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NONLOCAL NONLINEAR SYSTEMS OF TRANSPORT EQUATIONS IN WEIGHTED L^1 SPACES: AN OPERATOR THEORETIC APPROACH

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1. INTRODUCTION

This report is concerned with a nonlocal nonlinear transport system of the form

$$\text{(NNS)} \qquad \begin{cases} \partial_t \boldsymbol{u} + z'(t)\partial_x \boldsymbol{u} = \boldsymbol{\varphi}(t,x,\boldsymbol{u},z(t)), & (t,x) \in (0,T)\times\mathbb{R}, \\ z(t) = L\Big(\displaystyle\int_{-\infty}^{+\infty} \boldsymbol{w}(x)\cdot\boldsymbol{u}(t,x)dx\Big), & t\in[0,T]. \end{cases}$$

Here $\boldsymbol{u} \equiv (u^i)_{i=1}^N \colon [0,T]\times\mathbb{R}\to\mathbb{R}^N$ and $z\colon [0,T]\to\mathbb{R}$ are unknown, $0<T<\infty$ is arbitrary, N is a given positive integer and z' stands for the time derivative of z. The left-hand side of the evolution equation in (NNS) is called the material derivative of $\boldsymbol{u}$ and governed by a function $\boldsymbol{\varphi}\equiv(\varphi^i)_{i=1}^N\colon [0,T]\times\mathbb{R}\times\mathbb{E}\times\mathbb{R}\to\mathbb{R}^N$. The set $\mathbb{E}$ is defined as $\{\boldsymbol{v}=(v^i)_{i=1}^N\in\mathbb{R}^N \mid v^i\ge 0 \text{ and } \sum_{i=1}^N v^i \le 1\}$ and $\boldsymbol{\varphi}$ is assumed to be continuous in $(t,\boldsymbol{u},z)$; $\boldsymbol{\varphi}$ need not be continuous in x. The function z is represented as a nonlocal nonlinear term determined by an $\mathbb{R}$-valued, continuous and decreasing function L on an open interval (a,b) and an $\mathbb{R}^N$-valued weight function $\boldsymbol{w}\equiv(w^i)_{i=1}^N$ on $\mathbb{R}$. Accordingly, solutions $\boldsymbol{u}$ to (NNS) are sought in such a way that $\boldsymbol{u}(t,x)\in\mathbb{E}$ for a.e. $x\in\mathbb{R}$ and $a<\int_{-\infty}^{+\infty}\boldsymbol{w}(x)\cdot\boldsymbol{u}(t,x)dx<b$ for $t\in[0,T]$.

In case of $N=4$, Comincioli et al. [10] have shown the existence and uniqueness of classical solutions to (NNS) for the following case: The function $\boldsymbol{\varphi}$ has the form

$$\varphi^i(t,x,u^1,u^2,u^3,u^4,z) = \sum_{j=i\pm1}\big[a_{ij}(t,x)u^j - a_{ji}(t,x)u^i\big], \quad i=1,\,2,\,3,\,4,$$

which is linear in $\boldsymbol{u} = (u^1, u^2, u^3, u^4)$ and is smooth in (t, x), $\boldsymbol{w}(x) = (0, 0, x - \delta, x)$ (δ a given constant) and

$$L(\tau) = -\log(1+\tau) + \log\Big(1 + \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx\Big),$$

$a = -1$, $b = +\infty$, where $\boldsymbol{u}_0(\cdot)$ is an initial-function.

The system (NNS) is regarded as a mathematical model which describes the cross-bridge mechanism in muscle contraction, if N, φ, L and $\boldsymbol{w}$ are specified in an appropriate way and an initial condition

(IC) $$\boldsymbol{u}(0, x) = \boldsymbol{u}_0(x), \quad x \in \mathbb{R}$$

is imposed in such a way that the initial-function $\boldsymbol{u}_0$ is compactly supported and satisfies the compatibility condition

(I) $\boldsymbol{u}_0(x) \in \mathbb{E}$ a.e. in $\mathbb{R}$ and $a < \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{u}_0(x)dx < b$.

In order to formulate more reasonable models, it is preferable that the function φ and initial-function $\boldsymbol{u}_0$ should be nonsmooth and even discontinuous. Therefore it is not always expected to obtain classical solutions to the initial-value problem (NNS)–(IC).

The general class of (NNS) can be treated, but we here focus our attention on the so-called four-state cross-bridge model. Our objective are introduce a notion of weak solution to the evolution problem (NNS)–(IC) for the case $N = 4$ and discuss the uniqueness and global existence of the weak solutions under suitable assumptions on $\boldsymbol{w}$, φ, L and condition (I).

For the model equations for the two-state cross-bridge model and other models, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 19, 20] and the references therein.

The plan of this report is as follows. In Section 2 we state assumptions on the data of (NNS) and our main results. In Section 3 we investigate the semilinear evolution equation, which is the first equation for given $z(\cdot)$. In addition, we reduce the initial-value problem (NNS)–(IC) there. In Section 4 we demonstrate the existence result by the fixed point argument. In Section 5 we prove the uniqueness result.

We give only outlines of our discussion in the report. For the details and more general assumptions on the data of (NNS), we refer to [20].

2. Main results

In this section we mention assumptions on the data, definition of weak solution to (NNS)–(IC) and existence and uniqueness results through an abstract framework.

First, we put the following condition for the weight function $\boldsymbol{w} \equiv (w^1, w^2, w^3, w^4)$.

(W) $w^1(x) = w^2(x) \equiv 0$, and $w^3(x)$ and $w^4(x)$ are strictly increasing and bi-Lipschitz continuous over $\mathbb{R}$.

