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COMPLEX VECTOR BUNDLES AND JACOBI FORMS

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INTRODUCTION

In these notes we present a new link between the theory of automorphic forms and geometry. For an arbitrary compact manifold one can define its elliptic genus. It is a modular form in one variable with respect to a congruence subgroup of level 2 (see, for example, [L], [HBJ]). For a compact complex manifold one can define its elliptic genus as a function in two complex variables (see [K], [Hö]). In the last case the elliptic genus is the holomorphic Euler characteristic of a formal power series with vector bundle coefficients. If the first Chern class $c_1(M)$ of the complex manifold is equal to zero in $H^2(M,\mathbb{R})$, then the elliptic genus is a weak Jacobi modular form with integral Fourier coefficients of weight 0 and index $d/2$, where $d=\dim_{\mathbb{C}}(M)$. The same modular form appears in physic as the partition function of $N=2$ super-symmetric sigma model whose target space is Calabi–Yau manifold M (see [W], [EOTY], [KYY], [D]). We note that all "good" partition functions appeared in physic are automorphic forms with respect to some groups. This fact reflects that physical models have some additional symmetries. If $c_1(M)\neq 0$, then the elliptic genus of M is not automorphic form. In these notes we define *a modified Witten genus* or *automorphic correction of elliptic genus* of an arbitrary holomorphic vector bundle over a compact complex manifold and we study its properties.

We mainly present here automorphic aspects of the theory. In the proof of the theorem that the modified Witten genus is a Jacobi form we use a nice formula which relates the Jacobi theta-series, its logarithmic derivative, the quasi-modular Eisenstein series $G_2(\tau)$ and all derivatives of Weierstrass $\wp$-function (see Lemma 1.3 bellow). To get applications to the theory of complex manifolds we study $\mathbb{Z}$-structure of the graded ring of weak Jacobi forms with integral coefficients. We prove that the graded ring of Jacobi forms of weight 0 has four generators

$$J^{\mathbb{Z}}_{0,*}=\oplus_{m\geq 1}J^{\mathbb{Z}}_{0,m}=\mathbb{Z}[\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4}]$$

which satisfy the only relation $4\phi_{0,4}=\phi_{0,1}\phi_{0,3}-\phi_{0,2}^2$. The functions $\phi_{0,1},\dots,\phi_{0,4}$ are the fundamental Jacobi forms related to Calabi–Yau manifolds of dimension $d=2, 3, 4, 8$. The same Jacobi forms are generating functions for the multiplicities of all positive roots of the four generalized Lorentzian Kac–Moody Lie algebras of Borcherds type constructed in [GN1–GN4] (see also §3 of this paper).

The q^0-term of the Fourier expansion ($q=e^{2\pi i\tau}$) of the elliptic genus is essentially equal to the Hirzebruch χ_y-genus of the manifold. Thus we can analyze the arithmetic

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properties of the χ_y-genus of the complex manifold with $c_1(M) = 0$ and its special values such as signature ($y = 1$) and Euler number ($y = -1$) in terms of Jacobi modular forms. For example, we prove that the Euler number of a Calabi-Yau manifold M_d of dimension d satisfies

$$e(M_d) \equiv 0 \mod 8 \qquad \text{if} \quad d \equiv 2 \mod 8$$

(see Proposition 2.6). The special values of the generators of the Jacobi ring at $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are related to the Hauptmodules of the fields of modular functions. Using this fact we prove that

$$\chi_{y=\zeta_3}(M_d) \equiv 0 \mod 9 \qquad \text{if} \quad d \equiv 2 \mod 6$$

(see Proposition 2.7). Some other constructions (for example, $\hat{A}_2^{(2)}$-genus, the second quantized elliptic genus) and other applications to the theory of vector bundles one can find in my course given at RIMS, Kyoto University, on our joint seminar with K. Saito in 1998/99. I would like to take this opportunity to express my gratitude to all members of K. Saito's seminar. I am also grateful to the Research Institute for Mathematical Science of Kyoto University for hospitality.

§1. Automorphic correction of elliptic genus

Let M be an almost complex compact manifold M of (complex) dimension d and let E be a complex vector bundle over M. Let us fix two formal variables $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi iz)$, where $\tau \in \mathbb{H}_1$ (the upper half-plane) and $z \in \mathbb{C}$. One defines a formal power series $\mathbf{E}_{q,y} \in K(M)[[q, y^{\pm 1}]]$

$$\mathbf{E}_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} E^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-yq^n} E \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M \tag{1.1}$$

where T_M denotes the holomorphic tangent bundle of M and

$$\bigwedge\nolimits_x E = \sum_{k \geq 0} (\wedge^k E) x^k, \qquad S_x E = \sum_{k \geq 0} (S^k E) x^k$$

are formal power series with exterior powers and symmetric powers of a bundle E as coefficients. We propose the following

Definition 1.1. *Modified Witten genus* (MWG) of a complex vector bundle E of rank r over a compact (almost) complex manifold M of dimension d is defined as follows

$$\chi(M, E; \tau, z) = q^{(r-d)/12} y^{r/2} \int_M \exp\Big(\frac{1}{2}(c_1(E) - c_1(T_M))\Big) \cdot$$
$$\exp\Big((p_1(E) - p_1(T_M)) \cdot G_2(\tau)\Big) \exp\Big(-\frac{c_1(E)}{2\pi i} \frac{\vartheta_z}{\vartheta}(\tau, z)\Big) \operatorname{ch}(\mathbf{E}_{q,y}) \operatorname{td}(T_M)$$

where $c_1(E)$ and $p_1(E)$ are the first Chern and Pontryagin class of E, td is the Todd class, $\operatorname{ch}(\mathbf{E}_{q,y})$ is the Chern character which we apply to each coefficient of the formal power series and the integral $\int_M$ denotes the evaluation of the top degree differential form on the fundamental cycle of the manifold.

In the definition we use Jacobi theta-series of level two $\vartheta(\tau,z) = -i\vartheta_{11}(\tau,z)$:

$$\vartheta(\tau,z) = \sum_{n\equiv 1 \, mod\, 2} (-1)^{\frac{n-1}{2}} q^{\frac{n^2}{8}} y^{\frac{n}{2}} = -q^{1/8}y^{-1/2}\prod_{n\geq 1}(1-q^{n-1}y)(1-q^n y^{-1})(1-q^n),$$

$\vartheta_z(\tau,z) = \dfrac{\partial\vartheta}{\partial z}(\tau,z)$ and $G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)\, q^n$ is a quasi-modular Eisenstein series of weight 2 where $\sigma_1(n)$ is the sum of all positive divisors of n.

