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Some New Trends and Questions in White Noise Calculus

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Introduction

During the last decade the white noise calculus, launched out by T. Hida [8] in 1975, has developed considerably keeping much contact with various research fields and applications, see e.g., [10], [18], [22]. This short note is concerned with the recent achievements of operator theory on white noise spaces, in particular, from the viewpoints of quantum stochastic analysis and harmonic analysis on Gaussian space. Some open questions are also mentioned throughout the paper.

In Section 1 we give a quick survey of white noise distribution theory based on the recent work of Cochran, Kuo and Sengupta [6]. In Section 2 we report recent results [5] on white noise operators and their regularity property. We believe that our discussion can be improved according to the recent works by Asai, Kubo and Kuo [1], [2], where a new aspect is given to a method of constructing the space of white noise functions. Section 3 contains some results on differential equations for white noise operators in normal-ordered form. This class of differential equations includes quantum stochastic differential equations of Hudson–Parthasarathy type and those having singular coefficients such as higher powers of quantum white noises. In Section 4, we introduce a CKS-space over the complex Gaussian space and prove resolution of the identity via coherent states. Then, we discuss decompositions of a white noise function and of a white noise operator in terms of coherent states. These are infinite dimensional extensions of the well known results in quantum mechanics [7], [12], [30].

1 White Noise Distribution Theory

1.1 Weighted Fock Space

Let H be a Hilbert space and denote by $H^{\widehat{\otimes} n}$ the n-fold symmetric tensor power of H. For a sequence $\alpha=\{\alpha(n)\}_{n=0}^{\infty}$ of positive numbers we put

$$\Gamma_{\alpha}(H)=\left\{\phi=(f_n)_{n=0}^{\infty}\,;\, f_n\in H^{\widehat{\otimes} n},\ \|\phi\|^2\equiv\sum_{n=0}^{\infty} n!\,\alpha(n)\,|f_n|^2<\infty\right\},$$

where $|\cdot|$ stands for the norm of $H^{\widehat{\otimes} n}$. In an obvious manner $\Gamma_{\alpha}(H)$ becomes a Hilbert space and is called a *weighted Fock space.* The (Boson) Fock space is by definition the special case of $\alpha(n)\equiv 1$ and is denoted by $\Gamma(H)$.

For two sequences $\alpha = \{\alpha(n)\}$ and $\beta = \{\beta(n)\}$ we write $\beta \prec \alpha$ if there exists a positive number $C > 0$ such that $\beta(n) \le C\alpha(n)$. In that case we have a natural inclusion map $\Gamma_\alpha(H) \hookrightarrow \Gamma_\beta(H)$ which is continuous and has a dense image. In particular, $\Gamma_\alpha(H) \hookrightarrow \Gamma(H)$ for $1 \prec \alpha$.

1.2 CKS-Space

Although further abstraction is possible, for simplicity we consider the usual Gelfand triple:

$$E \equiv \mathcal{S}(\mathbf{R}) \subset H \equiv L^2(\mathbf{R}, dt) \subset E^* \equiv \mathcal{S}'(\mathbf{R}), \tag{1.1}$$

where $\mathcal{S}(\mathbf{R})$ is the space of rapidly decreasing functions and $\mathcal{S}'(\mathbf{R})$ the dual space, i.e., the space of tempered distributions. It is well known that

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

where $E_{\pm p}$ is the Hilbert space obtained by completing E with respect to the norm

$$|\,\xi\,|_{\pm p} = |\,A^{\pm p}\xi\,|_H, \qquad A = 1 + t^2 - \frac{d^2}{dt^2}.$$

Definition 1.1 A sequence $\alpha = \{\alpha(n)\}_{n=0}^\infty$ of positive numbers is called a *CKS-sequence* if it satisfies the following three conditions:

(A1) $\alpha(0) = 1$ and $1 \prec \alpha$ (set $\gamma_\alpha = \sup_n \alpha^{-1}(n) < \infty$);

(A2) the generating function $G_\alpha(t) = \displaystyle\sum_{n=0}^\infty \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;

(A3) the power series $\widetilde{G}_\alpha(t) = \displaystyle\sum_{n=0}^\infty t^n \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\}$ has a positive radius of convergence.

Given a CKS-sequence $\alpha = \{\alpha(n)\}_{n=0}^\infty$ we put

$$\Gamma_\alpha(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p).$$

It is proved that $\Gamma_\alpha(E)$ is a nuclear space, the topology of which is given by the family of norms:

$$\|\,\phi\,\|_{p,+}^2 = \sum_{n=0}^\infty n!\,\alpha(n)\,|\,f_n\,|_p^2, \qquad \phi = (f_n), \quad p \ge 0.$$

By a standard argument we see that

$$\Gamma_\alpha(E)^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{\alpha^{-1}}(E_{-p}),$$

where $\Gamma_\alpha(E)^*$ carries the strong dual topology and $\cong$ stands for a topological isomorphism.

For simplicity we write $\mathcal{W} = \mathcal{W}_\alpha$ for $\Gamma_\alpha(E)$. Thus we come to a Gelfand triple:

$$\mathcal{W} \subset \Gamma(H_{\mathbf{C}}) \subset \mathcal{W}^*, \tag{1.2}$$

which is referred to as the *Cochran–Kuo–Sengupta space* [6]. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Then

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in \mathcal{W}^*, \quad \phi = (f_n) \in \mathcal{W}, \tag{1.3}$$

and it holds that

$$| \langle\!\langle \Phi, \phi \rangle\!\rangle | \le \| \Phi \|_{-p,-} \| \phi \|_{p,+},$$

where

$$\| \Phi \|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} | F_n |_{-p}^2, \qquad \Phi = (F_n) \in \mathcal{W}^*.$$

The *Hida–Kubo–Takenaka space* [16] denoted by $\mathcal{W} = (E)$, and the *Kondratiev–Streit space* [14] denoted by $\mathcal{W} = (E)_\beta$, $0 \le \beta < 1$, are the CKS-spaces corresponding to $\alpha(n) \equiv 1$ and $\alpha(n) = (n!)^\beta$, respectively.