Each component $u^i(t,x)$ of the unknown function $\boldsymbol{u}(t,x)$ represents the density of cross-bridges of the position x in the ith state at t. In addition, it is required that the function $x \mapsto \boldsymbol{w}(x) \cdot \boldsymbol{u}(t,x)$ is integrable for the nonlocal term in (NNS) to make sense. Hence it is convenient to employ the following types of weighted L^1 spaces:

$$L^1(w^i) = \left\{ v \colon \mathbb{R} \to \mathbb{R} \,\middle|\, \text{measurable and } \int_{-\infty}^{+\infty} |v(x)|(1+|w^i(x)|)dx < \infty \right\},$$
$$|v|_{w^i} = \int_{-\infty}^{+\infty} |v(x)|(1+|w^i(x)|)dx.$$

In order to treat our problem in an operator theoretic fashion, we introduce the product space

$$X = L^1(w^1) \times L^1(w^2) \times L^1(w^3) \times L^1(w^4),$$
$$\|\boldsymbol{v}\| = |v^1|_{w^1} + |v^2|_{w^2} + |v^3|_{w^3} + |v^4|_{w^4} \text{ for } \boldsymbol{v} = (v^1, v^2, v^3, v^4) \in X.$$

Furthermore, we have to introduce the weighted Sobolev spaces $W^{1,1}(w^i)$ and the "weighted L^∞ spaces" $L^\infty(w^i)$:

$$W^{1,1}(w^i) := \{ v \in L^1(w^i) \mid v' \in L^1(w^i) \}, \quad |v|^{1,1}_{w^i} := |v|_{w^i} + |v'|_{w^i};$$
$$L^\infty(w^i) := \left\{ v \colon \mathbb{R} \to \mathbb{R} \,\middle|\, \text{measurable and } |v(x)| \le \frac{C}{1+|w(x)|} \text{ a.e. for some } C > 0 \right\},$$
$$\|v\|_{w^i} := \operatorname*{ess.\,sup}_{x \in \mathbb{R}} |v(x)|(1+|w(x)|).$$

We then refer to standard four-state linear models and consider a typical case in which the nonlinear function $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ is of the following form:

$$\varphi^i(t, x, u^1, u^2, u^3, u^4, z) = \sum_{j=i\pm1} \left[a_{ij}(t,x)(u^j)^{p_{ij}} - a_{ji}(t,x)(u^i)^{q_{ji}}\right], \quad i = 1,\ 2,\ 3,\ 4.$$

Here we introduce the cyclic rule on the indices: $i \equiv j \pmod 4$, that is, for instance, $5 \equiv 1$ and $0 \equiv 4$. Furthermore, the functions $a_{i,i\pm1}(t,x)$, $i = 1, 2, 3, 4$, have the forms

$$a_{i,i\pm1}(t,x) = \begin{cases} f_{i,i\pm1}(x), & i = 1,\ 2, \\ \gamma(t) f_{i,i\pm1}(x), & i = 3,\ 4, \end{cases}$$

and the functions $\gamma(t)$ and $f_{i,i\pm1}(x)$, $i = 1, 2, 3, 4$, satisfy the condition (F1) below:

(F1) γ, $f_{i,i\pm1}$ are all nonnegative, $\gamma \in C([0,T])$, $f_{i,i\pm1} \in L^1(w^i) \cap L^\infty(\mathbb{R})$, $i = 1, 2$, $f_{34} \in L^1(w^3) \cap L^\infty(\mathbb{R})$, $f_{32} \in L^1(w^3) \cap L^\infty(w^3)$, $f_{43} \in L^1(w^4) \cap L^\infty(\mathbb{R})$ and $f_{41} \in L^1(w^4) \cap L^\infty(w^4)$. Moreover, the powers $p_{i,i\pm1}$ and $q_{i,i\pm1}$ of nonlinearity satisfy $p_{i,i\pm1} \geq q_{i,i\pm1} \geq 1$, $i = 1, 2, 3, 4$.

On the function L, we impose the following condition which implies the maximal monotonicity of $-L^{-1}$ and is stronger than the local Lipschitz continuity of L:

(L) $-\infty \leq a < b \leq +\infty$, $L \in C(a,b)$ is strictly decreasing and satisfies $L(a+0) = +\infty$ and $L(b-0) = -\infty$. Furthermore, to each $r > 0$ there corresponds $\beta_r > 0$ such that

$$(1 + \lambda\beta_r)|L(\tau_1) - L(\tau_2)| \leq |L(\tau_1) - L(\tau_2) - \lambda(\tau_1 - \tau_2)|$$
$$\text{for } \lambda > 0 \text{ and } \tau_1,\ \tau_2 \in [L^{-1}(r), L^{-1}(-r)].$$

The above-mentioned evolution problem may be reformulated in an operator theoretic manner. To this end, we first define

$$(S(\sigma)v)(x) := v(x - \sigma) \text{ for } x \in \mathbb{R},\ v \in X,\ \sigma \in \mathbb{R}.$$

