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Exactly Solvable Chaos and Addition Theorems of Elliptic Functions

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Abstract

A unified view is given to recent developments about a systematic method of constructing rational mappings as ergodic transformations with non-uniform invariant measures on the unit interval $\boldsymbol{I} = [0,1]$. All of the rational ergodic mappings of $\boldsymbol{I}$ with explicit non-uniform invariant densities can be obtained by addition theorems of elliptic functions. It is shown here that the class of the rational ergodic mappings $\boldsymbol{I} \to \boldsymbol{I}$ are essentially same as the permutable rational functions obtained by J. F. Ritt.

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1 Introduction

Qualitative characterizing discrete-time dynamical systems have been recently gaining an attention. In continuous-time dynamical systems given by ordinary differential equations (ODE), the notion of integrability can serve as the sharp characteristic for general ODEs such that integrable ODEs are solvable in the sense that exact solutions are analytically obtained and non-integrable ODEs such as the three-body problems are not solvable and they show, in general, chaotic behavior. In contrast with ODE cases, the notion of integrability itself is still vague in discrete-time systems, that is, it is not so sharp as the case of ODEs. Quite recently, Hietarinta and Viallet show examples of discrete-time dynamical systems which pass the singularity confinement test (discrete-time version of Painléve test), but which nevertheless show chaotic behavior [5]. Thus, an issue is now focused on searching a characteristic distinguishing chaotic discrete-time systems from regular discrete-time dynamical systems (which have been thought as *integrable* systems. The purpose of the present paper is to present a class of discrete-time dynamical systems which have exact solutions, but which nevertheless are shown to be ergodic (thus, chaotic). Such dynamical systems (we call *exactly solvable chaos*) can be constructed in a systematic way, by utilizing addition theorems of elliptic functions. The main claim here is that solvable discrete-time dynamical systems with chaotic properties do not belong to exceptional classes but they are as *ubiquitous as multiplication formulas of elliptic functions*, which have interesting implications to the issue.

2 Construction

Let us consider an elliptic function $s(x)$ whose inverse function is defined by the following elliptic integral:

$$s^{-1}(x) = \int_0^x \frac{du}{\sqrt{a_0 + a_2u^2 + a_4u^4 + a_6u^6}} = \int_0^{x^2} \frac{dv}{2\sqrt{v(a_0 + a_2v + a_4v^2 + a_6v^3)}}, \tag{1}$$

where $0 \le x \le 1$ and $a_0, \cdots a_4$ are real-valued constants, which satisfy the following condition:

$$\begin{aligned} &a_0 + a_2u^2 + a_4u^4 + a_6u^6 > 0 \quad \text{for} \quad 0 < u < 1 \\ &a_0 + a_2 + a_4 + a_6 = 0, a_0 > 0. \end{aligned} \tag{2}$$

This elliptic function can be also considered as the solution $q(t)$ of a Hamiltonian system with a Hamiltonian

$$H = \frac{1}{2}p^2 + V(q) = 0, \tag{3}$$

where the potential function $V(q)$ is an even polynomial function of the form $V(q) = -(a_0 + a_2q^2 + a_4q^4 + a_6q^6)$ satisfying the conditions

$$V(0) = 0, \quad V(1) = 0, \quad V(q) < 0 \quad \text{for} \quad 0 < q < 1. \tag{4}$$

Thus on this condition about $a_0, \cdots, a_6$ in Eq. (2), this elliptic function $s(x)$ has a real period $4K$, where K is defined by the elliptic integral

$$K = \int_0^1 \frac{du}{\sqrt{a_0 + a_2u^2 + a_4u^4 + a_6u^6}}. \tag{5}$$

We have the relations

$$0 \leq s(x) \leq 1 \quad \text{for} \quad 0 \leq x \leq K, \tag{6}$$

and

$$s(0) = 0, \quad s(K) = 1, \quad s(2K) = 0, \quad s(3K) = -1, \quad s(4K) = 0. \tag{7}$$

