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Structure of unbounded viscosity solutions to semilinear degenerate elliptic equations

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1. Introduction.

We consider the Dirichlet problem for a semilinear degenerate elliptic equation **(DP)**:

(1) $$-g(|x|)\Delta u+f(|x|,u)=0 \quad \text{in} \quad \mathbf{R}^N$$

(2) $$\lim_{|x|\to\infty}\frac{u(x)}{h(|x|)}=1,$$

where $N\geq 2$,

$$g(|x|)=||x|-a_1|^{\lambda_1}\,||x|-a_2|^{\lambda_2}\cdots||x|-a_k|^{\lambda_k}\,(|x|+1)^{-\lambda^*},$$

$$0<a_1<a_2<\cdots<a_k,\quad 0<\lambda_i \quad (i=1,2,\cdots,k) \quad \text{and} \quad \lambda^*\geq 0,$$

Δ is the Laplacian, and $h(|x|)\in C(|x|>a_k)$ will be determined later.

We discuss the problem **(DP)** under the following assumptions:

(A.1) $f(t,y)\in C([0,\infty)\times\mathbf{R})$ is locally Lipschitz continuous in (t,y).
(A.2) For any $t>0$ fixed, $f(t,y)$ is strictly increasing in y.
(A.3) For any $t\in[0,\infty)$, there exists a continuous function $\varphi(t)$ such that $f(t,\varphi(t))=0$.

EXAMPLE

$$g(|x|)\Delta u=u\,|u|^{p-1}-f(|x|).$$

In this paper, we study **(DP)** in case $N=2$.
Our aim is to prove the following

A) For $\lambda_i>0$ $(i=1,2,\cdots,k)$ and $\lambda^*\geq 0$, there exists a unique standard and radial viscosity solution of **(DP)**.

B) If $\lambda_i \geq 1$ for all i $(i = 1, 2, \cdots, k)$, there exists a **unique** viscosity solution of **(DP)**.

From **A)**, every viscosity solution is radial and standard.

C) If $0 < \lambda_i < 1$ for some i $(i = 1, 2, \cdots, k)$, there exist infinitely many viscosity solutions of **(DP)**.

2. Structure of standard viscosity solutions.

Following Crandall and Huan [2], we call a viscosity solution u of (DP) a standard solution if $u(x) = \varphi(a_i)$ (i.e., $f(a_i, u(x)) = 0$) on $|x| = a_i$ $(i = 1, 2, \cdots, k)$.

In order to construct a standard viscosity solution we shall consider the following Dirichlet problems :

$$(\mathbf{P_0}) \quad \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 & \text{in} \quad B_{a_1}, \\ \qquad u(x) = b_1 & \text{on} \quad |x| = a_1; \end{cases}$$

$$(\mathbf{P_i}) \quad \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad A(a_i, a_{i+1}), \\ u(x) = b_i \quad \text{on} \quad |x| = a_i, \qquad u(x) = b_{i+1} \quad \text{on} \quad |x| = a_{i+1}; \end{cases}$$

$$(\mathbf{P_k}) \quad \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad A(a_k, \infty), \\ u(x) = b_k \quad \text{on} \quad |x| = a_k, \qquad \lim\limits_{|x| \to \infty} \dfrac{u(x)}{h(|x|)} = 1, \end{cases}$$

where $A(a_i, a_{i+1}) = \{x \in \mathbf{R}^N : a_i < |x| < a_{i+1}\}$, $i = 1, 2, \cdots, k-1$ and $b_i = \varphi(a_i)$ $(i = 1, 2, \cdots, k)$.

Let $u_0 \in C(\overline{B_{a_1}}) \bigcap C^2(B_{a_1})$ (resp. $u_i \in C(\overline{A(a_i, a_{i+1})}) \bigcap C^2(A(a_i, a_{i+1}))$) be a radial classical solution of $(\mathbf{P_0})$ (resp. $(\mathbf{P_i})$ $(i = 1, 2, \cdots, k)$).