Another concrete example of a CKS-spaces is constructed by means of the k-th order Bell numbers $\{B_k(n)\}$ defined by

$$G_{\mathrm{Bell}(k)}(t) = \frac{\overbrace{\exp(\exp(\cdots(\exp t)\cdots))}^{k-\text{times}}}{\exp(\exp(\cdots(\exp 0)\cdots))} = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} t^n, \tag{1.4}$$

see [6], [15]. In operator theory the second order Bell numbers play an important role.

1.3 Wiener–Itô–Segal Isomorphism

Let μ be the standard Gaussian measure on E^* and $L^2(E^*, \mu)$ the Hilbert space of $\mathbf{C}$-valued L^2-functions on E^*. The celebrated Wiener-Itô-Segal isomorphism is the unitary isomorphism between $\Gamma(H_{\mathbf{C}})$ and $L^2(E^*, \mu)$ uniquely determined by the correspondence

$$\left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right) \quad \leftrightarrow \quad \phi_\xi(x) = e^{\langle x, \xi\rangle - \langle \xi, \xi \rangle/2}, \qquad \xi \in E_{\mathbf{C}}.$$

The above element is called an *exponential vector.* It is known that $\{\phi_\xi ; \xi \in E_{\mathbf{C}}\}$ spans a dense subspace of $\mathcal{W} = \Gamma_\alpha(E_{\mathbf{C}})$ for any CKS-sequence α. The subspace of $L^2(E^*, \mu)$ corresponding to $\mathcal{W}$ in (1.2) is, by construction, consists of equivalence classes of L^2-functions on E^*. Nevertheless, since each equivalence class contains a unique continuous function on E^*, we may regard $\mathcal{W}$ as a space of continuous functions on E^*. More detailed properties of functions in $\mathcal{W}$ are examined in a similar manner as in [19], [29]. In this context elements of $\mathcal{W}$ and $\mathcal{W}^*$ are called a *white noise test function* and a *white noise distribution*, respectively.

1.4 Norm Estimates of Pointwise Multiplication

That the space of test functions is closed under pointwise multiplication is an important property and the norm estimate of $\phi\psi$ is interesting to study. In case of the Hida–Kubo–Takenaka space we proved in [22, Section 3.5] the following estimate: For any $\alpha, \beta, p \ge 0$ with $\rho^{2\alpha} + \rho^{\alpha+\beta+2p} + \rho^{2\beta} < 1$ it holds that

$$\| \phi\psi \|_p \le \frac{\sqrt{1 - \rho^{\alpha+\beta+2p}}}{1 - \rho^{2\alpha} - \rho^{\alpha+\beta+2p} - \rho^{2\beta}} \| \phi \|_{p+\alpha} \| \psi \|_{p+\beta}, \qquad \phi, \psi \in (E), \tag{1.5}$$

which seems to be the sharpest result up to now. However, it is believed that a further improvement is possible. In fact, in case of $p = 0$ a very sharp result is known in connection with Gaussian hypercontractivity [20] and is not covered by inequality (1.5). Moreover, a similar question for a general CKS-space remains open.

2 White Noise Operator Theory

A general theory for operators in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ has been extensively developed in [4], [22], [23], [27]. Since operators in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ are something like distributions, their regularity properties are of great interest. Here we survey the most recent results.

2.1 Weighted Fock Spaces Interpolating CKS-Space

Consider another Hilbert spaces $K^{\pm}$ which interpolate the Gelfand triple (1.1):

$$E \equiv \mathcal{S}(\mathbf{R}) \subset K^+ \subset H \equiv L^2(\mathbf{R}) \subset K^- \subset \mathcal{S}'(\mathbf{R}) \equiv E^*, \tag{2.1}$$

where $K^{\pm}$ are dual each other in such a way that the canonical bilinear form on $K^- \times K^+$ is compatible to the inner product of H. The norms of $K^{\pm}$ is denoted by $|\cdot|_{\pm}$. Moreover, we assume that the imbedding $K^+ \to H$ is a contraction: $|\xi|_0 \le |\xi|_+$, $\xi \in K^+$. For example, $K^+ = E_p$ has this property for any $p \ge 0$. In that case, the natural inclusion $\Gamma(K^+) \subset \Gamma(H)$ is defined and becomes a contraction again. Then we have the following

Lemma 2.1 *Let α be a CKS-sequence and β a positive sequence such that $1 \prec \beta \prec \alpha$. Then we have continuous inclusions:*

$$\mathcal{W}_\alpha \subset \Gamma_\beta(K_{\mathbf{C}}^+) \subset \Gamma(H_{\mathbf{C}}) \subset \Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-) \subset \mathcal{W}_\alpha^*. \tag{2.2}$$

Moreover, $\Gamma_\beta(K_{\mathbf{C}}^+)$ and $\Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-)$ are dual each other.

For simplicity the norms of $\Gamma_{\beta^{\pm}}(K_{\mathbf{C}}^{\pm})$ are denoted by $\|\cdot\|_{\pm}$, namely,

$$\|\phi\|_{\pm}^2 = \sum_{n=0}^{\infty} n!\, \beta^{\pm 1}(n)\, |f_n|_{\pm}^2, \qquad \phi = (f_n) \in \Gamma_{\beta^{\pm}}(K_{\mathbf{C}}^{\pm}).$$

2.2 Integral Kernel Operators

An integral kernel operator with kernel distribution $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ admits a formal integral expression:

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbf{R}^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \tag{2.3}$$

for the precise definition see [22]. We first recall that for any kernel $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ the integral kernel operator $\Xi_{l,m}(\kappa_{l,m})$ always belongs to $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. It is thus natural to answer regularity questions about such an integral kernel operator by indicating a space $\Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-)$ such that $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-))$, see (2.2).