Then the one-parameter family $\{S(\sigma)\}_{\sigma\in\mathbb{R}}$ is a C_0-group in X of type ω: $\|S(\sigma)\| \leq e^{\omega|\sigma|}$ for $\sigma \in \mathbb{R}$, where

$$\omega = \max_{1\leq i\leq 4} \operatorname{ess.sup}_{x\in\mathbb{R}} \frac{(w^i)'(x)}{1 + |w^i(x)|},$$

and its generator $-\Lambda\colon \mathcal{D}(\Lambda) \subset X \to X$ is given by

$$\begin{aligned} \mathcal{D}(\Lambda) &= W^{1,1}(w^1) \times W^{1,1}(w^2) \times W^{1,1}(w^3) \times W^{1,1}(w^4), \\ \Lambda \boldsymbol{v} &= ((v^1)', (v^2)', (v^3)', (v^4)') \text{ for } \boldsymbol{v} = (v^1, v^2, v^3, v^4) \in \mathcal{D}(\Lambda). \end{aligned} \tag{2.1}$$

We also define a continuous linear functional $\mathfrak{f}$ on X by

$$\mathfrak{f}(\boldsymbol{v}) = \int_{-\infty}^{+\infty} \boldsymbol{w}(x) \cdot \boldsymbol{v}(x) dx.$$

In addition, we put $D = \{\boldsymbol{v} \in X \mid \boldsymbol{v}(x) \in \mathbb{E} \text{ a.e.}\}$ and define a nonlinear mapping $F\colon [0,T] \times D \times \mathbb{R} \to X$ by

$$F(t, \boldsymbol{u}, z) = \varphi(t, \cdot, \boldsymbol{u}(\cdot), z) \text{ for } (t, \boldsymbol{u}, z) \in [0,T] \times D \times \mathbb{R}. \tag{2.2}$$

We then can rewrite (NNS) to the following nonlinear evolution system in X

$$\begin{cases} \boldsymbol{u}'(t) + z'(t)\Lambda \boldsymbol{u}(t) = F(t, \boldsymbol{u}(t), z(t)), & t \in (0,T), \\ z(t) = L(\mathfrak{f}(\boldsymbol{u}(t))), & t \in [0,T]. \end{cases}$$

We now formulate a notion of weak solution to the problem (NNS)–(IC).

Definition. A pair of functions $(z, \boldsymbol{u}) \in C([0,T]) \times C([0,T]; X)$ is called a *weak solution* to (NNS)–(IC), if $\boldsymbol{u}(t) \in D$ and $a < \mathfrak{f}(\boldsymbol{u}(t)) < b$ for $t \in [0,T]$, and

$$\begin{aligned} &\boldsymbol{u}(t) = S(z(t) - z(0))\boldsymbol{u}_0 + \int_0^t S(z(t) - z(\tau))F(\tau, \boldsymbol{u}(\tau), z(\tau))d\tau, \\ &z(t) = L(\mathfrak{f}(\boldsymbol{u}(t))), \qquad t \in [0,T], \end{aligned}$$

are satisfied.

Our existence theorem may be stated as follows:

Theorem 1 (existence). *Assume that* (W), (F1) *and* (L) *hold. Let* $\boldsymbol{u}_0 \in X$ *satisfy* (I). *Then there exists a weak solution* $(z, \boldsymbol{u})$ *to* (NNS)–(IC) *such that the functions* $z(t)$ *and* $\mathfrak{f}(\boldsymbol{u}(t))$ *are Lipschitz continuous on* $[0,T]$.

In order to obtain a uniqueness theorem, we necessitate imposing an additional condition on φ as stated below.

(F2) For any $r > 0$ there is a constant $C_r > 0$ such that

$$\int_{-\infty}^{+\infty} |f_{ij}(x + \sigma_1) - f_{ij}(x + \sigma_2)|(1 + |w^i(x)| + |w^j(x)|)dx \le C_r|\sigma_1 - \sigma_2|$$

$$\text{for } |\sigma_1|,\ |\sigma_2| \le r,\ j = i \pm 1 \text{ and } i = 1,\ 2,\ 3,\ 4.$$

Theorem 2 (uniqueness). *Assume* (F2) *in addition to* (W), (F1) *and* (L). *If* (z_j, u_j), $j = 1, 2$, *are weak solutions to* (NNS), *then we have*

$$|z_1 - z_2|_\infty \le C\|S(-z_1(0))u_1(0) - S(-z_2(0))u_2(0)\|, \tag{2.3}$$

where C *is a constant which may depend on* $|z_j|_\infty$, $\|u_j(0)\|$, $j = 1, 2$. *In particular, weak solutions to* (NNS) *are uniquely determined by the initial data.*

These theorems are proved in Sections 4 and 5.

3. Semilinear evolution equations

This section is devoted to solving the semilinear evolution equations in X for given function $z(\cdot)$:

$$u' + z'(t)\Lambda u = F(t, u, z(t)), \qquad t \in (0, T). \tag{SE;z}$$

Here Λ is the linear operator defined by (2.1) and F the nonlinear mapping defined by (2.2). We also reduce the evolution problem (NNS)–(IC) in the last of this section.

For each $z \in W^{1,\infty}(0, T)$ and almost all $t \in (0, T)$, define a linear operator $A_z(t)$ in X by

$$\mathcal{D}(A_z(t)) := \begin{cases} \mathcal{D}(\Lambda), & \text{if } z'(t) \ne 0, \\ X, & \text{if } z'(t) = 0, \end{cases} \qquad A_z(t) := -z'(t)\Lambda.$$

Moreover, for each $z \in C([0, T])$, put $U_z(t, s) = S(z(t) - z(s))$, t, $s \in [0, T]$, where $\{S(\sigma)\}_{\sigma\in\mathbb{R}}$ is the C_0-group generated by $-\Lambda$. Then we easily obtain the following proposition.

Proposition 3.1. *Let* $z \in C([0, T])$. *Then the two-parameter family* $\{U_z(t, s)\}_{t,s\in[0,T]}$ *of continuous linear operators in* X *satisfies the following properties.*

(i) $(t, s) \mapsto U_z(t, s)$ *is* X*-strongly continuous on* $[0, T] \times [0, T]$.