Put

$$\tilde{u}(x) = \begin{cases} u_0(x) & \text{for} \quad x \in \overline{B_{a_1}}, \\ u_1(x) & \text{for} \quad x \in \overline{A(a_1, a_2)} \\ \vdots & \\ u_k(x) & \text{for} \quad x \in \overline{A(a_k, \infty)}. \end{cases} \tag{3}$$

It is easy to verify, by the definition of viscosity solutions, that $\tilde{u}$ is a radial and standard viscosity solution of (**DP**). An easy caluculation shows that $u(x) = y(t)$ ($| x |= t$) is a radial classical solution of ($\mathbf{P_i}$) if and only if $y(t)$ is a classical solution of the following boundary value problem (denoted by ($\mathbf{BVP_i}$)):

$$(4) \qquad g(t)(\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt}) = f(t,y) \quad \text{in} \quad a_i < t < a_{i+1}$$

$$y(a_i) = b_i \quad \text{and} \quad y(a_{i+1}) = b_{i+1} \quad (i = 0, 1, \cdots, k),$$

where $y(a_0) = b_0$ and $y(a_{k+1}) = b_{k+1}$ are replaced by $\frac{dy}{dt}(0) = 0$ and $\lim_{t\to\infty} \frac{y(t)}{h(t)} = 1$, respectively.

From now on we briefly explain that the existence and uniqueness of classical solutions of ($\mathbf{BVP_i}$) ($i = 1, 2, \cdots, k$) play an essential role to prove our assertion stated in Introduction. Assume $\lambda_i \geq 1$ for all $i = 1, 2, \cdots, k$. Let $u(x)$ be an arbitrary viscosity solution of (**DP**). Define

$$\overline{U}(x) = \sup_{|y|=|x|} u(y) \quad \text{and} \quad \underline{U}(x) = \inf_{|y|=|x|} u(y).$$

We observe that $\overline{U}(x)$ (resp. $\underline{U}(x)$) is continuous and radial viscosity subsolution (resp. supersolution) and $\overline{U}(x) = \underline{U}(x) = b_i$ on $| x |= a_i$ (by $\lambda_i \geq 1$). By the well-known comparison theorem, we have

$$y_i(| x |) \leq \underline{U}(x) \leq \overline{U}(x) \leq y_i(| x |)$$

for $a_i \leq| x |\leq a_{i+1}$ ($i = 0, 1, 2, \cdots, k$), where y_i is the unique solution of ($\mathbf{BVP_i}$).

3. Existence and uniqueness for ($\mathbf{BVP_i}$).

In order to study ($\mathbf{BVP_i}$), we introduce the following integral equations:

$$(5) \qquad y(t) = \alpha + \int_0^t (\log t/s)sg(s)^{-1}f(s, y(s))ds,$$

$$(6) \qquad y(t) = \alpha + t_0\beta\log(t/t_0) + \int_{t_0}^t \log(t/s)sg(s)^{-1}f(s, y(s))ds,$$

where $0 < t_0 \notin \{a_1, a_2, \cdots, a_k\}, \alpha$ and β are real parameters. Applying a fixed point theorem, we can prove the local existence of solutions of (5) and (6).

First, to solve $(\mathbf{BVP_0})$, we define

$$S_0^+ = \{\alpha \in \mathbf{R};\ \lim_{t \to T_\alpha} y_\alpha(t) = +\infty\}$$

$$S_0 = \{\alpha \in \mathbf{R};\ \lim_{t \to a_1} y_\alpha(t) = \text{exists}\}$$

$$S_0^- = \{\alpha \in \mathbf{R};\ \lim_{t \to T_\alpha} y_\alpha(t) = -\infty\},$$

where y_α is a classical solution of (5) obtained by prolonging local solutions of (5) and (6).

We see that (i) in case $0 < \lambda_1 < 2$,

$$S_0^+ = [\overline{\alpha}, \infty), \quad S_0 = (\underline{\alpha}, \overline{\alpha}), \quad S_0^- = (-\infty, \underline{\alpha}]$$

$$\text{and} \quad \{y_\alpha(a_1) = \lim_{t \to a_1} y_\alpha(t); \alpha \in S_0\} = \mathbf{R};$$

and (ii) in case $\lambda_1 \geq 2$,

$$S_0^+ = (\alpha_0, \infty), \quad S_0 = \{\alpha_0\}, \quad S_0^- = (-\infty, \alpha_0) \quad \text{and} \quad y_{\alpha_0}(a_1) = b_1.$$

Consequently we have

Proposition 1. *There exists a unique classical solution y_0 of* $(\mathbf{BVP_0})$.