In addition to (A1)–(A3) we need another condition for the weight sequence $\alpha = \{\alpha(n)\}$:

(A4) there exists a constant $C_\alpha > 0$ such that $\alpha(n)\alpha(m) \le C_\alpha^{n+m}\alpha(n+m)$ for any n, m.

Then we have the following result, for the proof see [5].

Theorem 2.2 *Let α, β be two sequences of positive numbers such that $1 \prec \beta \prec \alpha$. Assume that α satisfies (A1)–(A4) and β satisfies (A4). Then for $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ the following three conditions are equivalent:*

(i) $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-))$;

(ii) $\kappa \in (K_{\mathbf{C}}^-)^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$;

(iii) $\kappa \otimes_m \in \mathcal{L}(E_{\mathbf{C}}^{\otimes m}, (K_{\mathbf{C}}^-)^{\otimes l})$;

Let $\{e_i\} \subset E$ be the orthonormal basis of $H = L^2(\mathbf{R})$ such that $Ae_i = (2i+2)e_i$ for $i = 0, 1, 2, \cdots$ as usual, and let $\{f_i\}$ be an orthonormal basis of $K^\pm$. For simplicity we put

$$f(\mathbf{i}) = f_{i_1} \otimes \cdots \otimes f_{i_l}, \qquad e(\mathbf{j}) = e_{j_1} \otimes \cdots \otimes e_{j_m},$$

and, for $p \in \mathbf{R}$ we define

$$|\,\kappa\,|_{l,m;\pm,p}^2 = \sum_{\mathbf{i},\mathbf{j}} |\,\langle \kappa,\, f(\mathbf{i}) \otimes e(\mathbf{j})\rangle_\pm\,|^2\,|\,e(\mathbf{j})\,|_p^2, \qquad \kappa \in (K_{\mathbf{C}}^\pm)^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*,$$

where $\langle \cdot,\, \cdot \rangle_\pm$ stands for the inner product of $(K_{\mathbf{C}}^\pm)^{\otimes l} \otimes H_{\mathbf{C}}^{\otimes m}$. With these notation conditions (i)–(iii) are also equivalent to

(iv) there exists $p \ge 0$ such that $|\,\kappa\,|_{l,m;-,-p} < \infty$.

The above condition is very useful together with Theorem 2.2 in many practical questions about convergence of operators.

2.3 Fock expansion

One of the most significant features of our approach is that every operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ admits the Fock expansion:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{2.4}$$

where the series converges in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$.

Although the class $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ contains all bounded operators on $\Gamma(H_{\mathbf{C}})$, we do not know how to characterize bounded operators in this context. In this connection we only mention the following

Proposition 2.3 *Let $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$ if and only if $\kappa \in H_{\mathbf{C}}^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$.*

This is obtained by specializing Theorem 2.2 to the case of $K_{\mathbf{C}}^\pm = H_{\mathbf{C}}$ and $\beta = 1$. For a general Ξ as in (2.4) we need control the convergence in terms of the weight sequence α. On the other hand, argument in [21] might be relevant to this question.

2.4 Wick Products and Wick Exponentials

The Wick product (or normal-ordered product) is to be defined for white noise operators as a unique extension of the relations:

$$a_s \diamond a_t = a_s a_t, \quad a_s^* \diamond a_t = a_s^* a_t, \quad a_s \diamond a_t^* = a_t^* a_s, \quad a_s^* \diamond a_t^* = a_s^* a_t^*.$$

In fact, for two integral kernel operators $\Xi_{l_1,m_1}(\kappa)$ and $\Xi_{l_2,m_2}(\lambda)$ the *Wick product* is defined by

$$\Xi_{l_1,m_1}(\kappa) \diamond \Xi_{l_2,m_2}(\lambda) = \Xi_{l_1+l_2,m_1+m_2}(\kappa \circ \lambda),$$

where $\kappa \circ \lambda \in (E_{\mathbf{C}}^{\otimes(l_1+l_2+m_1+m_2)})^*$ is given by

$$\begin{aligned} &\kappa \circ \lambda(s_1, \cdots, s_{l_1+l_2}, t_1, \cdots, t_{m_1+m_2}) \\ &\qquad = \kappa(s_1, \cdots, s_{l_1}, t_1, \cdots, t_{m_1})\lambda(s_{l_1+1}, \cdots, s_{l_1+l_2}, t_{m_1+1}, \cdots, t_{m_1+m_2}). \end{aligned}$$

Let $\mathcal{L}_{\mathrm{ff}}$ denote the space of finite sums of integral kernel operators. We note that $\mathcal{L}_{\mathrm{ff}} \subset \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ for any CKS-sequence α. The Wick product is defined in $\mathcal{L}_{\mathrm{ff}}$ by linearity. Note also that the Wick product is commutative.

With the help of Theorem 2.2 we obtain

Lemma 2.4 *Assume that two sequences α, β satisfy the same conditions as in Theorem 2.2. Let $\Xi \in \mathcal{L}_{\mathrm{ff}}$ be given by*

$$\Xi = \sum_{l,m=0}^{\text{finite}} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*.$$

Then $\Xi \in \mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\beta^{-1}}(K_{\mathbf{C}}^-))$ if and only if $\kappa_{l,m} \in (K_{\mathbf{C}}^-)^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$.

Let $\mathcal{L}_{1\mathrm{f}} \subset \mathcal{L}_{\mathrm{ff}}$ denote the subspace of operators which are of the form:

$$\Xi = \sum_{m=0}^{\text{finite}} \Xi_{0,m}(\kappa_{0,m}) + \sum_{m=0}^{\text{finite}} \Xi_{1,m}(\kappa_{1,m}),$$

i.e., each term of the Fock expansion of Ξ involves at most one creation operators.

In [5] we discussed conditions under which the Wick exponential:

$$\operatorname{wexp} \Xi = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n} \tag{2.5}$$

converges. Here we mention the following two results.