1.2. Witten genus. As a limit case of the definition above one obtains the Witten genus (see [W], [L], [HBJ]). Let assume that M admits a spin structure (i.e., the second Witney-Stiefel class $w_2(M)$ is zero or $c_1(T_M) \equiv 0 \mod 2$) and $p_1(M) = 0$. Let $E = M \times \mathbb{C}^r$ be the trivial vector bundle of rank r over M. Then $\operatorname{ch}(\bigwedge_x E) = (1+x)^r$ and

$$q^{r/12}y^{r/2}\operatorname{ch}\left(\bigotimes_{n=0}^{\infty}\bigwedge\nolimits_{-y^{-1}q^n}E^* \otimes \bigotimes_{n=1}^{\infty}\bigwedge\nolimits_{-yq^n}E\right) = \left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^r.$$

Thus

$$q^{d/12}\chi(M, M\times\mathbb{C}^r;\tau,z) = \frac{\vartheta(\tau,z)^r}{\eta(\tau)^r}\int_M \prod_{i=1}^{d}\frac{x_i/2}{\sinh(x_i/2)}\prod_{n=1}^{\infty}\frac{1}{(1-q^n e^{x_i})(1-q^n e^{-x_i})}$$

$$= \hat{A}(M, \bigotimes_{n=1}^{\infty} S_{q^n}(T_M\oplus T_M{}^*))\frac{\vartheta(\tau,z)^r}{\eta(\tau)^r} = \text{Witten genus}\,(M)\frac{\vartheta(\tau,z)^r}{\eta(\tau)^{r+2d}}.$$

If we take the trivial vector bundle of rank 0, then

$$\chi(M,0;\tau,z) = \chi(M;\tau) = \frac{\text{Witten genus}\,(M)}{\eta(\tau)^{2d}}.$$

This is an automorphic function in τ with respect to $SL_2(\mathbb{Z})$.

1.3. Elliptic genus of Calabi–Yau manifolds. This case is of some interest in physics. Let $E = T_M$ and $c_1(T_M) = 0$. Then there are no correction terms of type $\exp(\dots)$ in Definition 1.1. Thus the MEG of T_M is, up to the factor $y^{d/2}$, the Euler–Poicaré characteristic of the element $\mathbf{E}_{q,y}$. This function is called *elliptic genus* of the Calabi–Yau manifold M or genus one partition function of the super-symmetric (2,2)-sigma model whose target space is M:

$$\chi(M, T_M;\tau,z) = \text{Elliptic genus}\,(M;\tau,z) = y^{d/2}\int_M \operatorname{ch}(\mathbf{E}_{q,y})\operatorname{td}(T_M).$$

According to the Riemann–Roch–Hirzebruch theorem one can see that the q^0-term of $\chi(M;\,\tau,z)$ is essentially the Hirzebruch χ_y-genus of the manifold M:

$$\chi(M;\,\tau,z) = \sum_{p=0}^{d}(-1)^p\chi_p(M)\,y^{\frac{d}{2}-p} + \tag{1.2}$$

$$q\Big(\sum_{p=-1}^{d+1}(-1)^p y^{-p}\big(\chi_p(M,T_M^*) - \chi_{p-1}(M,T_M^*) + \chi_p(M,T_M) - \chi_{p+1}(M,T_M)\big)\Big) + \dots$$

where $\chi(M,E)=\sum_{q=0}^{d}(-1)^q \dim H^q(M,E)$ and $\chi^p(M,E)=\chi(M,\,\wedge^p T^*_M\otimes E)$ or, for a Kähler manifold, $\chi_p(M)=\sum_q(-1)^q h^{p,q}(M)$. We remark that in this case the Fourier coefficient of the elliptic genus is equal to the index of the Dirac operator twisted with a corresponding vector bundle coefficient of the formal power series $\mathbf{E}_{q,y}$.

It is known that the elliptic genus of a Calabi–Yau manifold is a modular form in variables τ and z (see [Hö], [KYY]), i.e., it is a weak Jacobi form of weight 0 and index $d/2$. *If $c_1(T_M)\neq 0$, then the elliptic genus of M defined above is not a modular form in τ and z.* We add the three correction factors in Definition 1.1 in order to obtain a function with a good behavior with respect to the modular transformations in τ and z. If $E=T_M$ and $c_1(T_M)\neq 0$, then the integral in Definition 1.1 contains the only correction term

$$\exp\Bigl(-\frac{c_1(T_M)}{2\pi i}\frac{\vartheta_z}{\vartheta}(\tau,z)\Bigr).$$

Thus the elliptic genus of M (as a function in two variables) is equal to the zeroth term in a sum of $d+1$ summands of the modified genus. These summands correspond to all powers of the first Chern class of M

$$\chi(M,T_M;\tau,z)=\text{Elliptic genus}\,(M;\tau,z)+\sum_{n=1}^{d}\Bigl(\int_M c_1(M)^n(\ldots)\Bigr).$$

In general the elliptic genus is not an automorphic form in two variables but the modified elliptic genus is. The main result of this section is

Theorem 1.2. *Let E be a complex (holomorphic) vector bundle of rank r over a compact complex manifold M of dimension d. Let $\chi(M,E;\tau,z)$ be the modified Witten genus. Then the product*

$$\chi(M,E;\tau,z)\left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^{d-r}$$

is a weak Jacobi form of weight 0 and index $d/2$. In particular, $\chi(M,E;\tau,z)$ is a weak Jacobi form if $\operatorname{rank}(E)\geq\dim(M)$.