Since the equality

$$s^{-1}(-x) = \int_0^{-x} \frac{du}{\sqrt{a_0 + a_2u^2 + a^4u^4 + a^6u^6}} = -s^{-1}(x) \equiv y \tag{8}$$

holds, then we have

$$s(y) = -x, \quad s(-y) = x \tag{9}$$

which means that $s^2(y)$ is an *even function* as $s^2(y) = s^2(-y)$ and has a real period $2K$. Furthermore, since any elliptic function has an *algebraic addition theorem* [1], there exists a polynomial function F in three variables such that the relation

$$F(s^2(x_1 + x_2), s^2(x_1), s^2(x_2)) = 0 \tag{10}$$

holds. As a special case of the addition theorems, there exists a polynomial function G in two variables such that

$$G(s^2(px), s^2(x)) = 0, \tag{11}$$

where p is a positive integer greater than the unity. This means that we have an algebraic mapping as

$$s^2(px) = f(s^2(x)), \tag{12}$$

where f is an algebraic function. If we set

$$X_{n+1} \equiv s^2(px), \quad X_n \equiv s^2(x), \tag{13}$$

we have a discrete-time dynamical system

$$X_{n+1} = f(X_n). \tag{14}$$

on the unit interval $\boldsymbol{I} = [0,1]$. It is easy to check that there exist p pre-images

$$0 < x_m = s^2[\frac{2mK + s^{-1}(\sqrt{y})}{p}] < 1, \quad 0 \le m \le p-1 \tag{15}$$

satisfying $y = f(x_m)$ for $0 < y < 1$. This dynamical system $X_{n+1} = f(X_n)$ has the following remarkable property:

Theorem 1 *A dynamical system $X_{n+1} = f(X_n)$ defined as the addition theorem (12) of an elliptic function $s^2(x)$ is ergodic with respect to an invariant measure $\mu(dx)$ which is absolutely continuous with respect to the Lebesgue measure and their density function $\rho(x)$ is given by the formula*

$$\rho(x) = \frac{1}{2K\sqrt{x(a_0 + a_2x + a_4x^2 + a_6x^3)}}. \tag{16}$$

(Proof of Theorem 1)

Let us consider the diffeomorphisms of $\boldsymbol{I} = [0,1]$ into itself given by

$$0 \le \phi(x) = \frac{1}{K}s^{-1}(\sqrt{x}) \le 1 \quad \text{for} \quad 0 \le x \le 1. \tag{17}$$

Using the relations

$$\begin{aligned}
&s^2(K\cdot p\theta) = f[s^2(K\theta)] \quad \text{for} \quad \theta \in [0, \tfrac{1}{p}],\\
&s^2[K(2-p\theta)] = f[s^2(K\theta)] \quad \text{for} \quad \theta \in [\tfrac{1}{p}, \tfrac{2}{p}],\\
&\cdots,\\
&s^2[K(-(-1)^i i + \tfrac{1-(-1)^i}{2} + (-1)^i p\theta)] = f[s^2(K\theta)] \quad \text{for} \quad \theta \in [\tfrac{i}{p}, \tfrac{i+1}{p}],\\
&\cdots,\\
&s^2[K((-1)^p(p-1) + \tfrac{1+(-1)^p}{2} + (-1)^{p-1}p\theta)] = f[s^2(K\theta)] \quad \text{for} \quad \theta \in [\tfrac{p-1}{p}, 1]
\end{aligned} \tag{18}$$

by the fact that $s^2(x)$ is an even function and it has a real period $2K$, we can derive the piecewise-linear map $\tilde{f}(x) = \phi\circ f\circ\phi^{-1}(x)$ on $\boldsymbol{I} = [0,1]$ as

$$\begin{aligned}
&\tilde{f}(x) = px \quad \text{for} \quad x \in [0, \tfrac{1}{p}]\\
&\cdots,\\
&\tilde{f}(x) = -(-1)^i i + \tfrac{1-(-1)^i}{2} + (-1)^i px \quad x \in [\tfrac{i}{p}, \tfrac{i+1}{p}]\\
&\cdots,\\
&\tilde{f}(x) = -(-1)^{p-1}(p-1) + \tfrac{1-(-1)^{p-1}}{2} + (-1)^{p-1}px \quad x \in [\tfrac{p-1}{p}, 1]
\end{aligned} \tag{19}$$

.