(ii) $U_z(t, s)U_z(s, r) = U_z(t, r)$, $U_z(s, s) = I$ *for any* r, s, $t \in [0, T]$.

(iii) $U_z(t, s)Y \subset Y$, *and* $(t, s) \mapsto U_z(t, s)$ *is* Y*-strongly continuous on* $[0, T] \times [0, T]$, *where* $Y := \mathcal{D}(\Lambda)$ *is endowed with the graph-norm of* Λ.

(iv) *If* $z \in W^{1,\infty}(0,T)$ *and* $u \in Y$, *then*

$$U_z(t,s)u - u = \int_s^t A_z(\tau)U_z(\tau,s)u d\tau = \int_s^t U_z(t,\tau)A_z(\tau)u d\tau, \quad (t,s) \in [0,T] \times [0,T].$$

(v) *The operator* $U_z(t,s)$ *is invertible and* $U_z(t,s)^{-1} = U_z(s,t)$ *for any* t, $s \in [0,T]$. *Thus,* $\{U_z(t,s)\}_{t,s\in[0,T]}$ *is a unique evolution operator in* X *generated by* $\{A_z(t)\}_{t\in[0,T]}$.

Let $0 \le s < \varsigma \le T$ and $z \in C([s,\varsigma])$. A function $u \in C([s,\varsigma];X)$ is said to be a *weak solution* to (SE;z) on $[s,\varsigma]$, if $u(t) \in D$ and the following integral equation is satisfied:

$$u(t) = S(z(t)-z(s))u(s) + \int_s^t S(z(t)-z(\tau))F(\tau,u(\tau),z(\tau))d\tau, \quad t \in [s,\varsigma].$$

We easily have the following proposition by (F1) and (2.2).

Proposition 3.2. *The continuous mapping* $F\colon [0,T] \times D \times \mathbb{R} \to X$ *defined by* (2.2) *has the following properties.*

(i) F *is Lipschitz continuous in* u: *there is a constant* K *such that*

$$\|F(t,u,z) - F(t,v,z)\| \le K\|u-v\| \text{ for } t \in [0,T],\ u,\ v \in D \text{ and } z \in \mathbb{R};$$

(ii) F *satisfies the so-called subtangential condition:*

$$\liminf_{h\downarrow 0} h^{-1}d(u + hF(t,u,z), D) = 0 \text{ for } (t,u,z) \in [0,T] \times D \times \mathbb{R},$$

where $d(v,D)$ *stands for the distance from* v *to* D, *that is,* $d(v,D) = \inf_{u\in D}\|v-u\|$;

(iii) F *grows at most linearly in* u: *there are a constant* M *and an* X*-valued function* $\mathcal{F} \in C([0,T];X_+)$ *such that*

$$-Mu \leqslant F(t,u,z) \leqslant \mathcal{F}(t) + Mu \text{ in } X \text{ for } (t,u,z) \in [0,T] \times D \times \mathbb{R}.$$

Here $\leqslant$ *denotes the standard order relation in* X *and* X_+ *the positive cone of* X.

Our first goal is to prove the following theorem.

Theorem 3.3. *Let $0 \le s < \varsigma \le T$, $z \in C([s,\varsigma])$ and $\boldsymbol{u}_s \in D$. Then the initial-value problem for* (SE;z) *on* $[s,\varsigma]$ *with initial condition $\boldsymbol{u}(s) = \boldsymbol{u}_s$ possesses a unique weak solution $\boldsymbol{u}_z$.*

Proof. We employ the method of characteristic line. Setting $\boldsymbol{v}_z(t) := S(-z(t))\boldsymbol{u}_z(t)$, we reduce the problem for (SE;z) with $\boldsymbol{u}_z(s) = \boldsymbol{u}_s$ to the initial-value problem for the following ordinary differential equation

$$\text{(ODE;}z\text{)} \qquad \boldsymbol{v}'(t) = S(-z(t))F(t, S(z(t))\boldsymbol{v}(t), z(t)), \quad t \in [s,\varsigma]$$

with initial data $S(-z(s))\boldsymbol{u}_s$ or equivalent integral equation

$$\boldsymbol{v}(t) = S(-z(s))\boldsymbol{u}_s + \int_s^t S(-z(\tau))F(\tau, S(z(\tau))\boldsymbol{v}(\tau), z(\tau)), \quad t \in [s,\varsigma].$$

Put $G(t,\boldsymbol{v}) := S(-z(t))F(t, S(z(t))\boldsymbol{v}, z(t))$ for $(t,\boldsymbol{v}) \in [s,\varsigma] \times D$. Then noting that $\{S(\sigma)\}_{\sigma\in\mathbb{R}}$ is a C_0-group in X, we can check that $G\colon [s,\varsigma] \times D \to X$ is continuous and quasi-dissipative in the following sense

$$(1 - \lambda C)\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \le \|\boldsymbol{v}_1 - \boldsymbol{v}_2 - \lambda[G(t,\boldsymbol{v}_1) - G(t,\boldsymbol{v}_2)]\| \text{ for } \lambda > 0,\ t \in [s,\varsigma],\ \boldsymbol{v}_1,\ \boldsymbol{v}_2 \in D.$$

