

Here $C(i)$ is generated by cycles on $\Delta^i \times U$ such that the closure in $\Delta^i \times X$ does not meet faces properly. The moving lemma of Bloch [3] states that $C(*)$ is acyclic. In particular, we get the following consequences[2]:

Corollary 1.2. *(Localization) There is a long exact sequence*

$$\ldots \longrightarrow H^{i-2c}(Z,\mathbb{Z}(n-c)) \longrightarrow H^i(X,\mathbb{Z}(n)) \longrightarrow H^i(U,\mathbb{Z}(n)) \longrightarrow \ldots$$

Corollary 1.3. *We have an isomorphism*

$$H^i(X, \mathbb{Z}(n)) \cong \mathbb{H}^i(X_{\mathrm{Zar}}, \mathbb{Z}(n)).$$

In particular, there is a hypercohomology spectral sequence

$$E_2^{s,t} = H^s(X_{\mathrm{Zar}}, H^t(\mathbb{Z}(n))) \Rightarrow H^{s+t}(X, \mathbb{Z}(n)).$$

2. The axioms of Beilinson and Lichtenbaum

Let $\epsilon : X_{\text{ét}} \longrightarrow X_{\mathrm{Zar}}$ be the morphism of sites induced by the identity map. The following is a list of the axioms of Beilinson [1] and Lichtenbaum [12], extended by Milne [14], which have been proven so far for Bloch's higher Chow groups:

Theorem 2.1. 1. *(Weight 0)*

$$\mathbb{Z}(0) \cong \mathbb{Z}.$$

2. *(Weight 1)*

$$\mathbb{Z}(1) \cong \mathbb{G}_m[-1].$$

3. *(Product) There is a pairing*

$$Z(n) \otimes^L Z(m) \longrightarrow \mathbb{Z}(n+m).$$

4. *(Gersten resolution) The cohomology sheaves $H^t(\mathbb{Z}(n))$ of the complex of Zariski sheaves $\mathbb{Z}(n)$ admit a resolution:*

$$0 \longrightarrow H^t(\mathbb{Z}(n)) \longrightarrow \bigoplus_{x \in X^{(0)}} (i_x)_* H^t(k(x), \mathbb{Z}(n)) \longrightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* H^{t-1}(k(x), \mathbb{Z}(n-1)) \longrightarrow \dots$$

In particular, the complex $\mathbb{Z}(n)$ is acyclic above degree n.

5. *(Milnor K-theory) We have an isomorphism*

$$K_n^M(k) \xrightarrow{\sim} H^n(k, \mathbb{Z}(n)).$$

6. *(Quillen K-theory) There is an Atiyah-Hirzebruch spectral sequence*

$$H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(X).$$

7. *(Purity) Let $Z \xrightarrow{i} X$ be a closed immersion of codimension c. Then there is a quasi-isomorphism of complexes on Z*

$$\tau_{\leq n+c} Ri^! \mathbb{Z}(n)^X_{\text{ét}} \cong \mathbb{Z}(n-c)^Z_{\text{ét}}[-2c].$$

8. *(Beilinson-Lichtenbaum conjecture, only 2- and p-primary part)*

$$\mathbb{Z}(n)_{\mathrm{Zar}} \cong \tau_{\leq n} R\epsilon_* \mathbb{Z}(n)_{\text{ét}}.$$

9. *(Hilbert 90, only 2- and p-primary part)*

$$R^{n+1}\epsilon_* \mathbb{Z}(n)_{\text{ét}} = 0.$$

10. *(mod m-sheaf) For $(m, char\ k) = 1$,*

$$\mathbb{Z}/m(n)_{\text{ét}} \cong \mu_m^{\otimes n}.$$

11. *(mod p-sheaf) For $p = char\ k$,*

$$\mathbb{Z}/p^r(n) \cong \epsilon_* \nu_{p^r}^n[-n]$$
$$\mathbb{Z}/p^r(n)_{\text{ét}} \cong \nu_{p^r}^n[-n].$$

Proof. The properties 1-4 have been proven in Bloch's original paper [2]. The connection to Milnor K-theory has been proven by Nesterenko and Suslin [15] and independently by Tataro.