It is noted here that when $p=2$, $\tilde{f}(x)$ is the *tent* map, given by

$$\begin{aligned}
&\tilde{f}(x) = 2x \quad x \in [0, \tfrac{1}{2}]\\
&\tilde{f}(x) = 2-2x \quad x \in [\tfrac{1}{2}, 1].
\end{aligned} \tag{20}$$

Clearly, the map $\tilde{f}(x)$ in Eq. (19) is ergodic with respect to the Lebesgue measure on $\boldsymbol{I} = [0, 1]$. Thus, f is also ergodic with respect to the measure

$$\mu(dx) = \frac{d\phi(x)}{dx}dx = \frac{1}{2K}\frac{dx}{\sqrt{x(a_0 + a_2 x + a_4 x^2 + a_6 x^3)}}. \tag{21}$$

The measure $\mu(dx)$ is absolutely continuous with respect to the Lebesgue measure, by which we can define the algebraic density function $\rho(x)$ in Eq.(16). **(End of proof)**

A simple corollary of the theorem is the following: According to the Pesin identity which can be applied to the dynamical systems with absolutely continuous measure[7], the Lyapunov characteristic exponent $\Lambda(\mu)$ for the measure of the map $X_{n+1} = f(X_n)$ is equivalent to the Kolmogorov-Sinai entropy $h(\mu) = \log p$, namely:

$$\Lambda(\mu) = \int_0^1 \log|\frac{df}{dx}| \cdot \rho(x)dx = h(\mu) = \log p. \tag{22}$$

In this sense, discrete-time dynamical systems $X_{n+1} = f(X_n)$ constitute a typical class of *chaotic* dynamical systems with a special property that their invariant density functions are explicitly given. Hence, we call this class of chaotic dynamical systems *exactly solvable chaos* [11]. We note here that dynamical systems of exactly solvable chaos can be *good* pseudo random-number generators[13], especially for Monte Carlo simulations from the exact ergodicity

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{1}{N} Q(x_n) = \int_0^1 Q(x)\rho(x)dx. \tag{23}$$

We remark here that we can generate infinitely concrete chaotic dynamical systems with the unique invariant measure (16) from addition theorems $s^2(px) = f[s^2(x)]$ of elliptic functions $s(x)$ for $p = 2, 3, \cdots$.

3 Examples of exactly solvable chaos

Historically, the most simple example of exactly solvable chaos is the logistic map $Y = 4X(1-X) \equiv f_0(X)$ on $\boldsymbol{I} = [0, 1]$ given by Ulam and von Neumann in the late 1940's[10].

Ulam and von Neumann show that $X_{n+1} = 4X_n(1 - X_n)$ is ergodic with respect to an invariant probability measure $\mu(dx) = \frac{dx}{\pi\sqrt{x(1-x)}}$. The Ulam-von Neumann map can be seen as a special case of exactly solvable chaos of the above theorem 1 when we consider $s(x)$ defined by

$$s^{-1}(x) = \sin^{-1}(x) = \int_0^x \frac{du}{\sqrt{1-u^2}}. \tag{24}$$

This corresponds to the case that

$$a_0 = 1, \quad a_2 = -1, \quad a_4 = a_6 = 0 \tag{25}$$

in Eq. (1). Thus, we have the relation $s(x) = \sin(x)$ which admits the duplication formula $\sin^2(2x) = 4\sin^2(x)(1 - \sin^2(x)) \equiv f(\sin^2(x))$ ($p = 2$ in Eq.(12)) gives the logistic map. Clearly, in this case, $K = \frac{\pi}{2}$. Thus, using Theorem 1, we have calculate the density function $\rho(x)$ as

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{26}$$

When $p = 3$, we have *the cubic map* as $X_{n+1} = X(3 - 4X)^2$ from the triplication formula.

$$\sin^2(3x) = f[\sin^2(x)] = \sin^2(x)[3 - 4\sin^2(x)]^2 \tag{27}$$

The cubic map can be regarded as a special class of Chebyshev maps obtained by using more general addition formulas as $\sin^2(px) = f(\sin^2(px))$[3].