Next, to solve $(\mathbf{BVP_i})$ $(i = 1, 2, \cdots, k-1)$, we fix $t_0 \in (a_i, a_{i+1})$ and define for each $\alpha \in \mathbf{R}$

$$B_i^+ = \{\beta \in \mathbf{R};\ \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = +\infty\}$$

$$B_i = \{\beta \in \mathbf{R};\ \lim_{t \downarrow a_i} y_{\alpha\beta}(t) = \text{exists}\}$$

$$B_i^- = \{\beta \in \mathbf{R};\ \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = -\infty\},$$

where $y_{\alpha\beta}(t)$ is a solution of (6) on $(T_{\alpha\beta}, t_0]$ $(a_i \leq T_{\alpha\beta} < t_0)$.

We can prove that (i) in case $0 < \lambda_i < 2$,

$$B_i^- = [\overline{\beta}, \infty), \quad B_i = (\underline{\beta}, \overline{\beta}), \quad B_i^+ = (-\infty, \underline{\beta}]$$

$$\text{and} \quad \{y_{\alpha\beta}(a_i) = \lim_{t \downarrow a_i} y_{\alpha\beta}(t); \beta \in B_i\} = \mathbf{R};$$

and (ii) in case $\lambda_i \geq 2$,

$B_i^- = (\beta_i, \infty)$, $B_i = \{\beta_i\}$, $B_0^+ = (-\infty, \beta_i)$ for some $\beta_i = \beta(\alpha)$ and $y_{\alpha\beta(\alpha)}(a_i) = b_i$.

And then we solve

$$(7) \qquad \begin{cases} g(t)(\dfrac{d^2y}{dt^2} + \dfrac{1}{t}\dfrac{dy}{dt}) = f(t,y) \quad \text{in} \quad [t_0, a_{i+1}) \\ y(t_0) = \alpha, \quad \dfrac{dy}{dt}(t_0) = \beta(\alpha). \end{cases}$$

Define

$$A_i^+ = \{\alpha \in \mathbf{R}; \lim_{t\uparrow T_\alpha} y_\alpha(t) = +\infty\}$$

$$A_i = \{\alpha \in \mathbf{R}; \lim_{t\uparrow a_{i+1}} y_\alpha(t) = \text{exists}\}$$

$$A_i^- = \{\alpha \in \mathbf{R}; \lim_{t\uparrow T_\alpha} y_\alpha(t) = -\infty\},$$

where $y_\alpha(t) := y_{\alpha\beta(\alpha)}(t)$ is a solution of (7) on $[t_0, T_\alpha)$ $(t_0 < T_\alpha \leq a_{i+1})$.

We observe that (i) in case $0 < \lambda_{i+1} < 2$,

$$A_i^+ = [\overline{\alpha}, \infty), \quad A_i = (\underline{\alpha}, \overline{\alpha}), \quad A_i^- = (-\infty, \underline{\alpha}]$$

$$\text{and} \quad \{y_\alpha(a_{i+1}) = \lim_{t\uparrow a_{i+1}} y_\alpha(t); \alpha \in A_i\} = \mathbf{R};$$

and (ii) in case $\lambda_{i+1} \geq 2$,

$A_i^+ = (\alpha_i, \infty)$, $A_i = \{\alpha_i\}$, $A_i^- = (-\infty, \alpha_i)$ for some α_i, and $y_{\alpha_i}(a_{i+1}) = b_{i+1}$.

Therefore we have

PROPOSITION 2. *There exists a unique classical solution $y_i(t)$ of* ($\mathbf{BVP_i}$) $(i = 1, 2, \cdots, k-1)$.

(Note that the uniqueness in Propositions 1 and 2 follows immediately from the maximum principle.)

Finally we shall prove the existence and uniqueness of solutions of ($\mathbf{BVP_k}$). It should be noted that we have to introduce several boundary conditions at ∞ corresponding to the structure of (1). To state our result, we introduce some notation:

$$\ell := \lambda_1 + \lambda_2 + \cdots + \lambda_k - \lambda^* \quad \text{and} \quad \gamma := (\ell - 2)/(p-1),$$

where $p > 1$ is assumed. For $(\mathbf{BVP_k})$, we make the following assumptions:

(A.4) $\lim_{|x|\to\infty} \varphi(|x|) = \infty$ and $\lim_{t\to\infty} \dfrac{t^\ell(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = 0.$

(A.5) There exist positive constants k_0 and K_0 such that

$$k_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1}) \le f(t, y_1) - f(t, y_2)$$

$$\le K_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1})$$

for every $y_1 > y_2$ and $t >> 1$.