The Atiyah-Hirzebruch spectral sequence has first been established by Bloch and Lichtenbaum [5] for fields, and this result has been extended to varieties by Levine [11].

As for properties 7-9, it suffices to prove them rationally, and with mod m and mod p^r coefficients. Rationally, motivic cohomology for the étale and Zariski topology agree, hence Hilbert 90 and the Beilinson-Lichtenbaum conjecture follow. Similarly, purity for the Zariski topology is nothing but localization, and implies étale purity rationally. The mod m and mod p part will be dealt with in the next sections.

3. Mod l

The identification of the étale mod m sheaf follows from the following rigidity result of Bloch [5]: If $\pi : X \longrightarrow k$ is the structure map, then we have an quasi-isomorphism of complexes of sheaves on X

$$\mathbb{Z}/m(n)^X_{\text{ét}} \cong \pi^*\mathbb{Z}/m(n)^k_{\text{ét}}.$$

This reduces the problem to calculating motivic cohomology of a separably closed field, which has been done by Suslin [17]. This implies purity mod m (even without the truncation) by the corresponding statement for étale cohomology.

Conjecture 3.1. *(Kato [4]) For any field F and m prime to the characteristic of F,*

$$K^M_n(F)/m \xrightarrow{\sim} H^n(F_{\text{ét}}, \mu_m^{\otimes n}).$$

Using properties 5) and 10), this can be restated as an isomorphism

$$H^n(F, \mathbb{Z}/m(n)) \xrightarrow{\sim} \mathbb{H}^n(F_{\text{ét}}, \mathbb{Z}/m(n)).$$

It has been shown by Suslin and Voevodsky [16] (assuming resolution of singularities) and by Levine and the author [8] (in general) that this implies

$$H^i(F, \mathbb{Z}/m(n)) \xrightarrow{\sim} \mathbb{H}^i(F_{\text{ét}}, \mathbb{Z}/m(n)) \quad \text{for} \quad i \leq n.$$

Note that this is wrong for $i > n$, as the left hand side vanishes, whereas the right hand group may not; for example, $\mathbb{H}^2(F_{\text{ét}}, \mathbb{Z}/m(1)) = {}_m\operatorname{Br} F$.

Comparing the Gersten resolutions of $H^t(\mathbb{Z}/m(n))$ and $R^t\epsilon_*\mu_m^{\otimes n}$ (Bloch-Ogus), we see that the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture mod m. On the other hand, Suslin and Voevodsky [16] show that the m-primary part of Hilbert 90 and the Beilinson-Lichtenbaum conjecture are equivalent. Finally, Voevodsky showed in [18] that the 2-primary part of Hilbert 90 holds.

4. Mod p

We will assume that the base field k is perfect. Recall the definition of the logarithmic de Rham-Witt sheaves $\nu^n_{p^r} = W_r\Omega^n_{X,\log}$ as the subsheaf of $W_r\Omega^n_X$ generated locally for the étale topology by $d\log \bar{x}_1 \wedge \cdots \wedge d\log \bar{x}_n$, where $\bar{x} \in W_r\mathcal{O}^*_X$

are Teichmüller lifts of units. For example, $\nu_{p^r}^0 = \mathbb{Z}/p^r$ and there is a short exact sequence

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{p^r} \mathbb{G}_m \xrightarrow{dlog} \nu_{p^r}^n \longrightarrow 0.$$

In this situation, the analogue of the conjecture of Kato is a theorem [4]:

Theorem 4.1. *For any field F of characteristic p, there is an isomorphism*

$$K_n^M(F)/p^r \xrightarrow{\sim} \nu_{p^r}^n(F).$$

Note that this can again be reformulated as an isomorphism

$$H^n(F, \mathbb{Z}/p^r(n)) \xrightarrow{\sim} H^n(F_{\text{ét}}, \nu_{p^r}^n[-n]).$$