The alternative attempts to generalize the Ulam-von Neumann map within a set of rational functions was made by Katsura and Fukuda in 1985[6]. The Katsura-Fukuda map is given by

$$Y = \frac{4X(1-X)(1-lX)}{(1-lX^2)^2} \equiv f_l(X) \tag{28}$$

for $0 \le l < 1$. Clearly, the Ulam-von Neumann map can be regarded as a special case of Katsura-Fukuda maps. In this case, the corresponding elliptic function $s(x)$ is Jacobi sn function whose inverse function is defined by

$$s^{-1}(x) = sn^{-1}(x; \sqrt{l}) = \int_0^x \frac{du}{\sqrt{(1-u^2)(1-lu^2)}}, \tag{29}$$

where $\sqrt{l}$ corresponds to the modulus of Jacobi elliptic functions. This corresponds to the case that

$$a_0 = 1, \quad a_2 = -l + 1, \quad , a_4 = l, \quad a_6 = 0 \tag{30}$$

in Eq. (1). Katsura and Fukuda show [6] that the map (28) has exact solutions $X_n = sn^2(2^n\theta; \sqrt{l})$. Using the idea of theorem 1, the author shows [11] that the Katsura-Fukuda maps (28) are also ergodic with respect to an invariant measures which can be written explicitly as

$$\mu(dx) = \rho(x)dx = \frac{dx}{2K(l)\sqrt{x(1-x)(1-lx)}}, \tag{31}$$

where $K(l)$ is the elliptic integral of the first kind given by $K(l) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-lu^2)}}$. This can be easily checked using the duplication formula[14] of the Jacobi sn elliptic function

$$sn(2u;\sqrt{l}) = \frac{2sn(u;\sqrt{l})\sqrt{(1-sn^2(u;\sqrt{l}))(1-lsn^2(u;\sqrt{l}))}}{(1-lsn^4(u;\sqrt{l}))}. \tag{32}$$

Recently, the more general classes of exactly solvable chaos are derived from elliptic functions $s(x)$ whose inverse functions are defined by

$$s^{-1}(x) = \int_0^x \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}}, \tag{33}$$

where the parameters l and m are arbitrary real numbers satisfying the condition $-\infty < m \leq l < 1$ [11]. This corresponds to the case that

$$a_0 = 1, \quad a_2 = -(l+m+1), \quad a_4 = lm + l + m, \quad a_6 = -lm \tag{34}$$

in Eq. (1). The associated mapping generated by the duplication theorem $s^2(2x) = f^{(2)}_{l,m}(s^2(x))$ is given by the following rational transformations

$$f^{(2)}_{l,m}(X) = \frac{4X(1-X)(1-lX)(1-mX)}{1+AX^2+BX^3+CX^4} \in I, \tag{35}$$

where $A = -2(l+m+lm), B = 8lm, C = l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2$, and $X \in I$[11]. Clearly, this mapping is a generalized version of Ulam-von Neumann map and Katsuda-Fukuda map. This two-parameter family of the dynamical systems $x_{n+1} = f^{(2)}_{l,m}(x_n)$ (35) are also ergodic with respect to an invariant measures given by

$$\mu(dx) = \rho(x)dx = \frac{dx}{2K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}, \tag{36}$$

where K is given by the integrals

$$K(l,m) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}}. \tag{37}$$

We can check this fact by directly computing the duplication formula of $s(x)$ whose inverse function is defined in Eq. (33). Let us represent $s(x)$ in terms of the Weierstrass elliptic functions[2].

The Weierstrass elliptic function $\wp(u)$ of $u \in \boldsymbol{C}$ is defined by

$$\wp(u) = \frac{1}{u^2} + {\sum_{j,k}}' \left\{ \frac{1}{(u-2j\omega_1-2k\omega_2)^2} - \frac{1}{(2j\omega_1+2k\omega_2)^2} \right\}, \tag{38}$$

where the symbol $\sum'$ means that the summation is made over all combinations of integers j and k, except for the combination $j = k = 0$, and $2\omega_1$ and $2\omega_2$ are periods of the function $\wp(u)$[14]. The Weierstrass elliptic function $\wp(u)$ satisfies the differential equation

$$(\frac{d\wp(x)}{dx})^2 = 4\wp^3(x) - g_2\wp(x) - g_3, \tag{39}$$

with the invariants

$$g_2(\omega_1, \omega_2) = 60\sum_{j,k}{}' \frac{1}{(j\omega_1 + k\omega_2)^4} \quad \text{and} \quad g_3(\omega_1, \omega_2) = 140\sum_{j,k}{}' \frac{1}{(j\omega_1 + k\omega_2)^6}. \tag{40}$$