Theorem 2.5 *Let $\Xi \in \mathcal{L}_{1\mathrm{f}}$ be of the form:*

$$\Xi = \sum_{i=1}^{k} \Xi_{l_i,m_i}(\kappa_{l_i,m_i}), \qquad \kappa_{l_i,m_i} \in (K_{\mathbf{C}}^-)^{\otimes l_i} \otimes (E_{\mathbf{C}}^{\otimes m_i})^*. \tag{2.6}$$

Assume that the weight sequence α *satisfies (A1)–(A4). If*

$$\limsup_{n\to\infty} \frac{R_n^{1/n}}{n} = 0, \tag{2.7}$$

where

$$R_n = R_n(m_1, \cdots, m_k) = \max \left\{ \frac{m^m}{\alpha(m)}; \begin{array}{l} r_1 \ge 0, \cdots, r_k \ge 0 \\ r_1 + \cdots + r_k = n \\ m = m_1 r_1 + \cdots + m_k r_k \end{array} \right\}, \tag{2.8}$$

then the Wick exponential (2.5) converges in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(K_{\mathbf{C}}^-))$.

Theorem 2.6 *Let* $\Xi \in \mathcal{L}_{\mathrm{ff}}$ *be of the form:*

$$\Xi = \sum_{i=1}^{k} \Xi_{l_i,m_i}(\kappa_{l_i,m_i}), \qquad \kappa_{l_i,m_i} \in (K_{\mathbf{C}}^-)^{\otimes l_i} \otimes (E_{\mathbf{C}}^{\otimes m_i})^*. \tag{2.9}$$

Assume that the weight sequence α *satisfies (A1)–(A4). If*

$$\limsup_{n\to\infty} \frac{\{R_n(l_1, \cdots, l_k) R_n(m_1, \cdots, m_k)\}^{1/2n}}{n} = 0, \tag{2.10}$$

where $R_n(l_1, \cdots, l_k)$ *and* $R_n(m_1, \cdots, m_k)$ *are defined as in (2.8), then the Wick exponential (2.5) converges in* $\mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\alpha^{-1}}(K_{\mathbf{C}}^-))$.

Simple sufficient conditions for (2.7) and (2.10) are stated in the following

Lemma 2.7 *Let* $\{\alpha(n)\}$ *be a positive sequence. If there exist constant numbers* $C > 1$ *and* $N \ge 1$ *such that*

$$n^n \le (C \log n)^n \alpha(n), \qquad n \ge N, \tag{2.11}$$

then condition (2.7) for any choice of finitely many $m_1, \cdots, m_k \ge 0$ *and condition (2.10) for any choice of finitely many* $l_1, \cdots, l_k, m_1, \cdots, m_k \ge 0$ *are satisfied.*

It is obviously necessary to simplify the above argument and to find sharper conditions. Nevertheless, as is proved in [6], for $k \ge 2$ the k-th order Bell numbers satisfy condition (2.11) and, therefore we have

Theorem 2.8 *Let* $\alpha = \mathrm{Bell}(2)$ *be the second order Bell numbers. Then, for any* $\Xi \in \mathcal{L}_{\mathrm{ff}}$ *the Wick exponential* $\mathrm{wexp}\,\Xi$ *converges in* $\mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\alpha^{-1}}(K_{\mathbf{C}}^-))$. *If in addition* $\Xi \in \mathcal{L}_{\mathrm{1f}}$, *then* $\mathrm{wexp}\,\Xi$ *converges in* $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(K_{\mathbf{C}}^-))$.

3 Normal-Ordered White Noise Equations

3.1 Wick Product in General

In order to formulate our initial value problem we need extend the Wick product for general operators. This is performed by means of the operator symbol. The *operator symbol* of $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbf{C}}.$$

Any operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is uniquely determined by its symbol since $\{\phi_\xi; \xi \in E_{\mathbf{C}}\}$ spans a dense subspace of $\mathcal{W}$. The next result is known as the characterization theorem for operator symbols, see [4], [22].

Theorem 3.1 *Assume that the weight sequence α satisfies (A1)–(A4) and that α is nondecreasing. Then, a function $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} \to \mathbf{C}$ is the operator symbol of a certain operator in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ if and only if*

(O1) *For each $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$, the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is an entire holomorphic function on $\mathbf{C} \times \mathbf{C}$;*

(O2) *There exist constant numbers $K \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi, \eta)|^2 \leq K G_\alpha(|\,\xi\,|_p^2) G_\alpha(|\,\eta\,|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

It follows from Theorem 3.1 that for two operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ there exists a unique $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$\widehat{\Xi}(\xi, \eta) = e^{-\langle \xi, \eta \rangle} \widehat{\Xi}_1(\xi, \eta) \widehat{\Xi}_2(\xi, \eta), \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{3.1}$$

The operator Ξ defined by (3.1) is called the *Wick product* of Ξ_1 and Ξ_2 and is denoted by $\Xi = \Xi_1 \diamond \Xi_2$.

3.2 Unique Existence Theorems

Now we consider an initial value problem for a normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi|_{t=0} = I \quad \text{(identity)}, \tag{3.2}$$

where t runs over an interval containing 0. Since the formal solution is given by

$$\Xi_t = \mathrm{wexp}\left(\int_0^t L_s \, ds \right) = \sum_{n=0}^\infty \frac{1}{n!} \left(\int_0^t L_s \, ds \right)^{\diamond n}, \tag{3.3}$$

our task is to prove the convergence of (3.3) in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, where $\mathcal{W} = \mathcal{W}_\alpha$ is properly chosen according to the coefficient $\{L_t\}$. Here we mention some results obtained in [24], [26]. For $L \in \mathcal{L}_{\mathrm{ff}}$ we put

$$\deg L = \max\{l + m\,;\, \Xi_{l,m}(\kappa_{l,m}) \neq 0\}, \qquad L = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m}).$$

Theorem 3.2 *Consider the initial value problem (3.2). If $\deg L_t \leq 2/(1-\beta)$, $0 \leq \beta < 1$, for all t, then the unique solution lies in $\mathcal{L}((E)_\beta, (E)_\beta^*)$. In particular, if $\deg L_t \leq 2$ for all t, then the unique solution lies in $\mathcal{L}((E), (E)^*)$.*

For a general coefficient $\{L_t\}$ the Bell numbers play a role.