First we recall the definition of Jacobi forms of the type we need in this paper. Let $t\geq 0$ and k be integral or half-integral. Let v be a character of finite order (or a multiplier system for half-integral k) of $SL_2(\mathbb{Z})$. A holomorphic function $\phi(\tau,z)$ on $\mathbb{H}_1\times\mathbb{C}$ is called a *weak Jacobi form of weight k and index t with character v* if it satisfies the functional equations

$$\phi\Bigl(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Bigr)=v(\gamma)(c\tau+d)^k\,e^{2\pi i t\frac{cz^2}{c\tau+d}}\,\phi(\tau,z)\qquad \Bigl(\gamma=\bigl(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr)\in SL_2(\mathbb{Z})\Bigr)\quad(1.3a)$$

and

$$\phi(\tau,z+\lambda\tau+\mu)=(-1)^{2t(\lambda+\mu)}\,e^{-2\pi i t(\lambda^2\tau+2\lambda z)}\phi(\tau,z)\qquad(\lambda,\mu\in\mathbb{Z})\quad(1.3b)$$

and $\phi(\tau,z)$ has the Fourier expansion of the type

$$\phi(\tau,z)=\sum_{n\geq 0\ l\in t+\mathbb{Z}} f(n,l)\,q^n y^l.$$

We denote the space of all week Jacobi forms of weight k, index t and character (or multiplier system) v by $J_{k,t}(v)$. The space $J_{k,t}(v)$ is finite dimensional (see [EZ]). The only difference with [EZ] is that we admit Jacobi forms of half-integral weight and half-integral index. One of the main examples of weak Jacobi forms of half-integral weight with trivial SL_2-character is the quotient of the Jacobi theta-series by the cube of the Dedekind η-function

$$\phi_{-1,1/2}(\tau,z)=\vartheta(\tau,z)/\eta(\tau)^3=(r^{1/2}-r^{-1/2})+q(\dots)\in J_{-1,\frac{1}{2}}.$$

Sketch of the proof of Theorem 1.2. To prove the theorem we represent $\chi(M,E;\tau,z)$ in terms of the theta-series. Let $c(E)$ be the total Chern class of the vector bundle E

$$c(E)=\sum_{i=0}^{r}c_i(E)=\prod_{i=1}^{r}(1+x_i)$$

where $x_i=2\pi i\xi_i$ $(1\le i\le r)$ are the formal Chern roots of E. We denote by $x'_j=2\pi i\zeta_j$ $(1\le j\le d)$ the Chern roots of T_M. We recall that

$$\mathrm{ch}\,(\textstyle\bigwedge_t E)=\displaystyle\prod_{i=1}^{r}(1+te^{x_i}),\qquad \mathrm{ch}\,(S_tE)=\prod_{i=1}^{r}\frac{1}{1-te^{x_i}}.$$

According to the last formulae we have

$$\mathrm{ch}\,(\mathbf{E}_{q,y})\,\mathrm{td}\,(T_M)=\prod_{n=1}^{\infty}\prod_{j=1}^{d}\prod_{i=1}^{r}\frac{(1-q^{n-1}y^{-1}e^{-x_i})(1-q^ny e^{x_i})}{(1-q^{n-1}e^{-x'_j})(1-q^ne^{x'_j})}x'_j.$$

Therefore

$$q^{(r-d)/12}y^{r/2}\exp\Bigl(\frac{1}{2}(c_1(E)-c_1(T_M))\Bigr)\mathrm{ch}\,(\mathbf{E}_{q,y})\mathrm{td}\,(T_M)=$$

$$(-1)^{r-d}\prod_{i=1}^{r}\frac{\vartheta(\tau,-z-\xi_i)}{\eta(\tau)}\prod_{j=1}^{d}\frac{\eta(\tau)}{\vartheta(\tau,-\zeta_j)}(2\pi i\zeta_j).\tag{1.4}$$

Puting the last expression under the integral we obtain the following formula for the modified elliptic genus

$$\chi(M,E;\tau,z)=\int_M\prod_{i=1}^{r}\exp\Bigl(-4\pi^2G_2(\tau)\xi_i^2-\frac{\vartheta_z}{\vartheta}(\tau,z)\xi_i\Bigr)\frac{\vartheta(\tau,z+\xi_i)}{\eta(\tau)}\times$$

$$\prod_{j=1}^{d}\exp(4\pi^2G_2(\tau)\zeta_i^2)\,\frac{\eta(\tau)}{\vartheta(\tau,\zeta_j)}(2\pi i\zeta_j).\tag{1.5}$$

We shall calculate the top differential form under the integral using Lemma 1.3 bellow. To formulate this lemma we need to recall the definition of the Weierstrass $\wp$-function

$$\wp(\tau,z)=z^{-2}+\sum_{\substack{\omega\in\mathbb{Z}\tau+\mathbb{Z}\\ \omega\ne0}}\bigl((z+\omega)^{-2}-\omega^{-2}\bigr)\in J_{2,0}^{mer}$$

which is a meromorphic Jacobi form of weight 2 and index 0 with pole of order 2 along $z\in\mathbb{Z}\tau+\mathbb{Z}$.

Lemma 1.3. *The following formula is valid*

$$\exp\Big(-4\pi^2 G_2(\tau)\xi^2 - \frac{\vartheta_z}{\vartheta}(\tau,z)\xi\Big)\frac{\vartheta(\tau,z+\xi)}{\eta(\tau)} = \exp\Big(-\sum_{n\geq 2}\wp^{(n-2)}(\tau,z)\frac{\xi^n}{n!}\Big)$$

where $\wp^{(n)}(\tau,z) = \dfrac{\partial^n}{\partial z^n}\wp(\tau,z)$.

Proof. The Jacobi form $\phi_{-1,\frac{1}{2}}$ has the following exponential representation as a Weierstrass σ-function (see, for example, review [Sk])

$$\phi_{-1,\frac{1}{2}}(\tau,z) = \frac{\vartheta(\tau,z)}{\eta(\tau)^3} = (2\pi i z)\exp\Big(\sum_{k\geq 1}\frac{2}{(2k)!}G_{2k}(\tau)(2\pi i z)^{2k}\Big) \tag{1.7}$$

where $G_{2k}(\tau) = -B_{2k}/4k + \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$ is the Eisenstein series of weight $2k$. (For each $\tau\in\mathbb{H}_1$ the product is normally convergent in $z\in\mathbb{C}$.) Since one can obtain the Weierstrass $\wp$-function as the second derivative of the Jacobi theta-series $\dfrac{\partial^2}{\partial z^2}\log\vartheta(\tau,z) = -\wp(\tau,z) + 8\pi^2 G_2(\tau)$, the identity (1.7) implies that

$$\wp^{(n-2)}(\tau,z) = \frac{(-1)^n(n-1)!}{z^n} + 2\sum_{k\geq 2,\,2k\geq n}(2\pi i z)^{2k}G_{2k}(\tau)\frac{z^{(2k-n)}}{(2k-n)!}.$$

After that the formula of the lemma follows by direct calculation.