Let e_1, e_2 and e_3 be the roots of $4z^3 - g_2 z - g_3 = 0$; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{41}$$

The *discriminant* Δ of the function $\wp(u)$ is defined by $\Delta = g_2^3 - 27g_3^2$. If $\Delta > 0$, all roots e_1, e_2 and e_3 of the equation $4z^3 - g_2 z - g_3 = 0$ are *real*. Thus, we can assume that $e_1 > e_2 > e_3$. In this case, ω_1 and ω_2 are given by the formula

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad \omega_2 = i\int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2 z - 4z^3}}. \tag{42}$$

Using a rational transformation of a variable as $v = -\frac{(1-l)(1-m)}{y - \frac{2l+2m-3lm-1}{3}} + 1$, we can rewrite $s^{-1}(x)$ as

$$s^{-1}(x) = \int_{\frac{2-l-m}{3}}^{\frac{2l+2m-3lm-1}{3} + \frac{(1-l)(1-m)}{1-x^2}} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}}, \tag{43}$$

where $g_2 = \frac{4(1-l+l^2-m+m^2-lm)}{3}, g_3 = \frac{4(2-l-m)(2l-m-1)(2m-l-1)}{27}$. We note here that $4y^3 - g_2 y - g_3$ can be factored as

$$4y^3 - g_2 y - g_3 = 4(y - \frac{2-l-m}{3})(y - \frac{2l-m-1}{3})(y - \frac{2m-l-1}{3}). \tag{44}$$

Thus, we set $e_1 = \frac{2-l-m}{3} > e_2 = \frac{2l-m-1}{3} > e_3 = \frac{2m-l-1}{3}$. Thus, $s(x)$ can be written explicitly in terms of the Weierstrass elliptic function as

$$s^2(x) = 1 - \frac{(1-l)(1-m)}{\wp(\omega_1 - x) - \frac{2l+2m-3lm-1}{3}} = \frac{sn^2(\sqrt{1-m}x; \sqrt{\frac{l-m}{1-m}})}{1 - m + msn^2(\sqrt{1-m}x; \sqrt{\frac{l-m}{1-m}})}. \tag{45}$$

The function $s^2(x)$ also has the same periods $2\omega_1$ and $2\omega_2$ of the Weierstrass elliptic function $\wp(x)$ and the real period $2\omega_1$ is given by the formula

$$2K(l,m) = 2\omega_1 = \frac{2K(\frac{l-m}{1-m})}{\sqrt{1-m}}. \tag{46}$$

Using the addition formula of Jacobi *sn* function, we finally obtain the *explicit* duplication formula of $s(x)$ as

$$s^2(2x) = \frac{4s^2(x)(1-s^2(x))(1-ls^2(x))(1-ms^2(x))}{1+As^4(x)+Bs^6(x)+Cs^8(x)}, \tag{47}$$

where $A = -2(l+m+lm), B = 8lm$ and $C = l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2$. Thus, we obtain the two-parameter family of generalized Ulam-von Neumann maps $X_{n+1} = f_{l,m}^{(2)}(X_n)(35)$.
In the same way, we can construct *generalized cubic maps* $f_{l,m}^{(3)}$ from the triplication formula $s^2(3x) = f_{l,m}^{(3)}(s^2(x))$ as

$$Y = f_{l,m}^{(3)}(X) = \frac{X(-3+4X+\sum_{i=1}^{4} A_i X^i)^2}{1+\sum_{i=2}^{9} B_i X^i}, \tag{48}$$