Here C is a constant which depends on $\sup_{\tau\in[s,\varsigma]} |z(\tau)|$. We also see that G satisfies the subtangential condition:

$$\liminf_{h\downarrow 0} h^{-1} d(\boldsymbol{v} + hG(t,\boldsymbol{v}), D) = 0 \text{ for } t \in [s,\varsigma],\ \boldsymbol{v} \in D,$$

by definition of G and Proposition 3.2 (i) and (ii). Hence we may apply [17, Corollary 1.1], and get a unique classical solution $\boldsymbol{v}_z \in C([s,\varsigma]; D) \cap C^1([s,\varsigma]; X)$ to the initial-value problem for (ODE;z) on $[s,\varsigma]$ under the initial condition $\boldsymbol{v}_z(s) = S(-z(s))\boldsymbol{u}_s$. The function $\boldsymbol{u}_z(t) := S(z(t))\boldsymbol{v}_z(t)$ gives a desired, unique weak solution to the initial-value problem for (SE;z). □

We next define a continuous linear functional $\mathfrak{g}$ on X as follows

$$\mathfrak{g}(\boldsymbol{v}) = -\int_{-\infty}^{+\infty} \boldsymbol{w}'(x) \cdot \boldsymbol{v}(x)dx,$$

where $w'(x) = ((w^1)'(x), (w^2)'(x), (w^3)'(x), (w^4)'(x))$. Then it is clear that $\mathfrak{g}$ is the unique extension of $\mathfrak{f}\Lambda$ to X, and that for each $v \in X$

$$\mathfrak{f}(S(\sigma)v) = \mathfrak{f}(v) - \int_0^\sigma \mathfrak{g}(S(\tau)v)d\tau, \quad \sigma \in \mathbb{R}. \tag{3.1}$$

Lemma 3.4. *Let $0 \le s < \varsigma \le T$, $u_s \in D$, and let $u_z \in C([s,\varsigma]; D)$ be a weak solution to the initial-value problem for (SE;z) on $[s,\varsigma]$ with $u_z(s) = u_s$. Then $z \mapsto \mathfrak{f}u_z$ is a continuous mapping from $C([s,\varsigma])$ into itself, where $C([s,\varsigma])$ is equipped with the supremum-norm $|\cdot|_\infty$. In addition, if $z \in W^{1,\infty}(s,\varsigma)$, then we have $\mathfrak{f}(u_z(\cdot)) \in W^{1,\infty}(s,\varsigma)$ and*

$$(\mathfrak{f}u_z)'(t) = -z'(t)\mathfrak{g}(u_z(t)) + \mathfrak{f}F(t, u_z(t), z(t)) \text{ a.e. } t \in (s,\varsigma).$$

Proof. Suppose that $z_n \to z$ in $C([s,\varsigma])$ and that u_z and u are weak solutions to (SE;z_n) and (SE;z) with $u_n(s) = u(s) = u_s$, respectively. Put $v_n(t) = S(-z_n(t))u_n(t)$ and $v(t) = S(-z(t))u(t)$. Then v_n (resp. v) is a unique solution to (ODE;z_n) with $v_n(s) = S(-z_n(s))u_s$ (resp. (ODE;z) with $v(s) = S(-z(s))u_s$) as stated in the proof of Theorem 3.3. By definition of F and Proposition 3.2 (i) we see that

$$\begin{aligned}
&\|v_n(t) - v(t)\| \\
&\le \|[S(-z_n(s)) - S(-z(s))]u_s\| + C\int_s^\varsigma \|[S(z_n(\tau)) - S(z(\tau))]v(\tau)\|d\tau \\
&+ \int_s^\varsigma \|[S(-z_n(\tau)) - S(-z(\tau))]F(\tau, S(z(\tau))v(\tau), z(\tau))\|d\tau + C\int_s^t \|v_n(\tau) - v(\tau)\|d\tau, \\
&\qquad t \in [s,\varsigma],
\end{aligned}$$

where C is a constant which depends on $\sup_m |z_m|_\infty$. Using Gronwall's Lemma, and then taking the limit, we know that $v_n \to v$ in $C([s,\varsigma]; X)$ as $n \to \infty$. Moreover, it follows from (3.1) that

$$\begin{aligned}
|\mathfrak{f}(u_n(t)) - \mathfrak{f}(u(t))| &= |\mathfrak{f}(S(z_n(t))v_n(t)) - \mathfrak{f}(S(z(t))v(t))| \\
&\le \|\mathfrak{f}\|e^{\omega\hat{r}}\|v_n(t) - v(t)\| + \|\mathfrak{g}\|e^{\omega\hat{r}}|z_n(t) - z(t)|\|v(t)\|, \ t \in [s,\varsigma].
\end{aligned}$$

Here $\|\mathfrak{f}\|$ and $\|\mathfrak{g}\|$ denote the operator-norm of the continuous linear functionals $\mathfrak{f}$ and $\mathfrak{g}$ and $\hat{r} := \sup_m |z_m|_\infty$. Then taking the supremum over $[s,\varsigma]$ and the limit as $n \to \infty$, we know that $\mathfrak{f}u_n \to \mathfrak{f}u$ in $C([s,\varsigma])$, so the mapping $z \mapsto \mathfrak{f}u_z$ is continuous.