Now we can finish the proof of Theorem 1.2. According to the formula of Lemma 1.3 the Chern roots x_i $(1\leq i\leq r)$ of the vector bundle E and the Chern roots x'_j $(1\leq j\leq d)$ of the manifold M can be splitted under the integral in (1.5), i.e.,

$$\chi(M,E;\tau,z) = \frac{\vartheta^r}{\eta^{r+2d}}\int_M P(E;\tau,z)\cdot W(M;\tau). \tag{1.8}$$

The first factor depends only on the vector bundle E

$$P(E;\tau,z) = \exp\Big(-\sum_{n\geq 2}\frac{\wp^{(n-2)}(\tau,z)}{(2\pi i)^n n!}\big(\sum_{i=1}^{r}x_i^n\big)\Big).$$

The second factor is the Witten factor

$$W(M;\tau) = \exp\Big(2\sum_{k\geq 2}\frac{G_{2k}(\tau)}{(2k)!}\big(\sum_{j=1}^{d}{x'_j}^{2k}\big)\Big)$$

which determines the Witten genus of the manifold M as a function in one variable τ (see §1.3). The derivation of order $(n-2)$ of the Weierstrass $\wp$-function is a meromorphic Jacobi form of weight n and index 0 with pole of order n along $z=0$. Thus the coefficient of a monomial in x_i, x'_j of the total degree d in (1.8) is a meromorphic Jacobi form of weight 0 and index $r/2$ with pole of order not bigger than $(d-r)$. Therefore the product $\vartheta(\tau,z)^{d-r}\chi(M,E,\tau,z)$ is holomorphic on $\mathbb{H}_1\times\mathbb{C}$. This is weak Jacobi form since its Fourier expansion does not contain negative powers of q. Theorem 1.2 is proved.

§2. $\mathbb{Z}$-STRUCTURE OF THE GRADED RING OF JACOBI FORMS AND THE SPECIAL VALUES OF THE ELLIPTIC GENUS

The structure over $\mathbb{C}$ of the graded ring of all weak Jacobi forms was determined in [EZ]. The elliptic genus of a Calabi–Yau manifold is a weak Jacobi form of weight 0 with integral Fourier coefficients. Thus one can put a question about $\mathbb{Z}$-structure of the graded ring

$$J_{0,*}^{\mathbb{Z}} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} J_{0,m}^{\mathbb{Z}}$$

of all Jacobi forms with integral Fourier coefficients. We introduce its ideal

$$J_{0,*}^{\mathbb{Z}}(q) = \{\phi \in J_{0,*}^{\mathbb{Z}} \mid \phi(\tau,z) = \sum_{n\geq 1,\ l\in\mathbb{Z}} a(n,l)q^n y^l\}$$

consisting of the Jacobi forms without q^0-term. Using standard considerations with divisors of one can prove

Lemma 2.1. *Let m be integral, then we have*

$$J_{2k,m+\frac12}^{\mathbb{Z}} = \phi_{0,\frac32}\cdot J_{2k,m-1}^{\mathbb{Z}}, \qquad J_{2k+1,m+\frac12}^{\mathbb{Z}} = \phi_{-1,\frac12}\cdot J_{2k+2,m}^{\mathbb{Z}}$$

where $\phi_{0,\frac32}(\tau,z) = \vartheta(\tau,2z)/\vartheta(\tau,z)$ and $\phi_{-1,\frac12} = \vartheta(\tau,z)/\eta(\tau)^3$. The ideal $J_{0,*}^{\mathbb{Z}}(q)$ is principal. It is generated by a weak Jacobi form of weight 0 and index 6

$$\xi_{0,6}(\tau,z) = \Delta(\tau)\phi_{-1,\frac12}(\tau,z)^{12} = \frac{\vartheta(\tau,z)^{12}}{\eta(\tau)^{12}} = q(y^{\frac12} - y^{-\frac12})^{12} + q^2(\dots).$$

There exists only one (up to a constant) weak Jacobi form of weight 0 and index 1

$$\phi_{0,1}(\tau,z) = -\frac{3}{\pi^2}\frac{\wp(\tau,z)\vartheta(\tau,z)^2}{\eta(\tau)^6} = (y+10+y^{-1}) + q(10y^{\pm 2} - 88y^{\pm 1} - 132) + \dots$$

(see [EZ]). In the theory of generalized Lorentizian Kac–Moody algebras (see [GN1–GN4]) we defined the following important Jacobi forms of small indices:

$$\begin{aligned}\phi_{0,2}(\tau,z) &= \tfrac12\eta(\tau)^{-4}\sum_{m,n\in\mathbb{Z}}(3m-n)\left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)q^{\frac{3m^2+n^2}{24}}y^{\frac{m+n}{2}}\\ &= (y+4+y^{-1}) + q(y^{\pm3} - 8y^{\pm2} - y^{\pm1} + 16) + q^2(\dots), \qquad (2.1)\\ \phi_{0,3}(\tau,z) &= \phi_{0,\frac32}^2(\tau,z) = (y+2+y^{-1}) + q(-2y^{\pm3} - 2y^{\pm2} + 2y^{\pm1} + 4) + q^2(\dots),\\ \phi_{0,4}(\tau,z) &= \frac{\vartheta(\tau,3z)}{\vartheta(\tau,z)} = (y+1+y^{-1}) - q(y^{\pm4} + y^{\pm3} - y^{\pm1} - 2) + q^2(\dots). \qquad (2.2)\end{aligned}$$

One can also represent these functions as symmetric polynomials in the quotients of the Jacobi theta-series $\vartheta_{ab}(\tau,z)/\vartheta_{ab}(\tau,0)$ of level 2. Let us put

$$\xi_{00} = \frac{\vartheta_{00}(\tau,z)}{\vartheta_{00}(\tau,0)},\quad \xi_{10} = \frac{\vartheta_{10}(\tau,z)}{\vartheta_{10}(\tau,0)},\quad \xi_{01} = \frac{\vartheta_{01}(\tau,z)}{\vartheta_{01}(\tau,0)}.$$

Then we have

$$\phi_{0,1}(\tau,z)=4(\xi_{00}^2+\xi_{10}^2+\xi_{01}^2),\qquad \phi_{0,\frac{3}{2}}(\tau,z)=4\xi_{00}\xi_{10}\xi_{01}$$
$$\phi_{0,2}(\tau,z)=2\big((\xi_{00}\xi_{10})^2+(\xi_{00}\xi_{01})^2+(\xi_{10}\xi_{01})^2\big).$$

(To check these formulae one should compare only q^0-terms of corresponding Jacobi forms.)

In the next theorem we construct a basis of the module $J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q)$ and we find generators of the graded ring $J_{0,*}$.