where $A_1, \cdots, A_4$ and $B_2, \cdots, B_9$ are given[11] by

$$\begin{aligned}
&A_1 = 4(l+m), A_2 = -6(l+m+lm), A_3 = 12lm,\\
&A_4 = l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2,\\
&B_2 = -12(l+m+lm),\\
&B_3 = 8(l+m+l^2+m^2+l^2m+lm^2+15lm),\\
&B_4 = 6(5l^2+5m^2-26lm-26l^2m-26lm^2+5l^2m^2),\\
&B_5 = 24(-2l^2-2m^2-2l^3-2m^3+4lm+7l^2m+7lm^2)\\
&+24(4l^3m+4lm^3+7l^2m^2-2l^3m^2-2l^2m^3),\\
&B_6 = 4(4l^2+4m^2+4l^4+4m^4+17l^3+17m^3-8lm)\\
&+4(-17l^2m-17lm^2-17l^3m-17lm^3-8l^4m-8lm^4)\\
&+4(4l^2m^4+4l^4m^2-17l^3m^2-17l^2m^3+17l^3m^3-54l^2m^2),\\
&B_7 = 24(-l^3-m^3-l^4-m^4+l^2m+lm^2-l^3m-lm^3)\\
&+24(l^4m+lm^4+4l^2m^2+4l^3m^2+4l^2m^3)\\
&+24(l^4m^2+l^2m^4-l^3m^3-l^4m^3-l^3m^4),\\
&B_8 = 3(3l^4+3m^4+4l^3m+4lm^3+4l^4m+4lm^4-14l^2m^2)\\
&+3(-4l^3m^2-4l^2m^3-4l^3m^3-14l^4m^2-14l^2m^4+4l^4m^3+4l^3m^4+3l^4m^4),\\
&B_9 = 8(-l^4m-lm^4+l^3m^2+l^2m^3+l^4m^2+l^2m^4-2l^3m^3+l^4m^3+l^3m^4-l^4m^4).
\end{aligned}$$

The generalized cubic map $f_{l,m}^{(3)}$ has the same invariant measure (36) as the one of the generalized Ulam-von Neumann map. Furthermore, since

$$\begin{aligned}
s^2(p_1p_2x) &= f_{l,m}^{(p_1)}(s^2(p_2x)) = f_{l,m}^{(p_1)} \circ f_{l,m}^{(p_2)}(s^2(x))\\
&= f_{l,m}^{(p_2)}(s^2(p_1x)) = f_{l,m}^{(p_2)} \circ f_{l,m}^{(p_1)}(s^2(x)) = f_{l,m}^{(p_1p_2)}(s^2(x)),
\end{aligned}$$

we have the following commutative relations

$$f_{l,m}^{(p_1p_2)}(X) = f_{l,m}^{(p_1)} \circ f_{l,m}^{(p_2)}(X) = f_{l,m}^{(p_2)} \circ f_{l,m}^{(p_1)}(X), \tag{49}$$

where p_1 and p_2 are positive integers and $f_{l,m}^{(1)}(X) \equiv X$. Thus, from the functional relations (49), all generalized models of exactly solvable chaos given by $Y = f_{l,m}^{(k)}(X)$ for $k \in \boldsymbol{Z}^+$ can be constructed by generalized models of exactly solvable chaos given by $Y = f_{l,m}^{(p)}(X)$ with p being primes. For examples, we can compute $Y = f_{l,m}^{(k)}(X)$ as follows:

$$\begin{aligned} f_{l,m}^{(4)}(X) &= f_{l,m}^{(2)} \circ f_{l,m}^{(2)}(X) \\ f_{l,m}^{(6)}(X) = f_{l,m}^{(2)} \circ f_{l,m}^{(3)}(X) &= f_{l,m}^{(3)} \circ f_{l,m}^{(2)}(X) \\ f_{l,m}^{(8)}(X) &= f_{l,m}^{(2)} \circ f_{l,m}^{(2)} \circ f_{l,m}^{(2)}(X) \\ f_{l,m}^{(9)}(X) &= f_{l,m}^{(3)} \circ f_{l,m}^{(3)}(X) \\ f_{l,m}^{(p_1p_2\cdots p_k)}(X) = f_{l,m}^{(p_1)} \circ f_{l,m}^{(p_2)} &\circ \cdots f_{l,m}^{(p_k)}(X). \end{aligned} \tag{50}$$

All of the above examples of exactly solvable chaos are *rational mappings* on the unit interval $\boldsymbol{I}$ and they can be summarized in the following table.

Explicit Invariant Densities of Rational Ergodic Mappings	
Rational Mappings	**Algebraic Densities**
Ulam-von Neumann map	$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$[10]
Chebyshev maps	$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$[3]
Katsura-Fukuda map	$\rho(x) = \frac{1}{K(l)\sqrt{x(1-x)(1-lx)}}$[11]
Generalized Ulam-von Neumann maps	$\rho(x) = \frac{1}{K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}$[11]
Generalized Chebyshev maps	$\rho(x) = \frac{1}{K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}$ [11]

4 Ritt and Weierstrass's theorems

The commutativity

$$f_p \circ f_q(x) = f_q \circ f_p(x) = f_{pq}(x), \tag{51}$$

for rational ergodic mappings $\{f_p\}$ comes from the fact that their mappings are derived from the multiplication formulas of a single elliptic function $s(x)$ as $s(pu) = f_p[s(u)]$. Thus, they have exact solutions of the form $X_n = s(p^n\theta)$.