Theorem 3.3 *Consider the initial value problem (3.2). If $t \mapsto L_t \in \mathcal{L}(\mathcal{W}_{\mathrm{Bell}(k)}, \mathcal{W}^*_{\mathrm{Bell}(k)})$ is continuous, the unique solution lies in $\mathcal{L}(\mathcal{W}_{\mathrm{Bell}(k+1)}, \mathcal{W}^*_{\mathrm{Bell}(k+1)})$.*

3.3 Regularity of Solutions

We continue the study of the initial value problem (3.2), where the coefficient $\{L_t\}$ has the following form:

$$L_t = \sum_{l,m=0}^{k} \Xi_{l,m}(\lambda_{l,m}(t)) \in \mathcal{L}_{\mathrm{ff}}. \tag{3.4}$$

Assume that $t \mapsto \lambda_{l,m}(t) \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ is continuous and put

$$\kappa_{l,m}(t) = \int_0^t \lambda_{l,m}(s)\, ds.$$

Note that $\kappa_{l,m}(t) \in (E_{\mathbf{C}}^{\otimes(l_i+m_i)})^*$. With these notations we may state

Theorem 3.4 *Let $\alpha = \mathrm{Bell}(2)$ be the second order Bell numbers. Let $L_t \in \mathcal{L}_{\mathrm{ff}}$ be given as in (3.4) such that $t \mapsto \kappa_{l,m}(t) \in (K_{\mathbf{C}}^-)^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$ is continuous for any l, m. Then the unique solution to (3.2) lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\alpha^{-1}}(K_{\mathbf{C}}^-))$. If in addition $L_t \in \mathcal{L}_{1\mathrm{f}}$, the solution lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(K_{\mathbf{C}}^-))$.*

The above is a direct consequence of Theorem 2.8.

Here are some examples. Let $\alpha = \mathrm{Bell}(2)$ be the second order Bell numbers. A quantum stochastic differential equation of Hudson–Parthasarathy type is equivalent to the initial value problem (3.2) with

$$L_t = L_1 a_t^* a_t + L_2 a_t + L_3 a_t^* + L_4.$$

It then follows from Theorem 3.4 that the solution lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(H_{\mathbf{C}}))$. More generally, the solution to the initial value problem (3.2) with

$$L_t = \sum_{n=0}^{\text{finite}} L_{0n} a_t^n + \sum_{n=0}^{\text{finite}} L_{1n} a_t^* a_t^n,$$

lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(H_{\mathbf{C}}))$, see also [5]. Next consider $\{L_t\}$ involving higher-derivatives of quantum white noises such as

$$L_t = \sum_{n=0}^{\text{finite}} L_{0n} \nabla_t^n a_t + \sum_{n=0}^{\text{finite}} L_{1n} a_t^* \nabla_t^n a_t.$$

In that case the solution to the initial value problem (3.2) lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma(E_{-p}))$ for some $p \geq 0$. Finally, the solution to the initial value problem (3.2) with $\{L_t\}$ involving higher powers of quantum white noises such as

$$L_t = \sum_{l,m=0}^{\text{finite}} L_{l,m} a_t^{*l} a_t^m$$

lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\alpha^{-1}}(E_{-p}))$ for some $p \geq 0$. In particular, the solution to

$$\frac{d\Xi}{dt} = (a_t^2 + a_t^{*2}) \diamond \Xi, \qquad \Xi|_{t=0} = I,$$

lies in $\mathcal{L}(\mathcal{W}_\alpha, \Gamma_{\alpha^{-1}}(E_{-p}))$ for $p > 1/2$.

4 Complex White Noise and Coherent State Representations

4.1 Complex Gaussian Space

Going back to the Gelfand triple $E \subset H \subset E^*$, we introduce the Gaussian measure on E^* with variance 1/2, denoted by $\mu_{1/\sqrt{2}}$, through the characteristic function:

$$e^{-|\xi|_0^2/4} = \int_{E^*} e^{i\langle x, \xi\rangle} \mu_{1/\sqrt{2}}(dx), \qquad \xi \in E.$$

Then, in view of the topological isomorphism $E_{\mathbf{C}}^* \cong E^* \times E^*$, we define a probability measure $\nu = \mu_{1/\sqrt{2}} \times \mu_{1/\sqrt{2}}$ on $E_{\mathbf{C}}^*$ by

$$\nu(dz) = \mu_{1/\sqrt{2}}(dx)\mu_{1/\sqrt{2}}(dy), \qquad z = x + iy \in E_{\mathbf{C}}^*.$$

Following Hida [9, Chapter 6] the probability space $(E_{\mathbf{C}}^*, \nu)$ is called the *complex Gaussian space*. The complex white noise has been discussed from various aspects in [11], [17], [31].

The following formulas are useful:

$$\int_{E^*} e^{\langle x, \xi\rangle} \mu_{1/\sqrt{2}}(dx) = e^{\langle \xi, \xi\rangle/4}, \qquad \xi \in E_{\mathbf{C}}, \tag{4.1}$$

$$\int_{E^*} \langle x,\, \eta\rangle\, e^{\langle x, \xi\rangle} \mu_{1/\sqrt{2}}(dx) = \frac{1}{2}\langle \xi,\, \eta\rangle\, e^{\langle \xi, \xi\rangle/4}, \qquad \xi, \eta \in E_{\mathbf{C}}, \tag{4.2}$$

$$\int_{E_{\mathbf{C}}^*} e^{\langle \overline{z}, \xi\rangle + \langle z, \eta\rangle}\, \nu(dz) = e^{\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}, \tag{4.3}$$

where $\overline{z} = x - iy$ for $z = x + iy \in E^* + iE^*$. Notice that $\langle \cdot,\, \cdot\rangle$ is the canonical $\mathbf{C}$-bilinear form on $E_{\mathbf{C}}^* \times E_{\mathbf{C}}$.