Theorem 2.2. 1. *Let m be a positive integer. The module*

$$J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q)=\mathbb{Z}\,[\psi_{0,m}^{(1)},\dots,\psi_{0,m}^{(m)}]$$

is a free $\mathbb{Z}$-module of rank m. Moreover we can chose a basis with the following q^0-terms

$$[\psi_{0,m}^{(n)}(\tau,z)]_{q^0}=y^n+n^2y+(2n^2-2)+n^2y^{-1}+y^{-n}\qquad (2\le n\le m),$$
$$[\psi_{0,m}^{(1)}]_{q^0}=\frac{1}{(12,m)}\left(my+(12-2t)+my^{-1}\right)$$

where $(12,m)$ is the greatest common divisor of 12 and m.

2. *The graded ring of all weak Jacobi forms of weight 0 with integral coefficients is finitely generated*

$$J_{0,*}^{\mathbb{Z}}=\bigoplus_m J_{0,m}^{\mathbb{Z}}=\mathbb{Z}\,[\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4}]$$

where $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ are algebraicly independent and

$$4\phi_{0,4}=\phi_{0,1}\phi_{0,3}-\phi_{0,2}^2.$$

The second claim of the theorem is a corollary of the first part which on can prove by induction on m and n. We give here only the formulae for the most important exceptional Jacobi forms having the q^0-term of type $y+c+y^{-1}$:

$$\phi_{0,6}(\tau,z)=\phi_{0,2}\phi_{0,4}-\phi_{0,3}^2=(y+y^{-1})+q(\dots),$$
$$\phi_{0,8}(\tau,z)=\phi_{0,2}\phi_{0,6}-\phi_{0,3}^2=(2y-1+2y^{-1})+q(\dots),$$
$$\phi_{0,12}(\tau,z)=\phi_{0,4}\phi_{0,8}-2\phi_{0,6}^2=(y-1+y^{-1})+q(\dots).$$

We note also that

$$\xi_{0,6}=-\phi_{0,1}^2\phi_{0,4}+9\phi_{0,1}\phi_{0,2}\phi_{0,3}-8\phi_{0,2}^3-27\phi_{0,3}^2. \tag{2.3}$$

To prove that $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{0,3}$ are algebraicly independent one has to consider values at $z=\frac{1}{2}$. We have

$$\phi_{0,2}(\tau,\frac{1}{2})\equiv 2,\quad \phi_{0,3}(\tau,\frac{1}{2})\equiv 0,\quad \phi_{0,4}(\tau,\frac{1}{2})\equiv -1.$$

(The two last identities follow from definition and the first one is a corollary of the torsion relation of the theorem.) The restriction of

$$\phi_{0,1}(\tau, \frac{1}{2}) = \alpha(\tau) = 8 + 2^8 q + 2^{11} q^2 + 11 \cdot 2^{10} q^3 + 3 \cdot 2^{14} q^4 + 359 \cdot 2^9 q^5 + \dots \qquad (2.4)$$

is a modular function with respect to $\Gamma_0(2)$ with a character of order 2.

We have also a result about the structure of the bigraded ring of all integral weak Jacobi forms

$$J^{\mathbb{Z}}_{*,*} = \bigoplus_{k\in\mathbb{Z},\, m\in\mathbb{Z}_{\geq 0}} J^{\mathbb{Z}}_{k,m}.$$

Theorem 2.3.

$$J^{w,\mathbb{Z}}_{*,*} = \mathbb{Z}[E_4(\tau), E_6(\tau), \Delta(\tau), E_{4,1}, E_{4,2}, E_{4,3}, E_{6,1}, E_{6,2}, E'_{6,3}, \phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}, \phi_{-2,1}]$$

where $\phi_{-2,1} = \vartheta^2/\eta^6$, $E_{4,1}, \dots E_{6,2}$ *are the Eisenstein–Jacobi series with the zeroth Fourier coefficient equals to* 1 *and* $E'_{6,3} = E_{6,3} + \frac{22}{61}\Delta_{12}\phi^3_{-2,1}$.

Using the result above we can analyze the value of the elliptic genus at the following special points $z = 0$ (Euler number), $z = \frac{1}{2}$ (signature), $z = \frac{\tau+1}{2}$ ($\hat{A}$-genus) and $z = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$. For this end we have to study the restriction of the generators of the graded ring of the integral week Jacobi forms. A special value of a Jacobi form is a modular form in τ. In the next lemma we give a little more precise statement than in [EZ, Theorem 1.3].

Lemma 2.4. *Let* $\phi \in J_{0,t}$ *(*$t \in \mathbb{Z}/2$*) and* $X = (\lambda, \mu) \in \mathbb{Q}^2$*. Then*

$$\phi|_X(\tau, 0) = \phi(\tau, \lambda\tau + \mu)\exp(2\pi i t(\lambda^2\tau + \lambda\mu))$$

is an automorphic form of weight 0 *with a character with respect to the subgroup*

$$\Gamma_X = \{M \in SL_2(\mathbb{Z}) \,|\, XM - X \in \mathbb{Z}^2\}.$$

It is easy to see that if $\phi \in J^{\mathbb{Z}}_{k,m}$ with integral m, then the form $\phi(\tau, \frac{1}{N})$ still has integral Fourier coefficients if $N = 1, \dots, 6$. In particular, the value of $\xi_6(\tau, z)$ at these points is related to the "Hauptmodule" for the corresponding group $\Gamma_0(N)$:

$$\xi_6(\tau, \frac{1}{2}) = 2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}, \qquad \xi_6(\tau, \frac{1}{4}) = 2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2},$$

$$\xi_6(\tau, \frac{1}{3}) = 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2}, \qquad \xi_6(\tau, \frac{1}{6}) = \left(\frac{\Delta(\tau)\Delta(6\tau)}{\Delta(2\tau)\Delta(3\tau)}\right)^{1/2}.$$

Let us analyze the corresponding values of the four generators $\phi_{0,n}$ of the graded ring $J^{\mathbb{Z}}_{0,*}$. From the definition (see (2.1)–(2.2)) and the identity $4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - \phi^2_{0,2}$ we obtain

$$\phi_{0,1}(\tau, 0) = 12, \quad \phi_{0,2}(\tau, 0) = 6, \quad \phi_{0,3}(\tau, 0) = 4, \quad \phi_{0,4}(\tau, 0) = 3 \qquad (2.5)$$

and

$$\begin{array}{lll}
\phi_{0,1}(\tau,\frac{1}{2})=\alpha(\tau) & \phi_{0,1}(\tau,\frac{1}{3})=\beta^2(\tau) & \phi_{0,1}(\tau,\frac{1}{4})=\frac{\gamma(\tau)^4+4}{\gamma(\tau)}\\
\phi_{0,2}(\tau,\frac{1}{2})=2 & \phi_{0,2}(\tau,\frac{1}{3})=\beta(\tau) & \phi_{0,2}(\tau,\frac{1}{4})=4\gamma^2(\tau)\\
\phi_{0,3}(\tau,\frac{1}{2})=0 & \phi_{0,3}(\tau,\frac{1}{3})=1 & \phi_{0,3}(\tau,\frac{1}{4})=2\gamma(\tau)\\
\phi_{0,4}(\tau,\frac{1}{2})=-1 & \phi_{0,4}(\tau,\frac{1}{3})=0 & \phi_{0,4}(\tau,\frac{1}{4})=1.
\end{array} \tag{2.6}$$