(A.6) $f(|x|, y)$ has the following form:

$$f(|x|, y) = y\,|y|^{p-1} - \varphi(|x|)\,|\varphi(|x|)|^{p-1}.$$

REMARK (i) It is easy to verify that $(\mathbf{A.6}) \Rightarrow (\mathbf{A.5}) \Rightarrow \{(\mathbf{A.1}), (\mathbf{A.2})\}$.

(ii) If $\lim_{t\to\infty} \dfrac{t^\ell(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = \delta > 0$ and $\lim_{t\to\infty} \dfrac{\varphi(t)}{t^\gamma} = \infty$, then $\varphi(t)$ blows up in a finite interval.

PROPOSITION 3. *Let $\ell \le 2$. Assume* (**A.4**) *and* (**A.5**). *Then there exists a unique solution of* $(\mathbf{BVP_k})$ *with $h(t) \approx \varphi(t)$. Moreover, if $h(t) \not\approx \varphi(t)$ then* $(\mathbf{BVP_k})$ *does not possess any solution, where $h(t) \approx \varphi(t)$ means that* $\lim_{t\to\infty} \dfrac{h(t)}{\varphi(t)} = 1.$

Sketch of proof of Proposition 3. Let $y_\alpha(t)$ be a classical solution of (4) in $[a_k, T_\alpha)$ satisfying $y_\alpha(a_k) = b_k$. Then, it is important to note that $\lim_{t\to T_\alpha} y_\alpha(t) = +\infty$ or $\lim_{t\to T_\alpha} y_\alpha(t) = -\infty$. (In other words, equation (4) does not possess any bounded solution.) Therefore, as before, we define

$$A^+ = \{\alpha \in \mathbf{R};\ \lim_{t\to T_\alpha} y_\alpha(t) = +\infty\}$$

$$A^- = \{\alpha \in \mathbf{R};\ \lim_{t\to T_\alpha} y_\alpha(t) = -\infty\},$$

where $T_\alpha \le \infty$. It is shown that $(1) A^+ \ne \emptyset, (2) A^- \ne \emptyset, (3) A^+ \bigcup A^- = \mathbf{R}$, and $(4) \alpha_1 < \alpha_2$ if $\alpha_1 \in A^-$ and $\alpha_2 \in A^+$. Hence, the cut $\overline{\alpha} = (A^-, A^+)$ is determined. Using (**A.4**), we have $A^- = (-\infty, \overline{\alpha}), A^+ = [\overline{\alpha}, \infty)$ and $T_{\overline{\alpha}} = \infty$. We can show that $\lim_{t\to\infty} \dfrac{y_{\overline{\alpha}}(t)}{\varphi(t)} = 1$ and the uniqueness of solutoins of $(\mathbf{BVP_k})$ with $h(t) \approx \varphi(t)$ holds.

In a similar spirit, we have

Proposition 4. *Let $\ell > 2$. Assume* (**A.4**), (**A.5**) *and* $\lim_{t\to\infty} \frac{\varphi(t)}{t^\gamma} = \infty$. *Then the assertions as in Proposition 3 are valid.*

Now, it remains to consider the case $\lim_{t\to\infty} \frac{\varphi(t)}{t^\gamma} = \kappa \quad (0 < \kappa < \infty)$ under the assumptions (**A.4**) and (**A.6**). In this case we may assume

$$g(t)^{-1} = t^{-\ell} + g_1(t)t^{-\ell}, \quad | g_1(t) | \leq K_1/t$$

$$\varphi(t)^p = \kappa^p t^{\gamma p} + \varphi_1(t)t^{\gamma p}, \quad | \varphi_1(t) | \leq K_1/t$$