4.2 CKS-Space over Complex Gaussian Space

As before let $\Gamma(H_{\mathbf{C}})$ be the Boson Fock space over $H_{\mathbf{C}}$. Then, through the celebrated Wiener–Itô–Segal isomorphism we have

$$L^2(E^*, \mu_{1/\sqrt{2}}) \cong \Gamma(H_{\mathbf{C}}), \tag{4.4}$$

where the isometric isomorphism is uniquely determined by the correspondence:

$$\psi_\xi(x) \equiv e^{\sqrt{2}\langle x, \xi\rangle - \langle \xi, \xi\rangle/2} \quad \leftrightarrow \quad \phi_\xi \equiv \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right), \qquad \xi \in E_{\mathbf{C}}. \tag{4.5}$$

The above ϕ_ξ is the usual *exponential vector* and is also called a *coherent state* (up to a normalizing fctor). In view of (1.2) we obtain

$$\mathcal{W} \otimes \mathcal{W} \subset \Gamma(H_{\mathbf{C}}) \otimes \Gamma(H_{\mathbf{C}}) \subset (\mathcal{W} \otimes \mathcal{W})^*. \tag{4.6}$$

On the other hand, identifying a function on $E_{\mathbf{C}}^*$ with one on $E^* \times E^*$ in such a way that

$$\phi \otimes \psi(x + iy) = \phi(x)\psi(y), \qquad x, y \in E^*, \quad \phi, \psi \in L^2(E^*, \mu_{1/\sqrt{2}}),$$

we see from (4.4) that

$$\Gamma(H_{\mathbf{C}}) \otimes \Gamma(H_{\mathbf{C}}) \cong L^2(E^*, \mu_{1/\sqrt{2}}) \otimes L^2(E^*, \mu_{\sqrt{2}}) \cong L^2(E_{\mathbf{C}}^*, \nu). \tag{4.7}$$

The Gelfand triple obtained by combining (4.6) and (4.7) is denoted by

$$\mathcal{D} \subset L^2(E_{\mathbf{C}}^*, \nu) \subset \mathcal{D}^*. \tag{4.8}$$

4.3 Resolution of the Identity via Coherent States

For any $z \in E_{\mathbf{C}}^*$ the same formula as in (4.5) is applied to defining an exponential vector $\phi_z \in \mathcal{W}^*$. We then put

$$Q_z\phi = \langle\langle \phi_{\bar{z}}, \phi \rangle\rangle \, \phi_z, \qquad \phi \in \mathcal{W}.$$

As is easily verified, $Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and its symbol is given by

$$\widehat{Q}_z(\xi, \eta) = \langle\langle Q_z\phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \phi_{\bar{z}}, \phi_\xi \rangle\rangle \, \langle\langle \phi_z, \phi_\eta \rangle\rangle = e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle}. \tag{4.9}$$

Note also that both maps $z \mapsto \phi_z \in \mathcal{W}^*$ and $z \mapsto Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ are continuous.

Theorem 4.1 *It holds that*

$$I = \int_{E_{\mathbf{C}}^*} Q_z \, \nu(dz), \tag{4.10}$$

where the integral is understood through the canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$.

PROOF. It follows by a standard argument that there exists an operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \int_{E_{\mathbf{C}}^*} \langle\langle Q_z\phi, \psi \rangle\rangle \, \nu(dz), \qquad \phi, \psi \in \mathcal{W},$$

for a precise argument see the proof of Lemma 4.4. Then, by using formulas (4.9) and (4.3), the symbol of Ξ is computed as follows:

$$\begin{aligned} \widehat{\Xi}(\xi, \eta) &= \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle = \int_{E_{\mathbf{C}}^*} \langle\langle Q_z\phi_\xi, \phi_\eta \rangle\rangle \, \nu(dz) \\ &= \int_{E_{\mathbf{C}}^*} e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} \, \nu(dz) = e^{\langle \xi, \eta \rangle} = \langle\langle \phi_\xi, \phi_\eta \rangle\rangle \,. \end{aligned}$$

Then the uniqueness of an operator symbol implies that $\Xi = I$. (General results on operator symbols in CKS-space are obtained in [4], [5], [27]. See also [22].) ∎

Theorem 4.1 is a infinite dimensional generalization of the well known formula:

$$I = \frac{1}{\pi} \int_{\mathbf{C}} |z\rangle\langle z| \, d^2z, \tag{4.11}$$

where $|z\rangle$ is the (normalized) coherent state. As an application of (4.11) one may prove that quantum mechanical density operators are represented as integrals over projection operators $|z\rangle\langle z|$. We shall discuss an infinite dimensional analogue.

4.4 Decomposition of White Noise Functions via Coherent States

Let $\xi \in E_{\mathbf{C}}$ and consider the exponential function $z \mapsto e^{\langle z, \xi \rangle}$, $z \in E_{\mathbf{C}}^*$. As is easily verified, $e^{\langle z, \xi \rangle} = \psi_{\xi/\sqrt{2}}(x)\psi_{i\xi/\sqrt{2}}(y)$ for $z = x + iy$, and hence $e^{\langle \cdot, \xi \rangle} = \psi_{\xi/\sqrt{2}} \otimes \psi_{i\xi/\sqrt{2}}$ belongs to $\mathcal{W} \otimes \mathcal{W} = \mathcal{D}$. Then by virtue of the characterization theorem for the S-transform in CKS-space [6] we come to the following

Theorem 4.2 *For any $w \in \mathcal{D}^*$ there exists a unique $\Phi \in \mathcal{W}^*$ such that*

$$\langle\!\langle \Phi, \phi_\xi \rangle\!\rangle = \langle\!\langle w, e^{\langle \cdot, \xi \rangle} \rangle\!\rangle \tag{4.12}$$