Comparing Gersten resolutions, we get $H^n(\mathbb{Z}/p^r(n)) \cong \nu_{p^r}^n$. As above (but with more technical difficulties), Levine and the author proved [7] that this implies

Theorem 4.2. *For any field F of characteristic p, $H^i(F, \mathbb{Z}/p^r(n)) = 0$ for $i \neq n$.*

Using the Gersten resolution, this determines $\mathbb{Z}/p^r(n)_{\text{Zar}}$, and using this result for all smooth schemes over k, we can identify $\mathbb{Z}/p^r(n)_{\text{ét}}$.

Since we shift by n, this implies the p-primary part of the Beilinson-Lichtenbaum conjecture:

$$\mathbb{Z}/p^r(n)_{\text{Zar}} = \nu_{p^r}^n[-n] = \tau_{\leq n} R\epsilon_* \nu_{p^r}^n[-n] = \tau_{\leq n} R\epsilon_* \mathbb{Z}/p^r(n)_{\text{ét}}.$$

Hilbert 90 is a consequence of this using a simple diagram chase [7]. Finally, the mod p purity statement follows from the analogous statement for the $\nu_{p^r}^n$, as proved by Milne [13] (note that in this case the truncation is necessary).

5. The motivation

The original motivation of Lichtenbaum for defining motivic complexes was to express values of zeta functions as an Euler characteristic. To give a flavour of this, we note the following result of Kahn [10]. Let X be a smooth projective variety over a finite field; let $\zeta(X, s)$ be its Artin-Hasse zeta function. Assume that the strong form of Tate's conjecture holds; i.e. the order a_n of vanishing of $\zeta(X, s)$ at $s = n$ equals the rank of the Chow group $\mathrm{CH}^n(X)$ for all smooth projective varieties over finite fields. Then we have

$$\zeta(X, s) = (1 - p^{n-s})^{a_n} \varphi(s),$$

and $\varphi(n)$ equals up to $\pm$ a power of p by

$$|R_n(X)|^{-1} \cdot \prod_{i \neq 2n, 2n+2} |\mathbb{H}^i(X_{\text{ét}}, \mathbb{Z}(n))|^{(-1)^i} \cdot |\mathbb{H}^{2n}(X_{\text{ét}}, \mathbb{Z}(n))_{\text{tors}}| |\mathbb{H}^{2n+2}(X_{\text{ét}}, \mathbb{Z}(n))_{\text{cotors}}|.$$

Here cotors means the quotient by the maximal divisible subgroup, and R_n is the discriminant of the pairing

$$\mathbb{H}^{2n}(X_{\text{ét}}, \mathbb{Z}(n))/\text{tors} \times \mathbb{H}^{2d-2n}(X_{\text{ét}}, \mathbb{Z}(d-n))/\text{tors} \longrightarrow \mathbb{H}^{2d}(X_{\text{ét}}, \mathbb{Z}(d))/\text{tors} \cong \mathbb{Z}_l.$$

There are predecessors of this result by various people for $s = 0, 1, 2$, using K-theory instead of motivic cohomology. Lichtenbaum realized that one has to use the weight-n-space of all K-groups in order to describe $\zeta(X, s)$ at $s = n$.

The original motivation of Beilinson [1] was to define a universal cohomology theory satisfying the axioms of Bloch-Ogus for a twisted Poincaré duality theory.

These motivic cohomology groups should be given as the extension groups in some conjectural category of mixed motives:

$$H^i(X,\mathbb{Z}(n)) = \mathrm{Hom}_{\mathcal{MM}}(\mathbb{Z}(0), M(X)\otimes\mathbb{Z}(n)[i]).$$

Here $M(X)$ is the motive associated to X, and $\mathbb{Z}(n)$ the n-fold tensor product of the Lefschetz motive.

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