Thus, the commutativity is the key for one-dimensional ergodic dynamical systems to have exact solutions.

With regard to the class of rational permutable functions in this way, Ritt showed in 1923[9] that rational functions constructed from the multiplication formulas of elliptic functions such as the above examples $\{f_p\}$ are *all* of rational commuting self-maps of the projected line except trivial cases. The trivial cases indicate the cases that rational permutable maps corresponds to the multiplication formulas of rational functions of e^z, $\cos(z)$. Consequently, permutable rational maps which are shown to be ergodic in the previous sections are essentially *all* of rational permutable ergodic mappings with exact solutions of the form $X_n = s(p^m\theta)$, with $s(x)$ being a meromorphic periodic function. Furthermore, if we try to construct ergodic dynamical systems with exact solutions without using addition theorems of elliptic functions, we must avoid some lack of analyticity on a map f due to the Weierstrass theorem on functions possessing an algebraic addition theorem as follows. The Weierstrass theorem says that any meromorphic function $s(u)$ possessing an algebraic addition theorem is either an elliptic function or is of the form $R(u)$ or $R(e^{\lambda u})$, where R is a rational function [8]. Thus, all of meromorphic and periodic functions $s(u)$ possessing an algebraic addition theorem belong to the class of elliptic functions. This is the reason why all of our ergodic rational transformations constructed here does not have a density function in a form $\rho(x) = \frac{1}{\sqrt{b_0+b_1x+b_2x^2+\cdots+b_mx^m}}$ with $m \geq 5$ whose integral defines the inverse function of a hyperelliptic function with genus $g(\geq 2)$, but has a density function in a form $\rho(x) = \frac{1}{\sqrt{b_0+b_1x+b_2x^2+b_3x^3+b_4x^4}}$, whose integral defines the inverse function of an elliptic function with genus $g(=1)$.

Of course, not all of ergodic transformations on $\boldsymbol{I}$ are in this category as follows:

Example 1: Gauss Map (1845)
The Gauss map $G : x \to 1/x - [1/x]$ which maps $\boldsymbol{I} - \{0\}$ onto $\boldsymbol{I}$ is known to be ergodic with respect to the absolutely continuous invariant measure $\mu(dx) = \frac{dx}{\ln 2(1+x)}$. However, the transformation itself is not a rational function.

Example 2: Bool Transformation (1857)
The transformation of Bool $B : x \to x - \frac{1}{x}$ which maps $\boldsymbol{R} - \{0\}$ onto $\boldsymbol{R}$ is known to be ergodic [4]. The transformation preserves infinite measures. However, the exact solutions X_n do not seem to exist.

Example 3:
The two onto mappings $S_2 : x \to 2x \mod 1$ and $S_3 : x \to 3x \mod 1$ are mutually permutable in the sense that $S_3 \circ S_2(x) = S_2 \circ S_3(x)$. They are ergodic with respect to the Lebesgue measure on $\boldsymbol{I}$. However, the transformations are not rational.

Example 4:

The two transformations $F_2 : x \to \frac{1}{2}(x - 1/x)$ and $F_3 : x \to \frac{x(x^2-3)}{3x^2-1}$ of the real line $\boldsymbol{R}$ are mutually permutable in the sense that $F_3 \circ F_2(x) = F_2 \circ F_3(x)$. They are ergodic with respect to a unique invariant measure $\mu(dx) = \frac{dx}{\pi(1+x^2)}$ (Cauchy distribution) which is absolutely continuous with respect to Lebesgue measure [12]. These properties comes from the fact that the transformations are equivalent to the multiplication formula of $\cot(x)$. Thus, these mappings have exact solutions $X_n = \cot(2^n\theta)$ and $X_n = \cot(3^n\theta)$. Hence, maps of example 4, although they are defined on the infinite support, can be considered to belong to the present category of exactly solvable chaos.