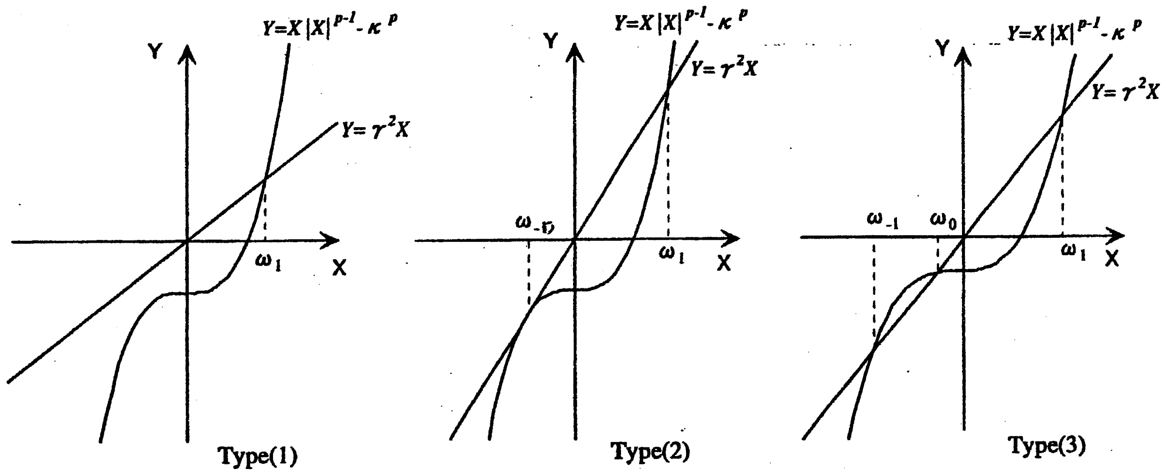
for every $t >> 1$. Putting $y(t) = t^\gamma w(t)$, we get a new **ODE** for $w(t)$:

$$\frac{d^2w}{dt^2}(t) + \frac{2\gamma+1}{t}\frac{dw}{dt}(t) = \frac{1}{t^2}\{w|w|^{p-1} - \kappa^p - \gamma^2 w\} + (\text{lower term}), \tag{8}$$

where

$$\text{lower term} = \frac{1}{t^2}\{g_1(t)(w|w|^{p-1} - \kappa^p) - (1 + g_1(t))\varphi_1(t)\}.$$

Then we have 3 types such that



We have to introduce various boundary functions $h(| x |)$ corresponding to Type (1) - Type (3). In what follows, we will focus on Type (3), because Type (3) is the most interesting case. In this case, we first note that every solution $w(t)$ of (8) with infinite life span converges to the one of $\{w_{-1}, w_0, w_1\}$. From this it follows that if $y(t)$ is a solution of (4) in (a_k, ∞) with infinite life span, then $y(t)/t^\gamma$ converges to the one of $\{w_{-1}, w_0, w_1\}$ as $t \to \infty$. Define

$$\begin{aligned}
A^+ &= \{\alpha \in \mathbf{R};\ \lim_{t\uparrow T_\alpha} y_\alpha(t) = +\infty \quad \text{and} \quad T_\alpha < \infty\} \\
A_1 &= \{\alpha \in \mathbf{R};\ \lim_{t\to\infty} y_\alpha(t)/t^\gamma = w_1\} \\
A_0 &= \{\alpha \in \mathbf{R};\ \lim_{t\to\infty} y_\alpha(t)/t^\gamma = w_0\} \\
A_{-1} &= \{\alpha \in \mathbf{R};\ \lim_{t\to\infty} y_\alpha(t)/t^\gamma = w_{-1}\} \\
A^- &= \{\alpha \in \mathbf{R};\ \lim_{t\uparrow T_\alpha} y_\alpha(t) = -\infty \quad \text{and} \quad T_\alpha < \infty\}.
\end{aligned}$$

LEMMA 5. *$A^- = (-\infty, \alpha_*)$, $A_{-1} = \{\alpha_*\}$, $A_0 = (\alpha_*, \alpha^*)$, $A_1 = \{\alpha^*\}$ and $A^+ = (\alpha^*, \infty)$.*