Since $\langle\!\langle \phi_z, \phi_\xi \rangle\!\rangle = e^{\langle z, \xi \rangle}$, identity (4.12) suggests us to adopt a formal integral expression:

$$\Phi = \int_{E_{\mathbf{C}}^*} w(z) \phi_z \, \nu(dz). \tag{4.13}$$

Conversely, it is not difficult to see that any $\Phi \in \mathcal{W}^*$ admits an expression as in (4.13). Moreover, it is noteworthy that (4.13) gives the inversion formula for the S-transform. Originally, the S-transform $S\Phi$ for $\Phi \in \mathcal{W}^*$ is a function on $E_{\mathbf{C}}$. However, for $\phi \in \mathcal{W}$ the S-transform $S\phi$ is naturally extended to a function of $E_{\mathbf{C}}^*$ by

$$S\phi(z) = \langle\!\langle \phi_z, \phi \rangle\!\rangle, \qquad z \in E_{\mathbf{C}}^*.$$

Theorem 4.3 *For $\phi \in \mathcal{W}$ it holds that*

$$\phi = \int_{E_{\mathbf{C}}^*} S\phi(\overline{z}) \phi_z \, \nu(dz). \tag{4.14}$$

PROOF. Write ψ for the right hand side of (4.14). It is easy to see that $\psi \in \mathcal{W}$ and the S-transform is computed as

$$\begin{aligned} S\psi(\xi) &= \int_{E_{\mathbf{C}}^*} S\phi(\overline{z}) S\phi_z(\xi) \, \nu(dz) \\ &= \int_{E_{\mathbf{C}}^*} \langle\!\langle \phi_{\overline{z}}, \phi \rangle\!\rangle \langle\!\langle \phi_z, \phi_\xi \rangle\!\rangle \, \nu(dz) = \int_{E_{\mathbf{C}}^*} \langle\!\langle Q_z \phi, \phi_\xi \rangle\!\rangle \, \nu(dz), \end{aligned}$$

where we used (4.9). It then follows from Theorem 4.1 that the last expression is equal to $\langle\!\langle \phi, \phi_\xi \rangle\!\rangle = S\phi(\xi)$. Then by the uniqueness of S-transform we conclude that $\phi = \psi$ as desired. ∎

The inversion formula for the S-transform was discussed in a different context by Kondratiev [13], see also [3, Chapter 2.5].

4.5 Diagonal Coherent State Represenations of Operators

We go back to the operator symbol of Q_z, see (4.9). Given $\xi, \eta \in E_{\mathbf{C}}$ we put

$$q_{\xi,\eta}(z) = \langle\!\langle Q_z \phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle \overline{z}, \xi \rangle + \langle z, \eta \rangle}, \qquad z \in E_{\mathbf{C}}^*. \tag{4.15}$$

Then, for $z = x + iy$ we obtain

$$q_{\xi,\eta}(x + iy) = e^{\langle x, \xi+\eta \rangle} e^{\langle y, i(-\xi+\eta) \rangle} = e^{\langle \xi, \eta \rangle} \psi_{(\xi+\eta)/\sqrt{2}}(x) \psi_{i(-\xi+\eta)/\sqrt{2}}(y). \tag{4.16}$$

Therefore $q_{\xi,\eta} = e^{\langle \xi, \eta \rangle} \psi_{(\xi+\eta)/\sqrt{2}} \otimes \psi_{i(-\xi+\eta)/\sqrt{2}}$ and belongs to $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{D}$.

Lemma 4.4 *For $w \in \mathcal{D}^*$ there is a unique operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that*

$$\langle\!\langle \Xi \phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle w, q_{\xi,\eta} \rangle\!\rangle, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{4.17}$$

PROOF. By definition we have

$$
\begin{aligned}
|\langle\langle w,\, q_{\xi,\eta}\rangle\rangle|^2 &= \left| e^{\langle \xi,\eta\rangle} \langle\langle w,\, \psi_{(\xi+\eta)/\sqrt{2}} \otimes \psi_{i(-\xi+\eta)/\sqrt{2}} \rangle\rangle \right|^2 \\
&\le e^{2|\langle \xi,\eta\rangle|} \,\| w \|_{-p}^2 \,\| \psi_{(\xi+\eta)/\sqrt{2}} \|_p^2 \,\| \psi_{i(-\xi+\eta)/\sqrt{2}} \|_p^2 \\
&= e^{2|\langle \xi,\eta\rangle|} \,\| w \|_{-p}^2 \, G_\alpha\Big(|\, (\xi+\eta)/\sqrt{2} \,|_p^2 \Big)\, G_\alpha\Big(|\, i(-\xi+\eta)/\sqrt{2} \,|_p^2 \Big),
\end{aligned}
$$

where $p \ge 0$ is chosen as $\| w \|_{-p} < \infty$. Then, after usual estimates fitting the generating function G_α, we conclude with the basis on characterization theorem of operator symbols (e.g., [22, §4.4], [27]) that the right hand side of (4.17) is the symbol of an operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. ∎

The operator Ξ defined as in (4.17) is written in a formal integral:

$$
\Xi = \int_{E_{\mathbf{C}}^*} w(z) Q_z \, \nu(dz) \tag{4.18}
$$

and is called a *diagonal coherent state representation.*

Theorem 4.5 *Every operator in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ admits a diagonal coherent state representation.*

PROOF. Given $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ we consider

$$
\Theta(\xi,\eta) = \langle\langle \Xi \phi_{(\xi+i\eta)/\sqrt{2}},\, \phi_{(\xi-i\eta)/\sqrt{2}} \rangle\rangle e^{-\langle \xi+i\eta,\, \xi-i\eta\rangle/2}, \qquad \xi,\eta \in E_{\mathbf{C}}.
$$