The automorphic functions $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ are automorphic forms of weight 0 with respect to the group Γ_0, $\Gamma_0^{(1)}(3)$ and $\Gamma_0^{(1)}(4)$ respectively. These functions have integral Fourier coefficients. The identity (2.3) gives us the following relations between the automorphic functions α, β and γ

$$2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}=\alpha(\tau)^2-64,\qquad 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2}=\beta(\tau)^3-27$$
$$2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2}=4(\left(\frac{\gamma(\tau)}{2}\right)^2-\left(\frac{2}{\gamma(\tau)}\right)^2).$$

It follows that

$$\alpha(\tau)-8\equiv 0 \mod 2^8,\qquad \beta(\tau)-3\equiv 0 \mod 3^3 \tag{2.7}$$

(compare with (2.4)). Using the definition of $\phi_{0,3}$ and $\gamma(\tau)$ and the relations between the Jacobi theta-series ϑ_{ab} of level 2 we have

$$\gamma(\tau)=\frac{\vartheta_{00}(2\tau)}{\vartheta_{01}(2\tau)}=\frac{\vartheta_{00}(2\tau,0)}{\vartheta_{01}(2\tau,0)}.$$

One can check that $\phi_{0,1}(\tau,2z)=\phi_{0,2}^2(\tau,z)-8\phi_{0,4}(\tau,z)$. Thus

$$\alpha(\tau)=16\gamma(\tau)^4-8=16\frac{\vartheta_{00}^4(2\tau)}{\vartheta_{01}^4(2\tau)}-8.$$

In particular *all Fourier coefficients of $\gamma(\tau)$ and $\alpha(\tau)$ are positive.*

Example 2.5. $\hat{A}$-genus. Let $X=(\frac{1}{N},\frac{1}{N})$. Then Γ_X (see Lemma 2.4) contains the principle congruence subgroup $\Gamma_1(N)$. In some cases Γ_X will be strictly larger. For example, if $X_2=(\frac{1}{2},\frac{1}{2})$, then

$$\phi|_{X_2}(\tau,0)=\phi(\tau,\frac{\tau+1}{2})\exp(\frac{\pi i}{2}(\tau+1))$$

is an automorphic form with respect of the so-called theta-group

$$\Gamma_\theta=\left\{M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})\mid M\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \text{or}\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\mod 2\right\}.$$

The corresponding character is given by $\epsilon_2(M) = \exp(2\pi i m(d+b-a-c)/4) = \pm 1$. This character is trivial if index m of Jacobi form is even. Let us consider Γ_θ-automorphic function

$$\hat\phi_m(\tau) = q^{-\frac{m}{4}}\phi_{0,m}(\tau, -\frac{\tau+1}{2}).$$

We have

$$\hat\phi_3 = 0, \quad \hat\phi_4 = -1, \quad \hat\phi_2 = -2, \quad \hat\xi_6 = \hat\phi_1^2 + 64 = \left(\frac{\vartheta_{00}}{\eta}\right)^{12}$$

where

$$\hat\phi_1(\tau) = 4\frac{\vartheta_{10}^4 - \vartheta_{01}^4}{\vartheta_{01}^2\vartheta_{10}^2} = -q^{-\frac14} + 20q^{\frac14} + \cdots \in \mathfrak{M}_0^{\mathbb{Z}}(\Gamma_0(2), \epsilon_2).$$

Now we analyze some special values of the elliptic genus. As it easy follows from (1.2) we get Euler number of a Calabi–Yau manifold M_d for $z=0$ (d is arbitrary) and and its signature for $z = \frac12$ (d is even):

$$\chi(M_d, \tau, 0) = e(M_d),$$
$$\chi(M_d, \tau, \frac12) = \sigma_M(\tau) = (-1)^{\frac{d}{2}} s(M_d) + q(\dots) \in \mathfrak{M}_0^{\mathbb{Z}}(\Gamma_0(2), v_2), \quad v_2(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) = e^{\pi i m \frac{c}{2}}.$$

The formulae (2.5) gives us some divisibility of Euler number of Calai–Yau manifolds. We note that the quotient $e(M)/24$ appears in physics as obstruction to cancelling the tadpole (see [SVW] where it was proved that $e(M_4) \equiv 0 \mod 6$).

Proposition 2.6. *Let M_d be an almost complex manifold of complex dimension d such that $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Then*

$$d \cdot e(M_d) \equiv 0 \mod 24.$$

If $c_1(M) = 0$ in $H^2(M, \mathbb{Z})$, then we have a more strong congruence

$$e(M) \equiv 0 \mod 8 \qquad \textit{if } d \equiv 2 \mod 8.$$

Proof. The first fact follows simply from (2.5). If $d \equiv 2 \mod 8$ one can write the elliptic genus as a polynom over $\mathbb{Z}$ in the generators ϕ

$$e(M_d) \equiv P(\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4})|_{z=0} \equiv c_{1,m}(\phi_{0,1}|_{z=0})(\phi_{0,4}|_{z=0})^{\frac{d-2}{8}} \mod 8.$$

If one put $z = -\frac{\tau+1}{2}$, i.e., $y = -q^{1/2}$, then one obtains that the series

$$\mathbb{E}_{q,-q^{1/2}} = \bigotimes_{n\ge1} \bigwedge\nolimits_{q^{n}/2} T_M \otimes \bigotimes_{n\ge1} \bigwedge\nolimits_{q^{n}/2} T_M^* \otimes \bigotimes_{n\ge1} S_{q^n}(T_M \oplus T_M^*)$$

is $*$-symmetric. According to the Serre duality all Fourier coefficients of $\hat\chi(M_d, \tau)$ are even. The constant $c_{1,m}$ from the last congruence is equal to the coefficient of $\hat\chi(M_d, \tau)$ at the minimal negative power of q. Therefore $c_{1,m}$ is even and we obtain divisibility of $e(M_{8m+2})$ by 8.