Using this lemma, we have

PROPOSITION 6. *Type (1) $\Longrightarrow$ $\exists$ a unique solution of* $(\mathbf{BVP_k})$ *with $h(t) \approx w_1 t^\gamma$.*
Type (2) $\Longrightarrow$ $\exists$ a unique solution of $(\mathbf{BVP_k})$ *with $h(t) \approx w_1 t^\gamma$ and $\exists$ infinitely many solutions of* $(\mathbf{BVP_k})$ *with $h(t) \approx w_0 t^\gamma$.*
Type (3) $\Longrightarrow$ $\exists$ a unique solution of $(\mathbf{BVP_k})$ *with $h(t) \approx w_1 t^\gamma$, $\exists$ infinitely many solutions of* $(\mathbf{BVP_k})$ *with $h(t) \approx w_0 t^\gamma$, and $\exists$ a unique solution of* $(\mathbf{BVP_k})$ *with $h(t) \approx w_{-1} t^\gamma$.*

REMARK In the case where $\lim_{t\to\infty} y(t)/t^\gamma = w_0$, if the above boundary condition is replaced by stronger another condition, then we can prove the uniqueness of solutions of $(\mathbf{BVP_k})$. In fact, let $\alpha \in A_0 = (\alpha_*, \alpha^*)$ and y_α be a solution of $(\mathbf{BVP_k})$. Then, for every $\sigma \in \mathbf{R}$, there exists a unique $q_\sigma(t) = O(t^{\gamma-1})$(as $t \to \infty$) such that

$$\lim_{t\to\infty} \frac{y_\alpha(t) - \{w_0 t^\gamma + q_\sigma(t)\}}{t^{\delta_1}} = \sigma,$$

where $\delta_1 = \sqrt{p|w_0|^{p-1}}$, $\delta_1 \neq 0$.
Let $\delta_1 = 0$. For for every $\sigma \in \mathbf{R}$, there exists a unique $q_\sigma(t) = O(t^{\gamma-1})$(as $t \to \infty$) such that

$$\lim_{t\to\infty} \frac{y_\alpha(t) - q_\sigma(t)}{\log t} = \sigma.$$

Main result

(I) Let $l \leq 2$ and $\lambda_i > 0 \quad (i = 1, 2, \cdots, k)$. Assume **(A.4)** and **(A.5)**. Then there exists a unique standard radial viscosity solution u of **(DP)** with boundary function

$h(|\ x\ |) \approx \varphi(|\ x\ |)$. Moreover, if $h(|\ x\ |) \not\approx \varphi(|\ x\ |)$ then **(DP)** does not possess any standard radial viscosity solution of **(DP)**.

(II) Let $l > 2$ and $\lambda_i > 0 \quad (i = 1, 2, \cdots, k)$. Assume **(A.4)**,**(A.5)** and $\lim_{t\to\infty} \varphi(t)/t^\gamma = +\infty$. Then the assertions of **(I)** are also valid.

(III) Let $l > 2$ and $\lambda_i > 0 \quad (i = 1, 2, \cdots, k)$. Assume **(A.4)**,**(A.6)** and $\lim_{t\to\infty} \varphi(t)/t^\gamma = \kappa(> 0)$. Then the same results for standard radial viscosity solutions of **(DP)** as those in Proposition 6 hold. Of course, boundary functions are replaced by

$$h(|\ x\ |) \approx w_1 \mid x \mid^\gamma \quad \text{in case Type(1);}$$

$$h(|\ x\ |) \approx w_i \mid x \mid^\gamma \quad (i \in \{0, 1\}) \quad \text{in case Type(2);}$$

$$h(|\ x\ |) \approx w_i \mid x \mid^\gamma \quad (i \in \{-1, 0, 1\}) \quad \text{in case Type(3).}$$

In particular, in the case where $h(|\ x\ |) \approx w_0 \mid x \mid^\gamma$, the boundary condition at ∞ is replaced by

$$\lim_{|x|\to\infty} \frac{u(x) - \{w_0 \mid x \mid^\gamma + q_\sigma(|\ x\ |)\}}{|\ x\ |^{\delta_1}} = \sigma,$$

where $\delta_1 = \sqrt{p|w_0|^{p-1}}, \delta_1 \neq 0$.
If $\delta_1 = 0$, then the boundary condition at ∞ is represented with

$$\lim_{|x|\to\infty} \frac{u(x) - q_\sigma(|\ x\ |)}{\log \mid x \mid} = \sigma.$$

(IV) If $\lambda_i \geq 1$ for all $i \in \{1, 2, \cdots, k\}$, then the uniqueness of viscosity solutions of **(DP)** holds. Hence, every viscosity solution of **(DP)** is radial.

References

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