Next, let $z \in W^{1,\infty}(s,\varsigma)$. It is clear that for $v \in X$

$$\frac{d}{dt}\mathfrak{f}(S(z(t))v) = -z'(t)\mathfrak{g}(S(z(t))v) \quad \text{a.e. } (s,\varsigma)$$

holds by (3.1). Since the function $v_z(t) = S(-z(t))u_z(t)$ is a classical solution to (ODE;z), we see that

$$\begin{aligned}(\mathfrak{f}u_z)'(t) &= (\mathfrak{f}S(z(t)))'v_z(t) + \mathfrak{f}S(z(t))v_z'(t) \\ &= -z'(t)\mathfrak{g}(S(z(t))v_z(t)) + \mathfrak{f}S(z(t))S(-z(t))F(t, S(z(t))v_z(t), z(t)) \\ &= -z'(t)\mathfrak{g}(u_z(t)) + \mathfrak{f}F(t, u_z(t), z(t)) \quad \text{a.e. } (s,\varsigma),\end{aligned}$$

and hence $(\mathfrak{f}u_z)'(\cdot) \in L^\infty(s,\varsigma)$. □

The remain of this section is devoted to the reduction of the initial-value problem for (NNS) to equivalent problems. Given $u_s \in X$, consider the following initial-value problems: Seek $z \in C([s,\varsigma])$ satisfying the following nonlinear constraint

(NC) $$a < \mathfrak{f}(u_z(t)) < b \text{ and } z(t) = L(\mathfrak{f}(u_z(t))), \quad t \in [s,\varsigma],$$

and $u_z(s) = u_s$; Seek $z \in C([s,\varsigma])$ satisfying the following functional equation

(FE) $$z(t) = (I - \lambda L^{-1})^{-1}(z(t) - \lambda\mathfrak{f}(u_z(t))), \quad t \in [s,\varsigma]$$

for some $\lambda > 0$, independent of t, and $u_z(s) = u_s$. Here u_z is a unique weak solution to the initial-value problem for (SE;z) on $[s,\varsigma]$ with $u_z(s) = u_s$, which is obtained in Theorem 3.3, and I is the identity operator in $\mathbb{R}$. Note that an inverese mapping $(I - \lambda L^{-1})^{-1}(\cdot)$ of $I - \lambda L^{-1}$ is defined on all of $\mathbb{R}$ as a single-valued function, since $-L^{-1}$ is maximal monotone.

Theorem 3.5. *Let* $0 \leq s < \varsigma \leq T$. *Under the initial condition* $\boldsymbol{u}(s) = \boldsymbol{u}_s$, *the initial-value problems for* (NNS), (NC) *and* (FE) *on* $[s, \varsigma]$ *are equivalent in the following sense:*
(i) *If* $(z, \boldsymbol{u})$ *is a weak solution to* (NNS), *then* z *is a solution to* (NC), *and* $\boldsymbol{u} \equiv \boldsymbol{u}_z$;
(ii) *If* z *is a solution to* (NC), *then* $(z, \boldsymbol{u}_z)$ *is a weak solution to* (NNS);
(iii) z *is a solution to* (NC) *if and only if this function is a solution to* (FE).
Here $\boldsymbol{u}_z$ *is a unique weak solution to the initial-value problem for* (SE;z) *on* $[s, \varsigma]$ *with initial data* $\boldsymbol{u}_s$, *which is obtained in Theorem* 3.3.

Proof. We easily see from definitions of solutions and Theorem 3.3 that (i) and (ii) hold. (iii) If $z \in C([s, \varsigma])$ satisfies that $a < \mathfrak{f}(\boldsymbol{u}_z(t)) < b$ and $z(t) = L(\mathfrak{f}(\boldsymbol{u}_z(t)))$ for $t \in [s, \varsigma]$, then $a < \mathfrak{f}(\boldsymbol{u}_z(t)) < b$ and $z(t) - \lambda \mathfrak{f}(\boldsymbol{u}_z(t)) = (I - \lambda L^{-1})(z(t))$ on $[s, \varsigma]$ for all $\lambda > 0$. Here note that $L \colon (a, b) \to \mathbb{R}$ is a bijection by (L). Therefore, it follows that $(I - \lambda L^{-1})^{-1}(z(t) - \lambda \mathfrak{f}(\boldsymbol{u}_z(t))) = z(t)$ on $[s, \varsigma]$ for all $\lambda > 0$. Conversely, if $z \in C([s, \varsigma])$ satisfies $(I - \lambda_0 L^{-1})^{-1}(z(t) - \lambda_0 \mathfrak{f}(\boldsymbol{u}_z(t))) = z(t)$ on $[s, \varsigma]$ for some $\lambda_0 > 0$, then it is evident that $a < \mathfrak{f}(\boldsymbol{u}_z(t)) < b$ and $z(t) = L(\mathfrak{f}(\boldsymbol{u}_z(t)))$ for $t \in [s, \varsigma]$. (Thus, $z(\cdot)$ satisfies $(I - \lambda L^{-1})^{-1}(z(t) - \lambda \mathfrak{f}(\boldsymbol{u}_z(t))) = z(t)$ on $[s, \varsigma]$ for all $\lambda > 0$.) □

Remark 3.6. We observe from the above theorem that if $(z, \boldsymbol{u}_z)$ is a weak solution to (NNS), then z is a fixed point of the mapping $z \mapsto (I - \lambda L^{-1})^{-1}(z(\cdot) - \lambda \mathfrak{f}(\boldsymbol{u}_z(\cdot)))$, and the converse is also true.

4. Fixed point argument

In this section we give sketch of proof of Theorem 1 by using Schauder's Fixed Point Theorem step by step in time.