We note that divisibility of $de(M)$ by 3 was proved by F. Hirzebruch in 1960. For a hyper-Kähler compact manifold the claim of the proposition above was proved by S. Salamon in [S]. After my talk on the elliptic genus at a seminar of MPI in Bonn in April 1997 Professor F. Hirzebruch informed me that the result of Proposition 2.6 was known for him (non-published). Using some natural examples he also proved that this property of divisibility of the Euler number modulo 24 is strict (see [H2]).

Formulae (2.6) provide us with a formula for the signature $\chi(M_d;\tau,\frac{1}{2})$ as a polynom in $\alpha(\tau)$. As a corollary of (2.6) and Theorem 2.2 we have that for an arbitrary Jacobi form of integral index

$$\phi_{0,4m}(\tau,\frac{1}{2}) = c + 2^{13}q(\dots) \qquad \phi_{0,4m+1}(\tau,\frac{1}{2}) = 8c + 2^{8}q(\dots)$$
$$\phi_{0,4m+2}(\tau,\frac{1}{2}) = 2c + 2^{12}q(\dots) \qquad \phi_{0,4m+3}(\tau,\frac{1}{2}) = 16c + 2^{9}q(\dots).$$

Similar to the proof of Proposition 2.4 we obtain a better congruence for the signature of a manifold with dim$\equiv 2 \mod 8$ and $c_1(M)=0$:

$$\chi(M_{8m+2};\tau,z) = 16c + 2^{9}q(\dots). \tag{2.9}$$

It is interesting that the values of the Hirzebruch y-genus at $y=e^{2\pi i/3}$ and $y=i$ also have some properties of divisibility. For $z=\frac{1}{3}$ (resp. $z=\frac{1}{4}$) we can write $\phi_{0,m}(\tau,\frac{1}{3})$ (resp. $\phi_{0,m}(\tau,\frac{1}{4})$) as a polynom in $\beta(\tau)=3+27(q+\dots)$ (resp. in $\gamma(\tau)^{\pm 1}$). This gives us the following results

$$\phi_{0,3m}(\tau,\frac{1}{3}) = c + 3^{6}q(\dots), \qquad \phi_{0,3m+1}(\tau,\frac{1}{3}) = 9c + 3^{4}q(\dots)$$
$$\phi_{0,3m+2}(\tau,\frac{1}{3}) = 3c + 3^{3}q(\dots).$$

Thus we have

Proposition 2.7. *If $c_1(M)=0$ (over $\mathbb{R}$), then*

$$\chi(M_{6m};\tau,\frac{1}{3}) \equiv c_1 \mod 3^{6}, \qquad \chi(M_{6m+2};\tau,\frac{1}{3}) \equiv 9c_2 \mod 3^{4},$$
$$\chi(M_{6m+4};\tau,\frac{1}{3}) \equiv 3c_3 \mod 3^{3}.$$

where $c_1,c_2,c_3\in\mathbb{Z}$. For $z=\frac{1}{4}$ we have:

$$\chi(M_{8m+2};\tau,\frac{1}{4}) = 4c + 2^{4}q(\dots), \qquad \phi_{0,4m+2}(\tau,\frac{1}{4}) = 4c + 2^{5}q(\dots)$$
$$\phi_{0,4m+3}(\tau,\frac{1}{4}) = 2c + 2^{8}q(\dots).$$

§3. SQEG AND HYPERBOLIC ROOT SYSTEMS

We can consider n-fold symmetric product of the manifold M, i.e., the orbifold space $M^{[n]} = M^n/S_n$, where S_n is the symmetric group of n elements. This is a singular manifold but one can define the string orbifold elliptic genus of $M^{[n]}$ (see for details the talk of R. Dijkgraaf at ICM-1998 in Berlin [D]). Using some arguments from the conformal field theory on orbifolds it was proved in [DVV] and [DMVV] that the string elliptic genus of the second quantization $\cup_{n\ge 1} M^{[n]}$ of a Calabi–Yau manifold M coincides with the second quantized elliptic genus of the given manifold:

$$\sum_{n=0}^{\infty} p^n \chi_{orb}(M^{[n]};\, q, y) = \prod_{m\ge 0,\, l,\, n>0} \frac{1}{(1-q^m y^l p^n)^{f(mn,l)}} \tag{3.1}$$

where $\chi(M,\tau,z) = \sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\,\mathbb{Z}/2)} f(m,l) q^m y^l$ is the elliptic genus of M.

For a $K3$ surface, the product in the left hand side of (3.1) is essentially the power -2 of the infinite product expansion of the product of all even theta-constants (see [GN1]). Following [DVV, §4] we call the product in (3.1) the *second-quantized elliptic genus* (SQEG) of the manifold M.

Theorem 3.1. *Let $M = M_d$ be a compact complex manifold of dimension d with trivial $c_1(M)$,*

$$\chi(M;\, \tau, z) = \sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\,\mathbb{Z}/2)} f(m,l) q^m y^l$$

be its elliptic genus and $\mathrm{SQEG}(M;\, Z)$ *($Z \in \mathbb{H}_2$) be its second quantized elliptic genus. We define a factor*

$$H(M;\, Z) = \begin{cases} \eta(\tau)^{-\frac12(e-3\chi'_{d_0})} \prod_{p=1}^{d_0} \left(\vartheta(\tau, pz)\, e^{\pi i p^2 \omega}\right)^{-\chi'_{d_0-p}} & \text{if } d = 2d_0 \\ \eta(\tau)^{-\frac12 e} \prod_{p=1}^{d_0} \left(\vartheta(\tau, \frac{2p-1}{2} z)\, e^{\frac14 \pi i (2p-1)^2 \omega}\right)^{-\chi'_{d_0-p+1}} & \text{if } d = 2d_0+1 \end{cases}$$

where $e = e(M)$ is Euler number of M and $\chi'_p = (-1)^p \chi_p(M)$ (see (1.2)). Then the product

$$E(M;\, Z) = \Psi(M;\, Z) \cdot \mathrm{SQEG}(M;\, Z) \qquad (d = 2d_0)$$
$$E^{(2)}(M;\, Z) = (E|\Lambda_2)(M;\, Z) \qquad (d = 2d_0+1)$$

determines a Siegel automorphic form of weight $-\frac12 \chi'_{d_0}(M)$ if d is even and of of weight 0 if d is odd with a character or a multiplier system of order $24/(24,e)$ with respect to a double extension of the paramodular group Γ_d^+ (resp. Γ_{2d}^+), if d is even (resp. d is odd).