It then follows from the characterization theorem for operator symbols that there exists $W \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$
\Theta(\xi,\eta) = \langle\langle W\phi_\xi,\, \phi_\eta \rangle\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}.
$$

Then, changing the parameters, we have

$$
\langle\langle \Xi\phi_\xi,\, \phi_\eta \rangle\rangle = \langle\langle W\phi_{(\xi+\eta)/\sqrt{2}},\, \phi_{i(-\xi+\eta)/\sqrt{2}} \rangle\rangle \, e^{\langle \xi,\eta\rangle}, \qquad \xi,\eta \in E_{\mathbf{C}}.
$$

In view of the canonical isomorphism $(\mathcal{W} \otimes \mathcal{W})^* \cong \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, we choose $w \in (\mathcal{W} \otimes \mathcal{W})^*$ such that

$$
\langle\langle \Xi\phi_\xi,\, \phi_\eta \rangle\rangle = \langle\langle w,\, \phi_{(\xi+\eta)/\sqrt{2}} \otimes \phi_{i(-\xi+\eta)/\sqrt{2}} \rangle\rangle \, e^{\langle \xi,\eta\rangle}. \tag{4.19}
$$

Now, taking (4.5) into account, we consider the functional realization of the right hand side of (4.19). As is seen already in (4.15) and (4.16), we have

$$
\psi_{(\xi+\eta)/\sqrt{2}}(x)\psi_{i(-\xi+\eta)/\sqrt{2}}(y) e^{\langle \xi,\eta\rangle} = q_{\xi,\eta}(x+iy) = \langle\langle Q_z\phi_\xi,\, \phi_\eta \rangle\rangle, \qquad z = x+iy.
$$

Hence (4.19) is written in a formal integral

$$
\langle\langle \Xi\phi_\xi,\, \phi_\eta \rangle\rangle = \int_{E_{\mathbf{C}}^*} w(z)\, \langle\langle Q_z\phi_\xi,\, \phi_\eta \rangle\rangle \, \nu(dz), \tag{4.20}
$$

which means that Ξ admits a diagonal coherent state representation as in (4.18). ∎

Here is a simple example. For $e \in E$ let D_e denote the differential operator along the direction e, that is,

$$D_e = \Xi_{0,1}(e) = \int_{\mathbf{R}} e(t) a_t \, dt.$$

The diagonal coherent state representation is given by

$$D_e = \int_{E^*_{\mathbf{C}}} \langle z, e \rangle \, Q_z \, \nu(dz),$$

which is verified by a direct computation or by observing

$$D_e Q_z = \langle z, e \rangle \, Q_z.$$

Similarly, we have

$$D^*_e = \int_{E^*_{\mathbf{C}}} \langle \bar{z}, e \rangle \, Q_z \, \nu(dz), \qquad D_e D^*_e = \int_{E^*_{\mathbf{C}}} \langle \bar{z}, e \rangle \langle z, e \rangle \, Q_z \, \nu(dz).$$

Since the exponential vectors $\{\phi_\xi\}$ are overcomplete, the uniqueness of a diagonal coherent state representation does not hold. In fact, in view of (4.15) and (4.17) we see that for $w \in \mathcal{D}^*$

$$\int_{E^*_{\mathbf{C}}} w(z) Q_z \, \nu(dz) = 0 \quad \Longleftrightarrow \quad \int_{E^*_{\mathbf{C}}} w(z) e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} \nu(dz) = 0 \quad \text{for all} \quad \xi, \eta \in E_{\mathbf{C}}.$$

Relation between the diagonal coherent state representation and the Fock expansion of a white noise operator has not yet discussed satisfactorily.

4.6 Some Applications

In harmonic analysis on Gaussian space rotation-invariant operators are interesting to investigate. For example, by using the Fock expansion we characterized the number operator and the Gross Laplacian by their rotation-invariance [22]. In this connection we only mention the following

Lemma 4.6 *Let $w \in \mathcal{D}^*$ and assume that $\Gamma(g^*) w = w$ for any $g \in O(E; H)$. Then,*

$$\Xi = \int_{E^*_{\mathbf{C}}} w(z) Q_z \, \nu(dz)$$

is rotation-invariant.

The diagonal coherent state representation seems useful in computing the trace of an operator Ξ. Note first that for $z \in E_{\mathbf{C}}$ we have $\phi_z \in \mathcal{W}$ and $Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W})$. Then, by the definition of the trace we have

$$\operatorname{Tr} Q_z = \sum_i \langle\!\langle \overline{f_i}, \, Q_z f_i \rangle\!\rangle = \sum_i \langle\!\langle \phi_{\bar{z}}, \, f_i \rangle\!\rangle \langle\!\langle \overline{f_i}, \, \phi_z \rangle\!\rangle = \langle\!\langle \phi_{\bar{z}}, \, \phi_z \rangle\!\rangle = e^{\langle \bar{z}, z \rangle},$$

where $\{f_i\}$ is a complete orthonormal basis. Since $\widehat{Q}_z(\xi,\eta) = e^{\langle\bar{z},\xi\rangle+\langle z,\eta\rangle}$ by (4.9), applying (4.3) we have

$$\int_{E_{\mathbf{C}}^*} \widehat{Q}_z(\bar{\xi},\xi)\,\nu(d\xi) = \int_{E_{\mathbf{C}}^*} e^{\langle\bar{z},\bar{\xi}\rangle+\langle z,\xi\rangle}\,\nu(d\xi) = e^{\langle\bar{z},z\rangle}.$$

Therefore,

$$\operatorname{Tr} Q_z = \int_{E_{\mathbf{C}}^*} \widehat{Q}_z(\bar{\xi},\xi)\,\nu(d\xi), \qquad z \in E_{\mathbf{C}}. \tag{4.21}$$

It is an interesting question to extend (4.21) to a larger class of operators. In this context one may hope that the trace class operators on $\Gamma(H_{\mathbf{C}})$ are characterized.

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