We again have to define continuous linear functionals on X:

$$\mathfrak{h}(\boldsymbol{v}) = \sum_{i=3,4} \int_{-\infty}^{+\infty} v^i(x)dx, \quad \bar{\mathfrak{f}}(\boldsymbol{v}) = \sum_{i=3,4} \int_{-\infty}^{+\infty} |w^i(x)| v^i(x)dx \text{ for } \boldsymbol{v} = (v^1, v^2, v^3, v^4) \in X.$$

Then it is evident that

$$0 < C_1 \mathfrak{h}(\boldsymbol{v}) \leq -\mathfrak{g}(\boldsymbol{v}) \leq C_2 \mathfrak{h}(\boldsymbol{v}) \text{ whenever } \boldsymbol{v} = (v^1, v^2, v^3, v^4) \in X_+ \text{ and } (v^3, v^4) \neq 0,$$

where C_1 $=\min_{i=3,4}$ ess.inf$_{x\in\mathbb{R}}(w^i)'(x)$ and C_2 $=\max_{i=3,4}$ ess.sup$_{x\in\mathbb{R}}(w^i)'(x)$. In addition, put $\xi(t)=\sum_{i=3,4}\int_{-\infty}^{+\infty}|w^i(x)|\big(a_{i,i+1}(t,x)+a_{i,i-1}(t,x)\big)dx$. Then we have

$$|\mathfrak{f}F(t,u,z)|\le\xi(t)+M\bar{\mathfrak{f}}(u)\text{ for }(t,u,z)\in[0,T]\times D\times\mathbb{R}.$$

Here M is the same constant appeared in Proposition 3.2 (iii).

After a little long calculation we have the following technical estimates.

Lemma 4.1. *Let* $0\le s<\varsigma\le T$, $z\in C([s,\varsigma])$ *and* u_z *a weak solution to* (SE;z) *on* $[s,\varsigma]$. *Then we have:*

(i) $e^{-M(t-s)}\mathfrak{h}(u_z(s))\le\mathfrak{h}(u_z(t))\le e^{M(t-s)}\big(\mathfrak{h}(u_z(s))+\int_s^t\mathfrak{h}(\mathcal{F}(\tau))d\tau\big)$, $t\in[s,\varsigma]$.

(ii) $\mathfrak{g}(u_z(t))\le -C_1e^{-M(t-s)}\mathfrak{h}(u_z(s))$, $t\in[s,\varsigma]$.

(iii) *If* $z\in W^{1,\infty}(s,\varsigma)$, *then*

$$\bar{\mathfrak{f}}(u_z(t))\le e^{M(t-s)}\Big[\bar{\mathfrak{f}}(u_z(s))+\int_s^t\bar{\mathfrak{f}}(\mathcal{F}(\tau))d\tau$$
$$+C_2|z|_\infty(t-s)e^{M(t-s)}\Big(\mathfrak{h}(u_z(s))+\int_s^t\mathfrak{h}(\mathcal{F}(\tau))d\tau\Big)\Big],\ t\in[s,\varsigma].$$

Sketch of proof of Theorem 1. Owing to Theorem 3.5, it suffices to show an existence of a solution to (FE). We divided the proof into two steps.

Let $u_0\in D$ satisfy $a<\mathfrak{f}(u_0)<b$.

Step 1. In this step we assume that $u_0=(u_0^1,u_0^2,u_0^3,u_0^4)$ satisfies $(u_0^3,u_0^4)\ne 0$. Put

$$\lambda_1=\Big[C_2e^{MT}\Big(\mathfrak{h}(u_0)+\int_0^T\mathfrak{h}(\mathcal{F}(\tau))d\tau\Big)\Big]^{-1},\qquad \varrho_1=C_1e^{-MT}\mathfrak{h}(u_0),$$
$$\kappa_1=|\xi|_{L^\infty(0,T)}+Me^{MT}\Big(\bar{\mathfrak{f}}(u_0)+\int_0^T\bar{\mathfrak{f}}(\mathcal{F}(\tau))d\tau\Big)+Me^{MT}\lambda_1^{-1},$$
$$d_1=\varrho_1^{-1}\kappa_1,\qquad \varsigma_1=\min\{d_1^{-1},T\}.$$

Then $0<\varsigma_1\le T$ and $\varsigma_1\le d_1^{-1}$.

We define an operator $\Psi\colon\mathcal{K}_1\to C([0,\varsigma_1])$ by

$$\mathcal{K}_1=\{\zeta\in W^{1,\infty}(0,\varsigma_1)\mid\zeta(0)=L(\mathfrak{f}(u_0)),\ |\zeta'|_\infty\le d_1\},\tag{4.1}$$

$$(\Psi\zeta)(t)=(I-\lambda_1L^{-1})^{-1}(\zeta(t)-\lambda_1\mathfrak{f}(u_\zeta(t))),\ t\in[0,\varsigma_1]\text{ for }\zeta\in\mathcal{K}_1.\tag{4.2}$$

Here u_ζ is a unique weak solution to the initial-value problem for (SE;ζ) on $[0, \varsigma_1]$ with initial data u_0. It is easy to check that $\mathcal{K}_1$ is a compact, convex subset of $C([0, \varsigma_1])$ equipped with $|\cdot|_\infty$. Use Ascoli-Arzelà's Theorem to see the compactness.

We next show

Lemma 4.2. *The mapping $\Psi\colon \mathcal{K}_1 \to C([0, \varsigma_1])$ is well-defined and continuous.*

Proof. Since $-L^{-1}$ is maximal monotone in $\mathbb{R}$ by (L), the resolvent $(I - \lambda_1 L^{-1})^{-1}(\cdot)$ is defined on $\mathbb{R}$ as a single-valued function and is a contraction operator in $\mathbb{R}$:

$$(4.3) \qquad |(I - \lambda_1 L^{-1})^{-1}(\zeta_1) - (I - \lambda_1 L^{-1})^{-1}(\zeta_2)| \le |\zeta_1 - \zeta_2| \text{ for } \zeta_1,\ \zeta_2 \in \mathbb{R}.$$

Hence for $z \in \mathcal{K}_1$ we see that $(\Psi z)(\cdot) \in W^{1,\infty}(0, \varsigma_1)$ by definition of Ψ and Lemma 3.4. In particular, $\Psi\colon \mathcal{K}_1 \to C([0, \varsigma_1])$ is well-defined.