The case of CY$_4$. The basic Jacobi modular forms for this dimension are the Jacobi forms $\phi_{0,2}$ and $\psi_{0,2}^{(2)}$ (see Theorem 2.2, part 1). They correspond to the following cusp forms for the paramodular group Γ_2 (see [GN1] and [GN4]):

$$\Delta_2(Z) = \text{Exp-Lift}(\phi_{0,2}(\tau,z)) = \text{Lift}(\eta(\tau)^3 \vartheta(\tau,z))$$
$$= \sum_{N\ge 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z} \\ n,\, m\equiv 1 \bmod 4 \\ 2nm-l^2=N^2}} N\left(\frac{-4}{Nl}\right) \sum_{a\,|\,(n,l,m)} \left(\frac{-4}{a}\right) q^{n/4} y^{l/2} s^{m/2} \in \mathfrak{M}_2^{cusp}(\Gamma_2, v_\eta^6 \times v_H)$$

and

$$\Delta_{11}(Z) = \mathrm{Lift}(\eta(\tau)^{21}\vartheta(\tau,2z)) = \mathrm{Exp\text{-}Lift}(\psi_{0,2}^{(2)}(\tau,z)) \in \mathfrak{M}_{11}(\Gamma_2).$$

For an arbitrary Calabi–Yau 4-fold M_4 we have the following formula for its SQEG

$$E(M_4;\, Z) = \Delta_{11}(Z)^{-\chi_0(M)}\Delta_2(Z)^{\chi_1(M)}. \tag{3.2}$$

We note that $\Delta_2(Z)^4$ is the first Γ_2-cusp form with trivial character and $\Delta_{11}(Z)$ is the first cusp form of odd weight with respect to Γ_2.

The Fourier expansion of the cusp forms $\Delta_2(Z)$, $\Delta_{11}(Z)$ and $\frac{\Delta_{11}(Z)}{\Delta_2(Z)}$ coincide with the Weyl–Kac–Borcherds denominator formula of generalized Kac–Moody super-algebras with a system of simple real roots of hyperbolic type determined by Cartan matrix $A_{1,II}$, $A_{2,II}$ and $A_{2,0}$ respectively:

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2\\ -2 & 2 & -2 & -6\\ -6 & -2 & 2 & -2\\ -2 & -6 & -2 & 2 \end{pmatrix},\quad A_{2,0} = \begin{pmatrix} 2 & -2 & -2\\ -2 & 2 & 0\\ -2 & 0 & 2 \end{pmatrix},\quad A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0\\ -2 & 2 & 0 & -4\\ -4 & 0 & 2 & -2\\ 0 & -4 & -2 & 2 \end{pmatrix}$$

(see [GN1]–[GN4]). Thus, the formula (3.2) gives us three particular cases of Calabi–Yau 4-folds of Kac–Moody type when the second quantized elliptic genus is a power of the denominator function of the corresponding Lorentzian Kac–Moody algebra:

$$\begin{aligned} E(M_4;\, Z) &= \Delta_{11}(Z)^{-\chi_0} && \text{if } \chi_1 = 0\\ E(M_4;\, Z) &= \left(\frac{\Delta_{11}(Z)}{\Delta_2(Z)}\right)^{-\chi_0} && \text{if } \chi_0(M) = -\chi_1(M)\\ E(M_4;\, Z) &= \Delta_2(Z)^{\chi_1} && \text{if } \chi_0(M) = 0. \end{aligned}$$

For more details and for the cases of $d > 4$ see [G1].

References

[B] R. Borcherds, *Automorphic forms on $O_{s+2,2}$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[D] R. Dijkgraaf, *The Mathematics of Fivebranes*, Documenta Mathem. ICM-1998 (1998).

[DVV] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Counting dyons in $N = 4$ string theory*, Nucl. Phys. **B484 (1997)** (1997), 543–561.

[DMVV] R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. **185** (1997), 197–209.

[EZ] M. Eichler, D. Zagier, *The theory of Jacobi forms*, Progress in Math. 55, Birkhäuser, 1985.

[EOTY] T. Eguchi, H. Ooguri, A. Taormina, S.-K. Yang,, *Superconformal Algebras and String Compactification on Manifolds with $SU(N)$ Holonomy*, Nucl. Phys. **B315** (1989), 193.

[G1] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms*, Preprint (1999).

[G2] V. Gritsenko, *Modified Witten genus*, Preprint MPI (1999).

[GH] V. Gritsenko, K. Hulek, *Commutator coverings of Siegel threefolds*, Duke Math. J. **94** (1998), 509–542.

[GN1] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras*, Amer. J. Math. **119** (1997), 181–224.

[GN2] V.A. Gritsenko, V.V. Nikulin, *The Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras*, Matem. Sbornik **187** (1996), 1601–1643.

[GN3] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part I*, International J. of Mathem. **9** (1998), 153–199.

[GN4] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part II*, International J. of Mathem. **9** (1998), 201–275.

[H1] F. Hirzebruch, *Elliptic genera of level N for complex manifolds*, Differential geometrical Methods in Theoretical Physics (K. Bleuler, M. Werner, eds.), Kluwer Acad. Publ., 1988, pp. 37–63; Appendix III to [HBJ].

[H2] F. Hirzebruch, *Letter to V. Gritsenko from 11 August 1997.*

[HBJ] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and Modular forms*, Aspects of Math. **E20** Vieweg–Verlag, 1992.

[Hö] G. Höhn, *Komplex elliptische Geschlecter und S^1-äquivariante Kobordismustheorie*, Diplomarbeit (1991), Bonn.

[K] I. Krichever, *Generalized elliptic genera and Baker–Akhiezer functions*, Mat. Zametki **47** (1990), 34–45.

[KYY] T. Kawai, Y. Yamada, S.-K. Yang, *Elliptic Genera and N=2 Superconformal Field Theory*, Nucl. Phys. **B414** (1994), 191-212.

[L] P.S. Landweber Ed., *Elliptic Curves and Modular Forms in Algebraic Topology*, Springer-Verlag, 1988.

[S] S. M. Salamon, *On the cohomology of Kähler and hyper-Kähler manifolds*, Topology **35** (1996), 137–155.

[SVW] S. Sethi, C. Vafa, E. Witten, *Constraints on low-dimensional string compactifications*, Nucl. Phys. **B480** (1996), 213–224.

[Sk] N-P. Skoruppa, *Modular forms*; Appendix I in [HBJ], pp. 121–161.

[W] E. Witten, *Elliptic genera and Quantum Field Theory*, Commun. Math. Phys. **109** (1987), 525–536.

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