5 Classification problem of exactly solvable chaos

It is known that any elliptic function $s(x)$ can be expressed in terms of Weierstrassian elliptic functions $\wp(x)$ and $\wp'(x)$ with the same periods, the expression being rational in $\wp(x)$ and linear in $\wp'(x)$[14]. The invariants $g_2(\omega_1, \omega_2)$ and $g_3(\omega_1, \omega_2)$ of Weierstrassian elliptic functions $\wp(x)$ are not changed under a transformation

$$\omega_1' = a\omega_1 + b\omega_2, \quad \omega_2' = c\omega_1 + d\omega_2, \tag{52}$$

where a, b, c and d are integers satisfying $ad - bc = 1$. An elliptic modular function J of $\wp(x)$ is given by

$$J(\tau) = \frac{g_2^3}{g_2^3 - 27g_3^2}, \tag{53}$$

where $\tau = \frac{\omega_2}{\omega_1}$. Since the elliptic modular function $J(\tau)$ in Eq.(53) is invariant under a certain condition; namely,

$$J(\frac{c + d\tau}{a + b\tau}) = J(\tau), \quad ad - bc = 1, \tag{54}$$

where a, b, c and d are integers, $J(\tau)$ of an elliptic function $s(x)$ can serve as a characteristic of exactly solvable chaos. For an example, let us consider the classification problem of the two-parameter family of the generalized Ulam-von Neumann maps $X_{n+1} = f_{l,m}^{(2)}(X_n)$. In this case, it is easy to check that

$$J(l, m) = \frac{4[(1-m)(1-l) + (l-m)^2]^3}{27(1-l)^2(1-m)^2(l-m)^2}. \tag{55}$$

Thus, the equality $J(l, m) = J(l', m')$ gives us the solutions of the classification problem as follows:

$$\lambda = \lambda', \lambda = 1 - \lambda', \lambda = \frac{1}{1-\lambda'}, \lambda = \frac{1}{\lambda'}, \lambda = \frac{\lambda'}{\lambda' - 1}, \lambda = \frac{\lambda' - 1}{\lambda'}, \tag{56}$$

where $\lambda \equiv \frac{l-m}{1-m}$ and $\lambda' \equiv \frac{l'-m'}{1-m'}$. This means that when one of the conditions in Eq.(56) is satisfied, dynamical systems $X_{n+1} = f^{(2)}_{l,m}(X_n)$ and $Y_{n+1} = f^{(2)}_{l',m'}(Y_n)$ have an algebraic relation such that the relation $H[f^{(2)}_{l,m}(X), f^{(2)}_{l',m'}(X)] = 0$ holds, where H is a polynomial function in two variables. In this way, we can solve the classification problem of exactly solvable chaos $X_{n+1} = f(X_n)$.

6 Summary and Discussions

Rational ergodic maps on the unit interval are systematically constructed from addition theorems of elliptic functions in a unified manner. According to the classical Weierstrass's theorem about meromorphic functions possessing algebraic addition theorems and Ritt's theorem about permutable rational functions, such rational mappings due to addition theorems of elliptic functions are essentially *all* rational ergodic transformations that have exact solutions as well as exact density functions. Therefore, in discrete-time systems, solvable models showing chaotic behavior do not belong to an exceptional class but they are ubiquitous as the presence of multiplication theorems of elliptic functions. This implies the gap between the notion of solvability and integrability in discrete-time dynamical systems as follows: The notion of solvability in discrete-time systems can be compatible with chaotic behavior which are commonly believed to indicate non-integrability of the systems, whereas the notion of the solvability is essentially same as that of the integrability in continuous-time systems (ODE). Thus, further studies of asking what is an essential characteristic distinguishing chaotic discrete-time systems from regular discrete-time systems are clearly needed for the settlement of the issue.

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[1] We say that a function $w(u)$ has an *algebraic addition theorem* if there exists a polynomial function G in three variables such that the relation $G(w(u_1+u_2), w(u_1), w(u_2)) = 0$ holds for an arbitrary set of the variables u_1 and u_2.

[2] Any elliptic function $w(u)$ has an algebraic relation with the Weierstrass elliptic function $\wp(u)$ with the same periods as $G(w, \wp) = 0$, G being a rational function in two variables.

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