To see the continuity of Ψ, let z_n, $z \in \mathcal{K}_1$ and $|z_n - z|_\infty \to 0$. Then it follows from (4.3) and Lemma 3.4 that

$$|\Psi z_n - \Psi z|_\infty \le |z_n - z|_\infty + \lambda_1 |\mathfrak{f} u_{z_n} - \mathfrak{f} u_z|_\infty \to 0.$$

Consequently, Ψ is continuous. □

Furthermore, we obtain

Lemma 4.3. *The mapping Ψ has values in $\mathcal{K}_1$, that is, $\Psi \mathcal{K}_1 \subset \mathcal{K}_1$.*

Proof. Let $z \in \mathcal{K}_1$. We have shown that $\Psi z \in W^{1,\infty}(0, \varsigma_1)$ in the proof of the previous lemma. Since $u_z(0) = u_0$ and $L^{-1}(z(0)) = \mathfrak{f}(u_0)$, we see $(\Psi z)(0) = (I - \lambda_1 L^{-1})^{-1}(z(0) - \lambda_1 \mathfrak{f}(u_0)) = z(0) = L(\mathfrak{f}(u_0))$.

Let us show that $|(\Psi z)'|_\infty \le d$. Let $0 \le t_1 < t_2 \le \varsigma_1$. Then it follows from (4.3) and Lemma 3.4 that

$$|(\Psi z)(t_1) - (\Psi z)(t_2)| \le \int_{t_1}^{t_2} \Big[|z'(\tau)| |1 + \lambda_1 \mathfrak{g}(u_z(\tau))| + \lambda_1 |\mathfrak{f} F(\tau, u_z(\tau), z(\tau))| \Big] d\tau.$$

Using Lemma 4.1 (i) and (ii), we see that

$$0 \le 1 + \lambda_1 \mathfrak{g}(u_z(t)) \le 1 - \lambda_1 \varrho_1, \quad t \in [0, \varsigma_1].$$

Moreover, we get that

$$|\mathfrak{f}F(t,u_z(t),z(t))| \le \kappa_1, \quad t\in[0,\varsigma_1],$$

by Lemma 4.1 (iii). Consequently, we have

$$|(\Psi z)(t_1)-(\Psi z)(t_2)| \le [d_1(1-\lambda_1\varrho_1)+\lambda_1\kappa_1](t_2-t_1) = d_1(t_2-t_1),$$

which implies $|(\Psi z)'|_\infty \le d_1$ as desired. □

Since Lemmas 4.2 and 4.3 allow us to apply Schauder's Fixed Point Theorem, we get a fixed point $\hat{z}\in\mathcal{K}_1$ of Ψ. This $\hat{z}$ is a solution to (FE) on $[0,\varsigma_1]$ with $u_{\hat{z}}(0)=u_0$. It is clear from Lemma 3.4 that $\mathfrak{f}(u_{\hat{z}}(\cdot))\in W^{1,\infty}(0,\varsigma_1)$. If $\varsigma_1=T$, then $\hat{z}$ is a global solution.

Let $\varsigma_1<T$. Put

$$\lambda_2=\left[C_2e^{M(T-\varsigma_1)}\Big(\mathfrak{h}(u_{\hat{z}}(\varsigma_1))+\int_{\varsigma_1}^T \mathfrak{h}(\mathcal{F}(\tau))d\tau\Big)\right]^{-1}, \qquad \varrho_2=C_1e^{-M(T-\varsigma_1)}\mathfrak{h}(u_{\hat{z}}(\varsigma_1)),$$

$$\kappa_2=|\xi|_{L^\infty(0,T)}+Me^{M(T-\varsigma_1)}\Big(\bar{\mathfrak{f}}(u_{\hat{z}}(\varsigma_1))+\int_{\varsigma_1}^T\bar{\mathfrak{f}}(\mathcal{F}(\tau))d\tau\Big)+Me^{MT}\lambda_2^{-1},$$

$$d_2=\varrho_2^{-1}\kappa_2, \qquad \varsigma_2=\min\{\varsigma_1+d_2^{-1},T\},$$

and define

$$\mathcal{K}_2=\{\zeta\in W^{1,\infty}(\varsigma_1,\varsigma_2)\mid \zeta(\varsigma_1)=\hat{z}(\varsigma_1),\ |\zeta'|_\infty\le d_2\},$$

$$(\Psi\zeta)(t)=(I-\lambda_2L^{-1})^{-1}(\zeta(t)-\lambda_2\mathfrak{f}(u_\zeta(t))),\ t\in[\varsigma_1,\varsigma_2] \text{ for } \zeta\in\mathcal{K}_2.$$

Then in a way similar to the above, we may apply Schauder's Fixed Point Theorem, and obtain a solution $\bar{z}\in W^{1,\infty}(\varsigma_1,\varsigma_2)$ on $[\varsigma_1,\varsigma_2]$ with $u_{\bar{z}}(\varsigma_1)=u_{\hat{z}}(\varsigma_1)$. Setting

$$z(t)=\begin{cases}\hat{z}(t), & \text{if } t\in[0,\varsigma_1],\\ \bar{z}(t), & \text{if } t\in(\varsigma_1,\varsigma_2],\end{cases}$$

we easily see that

$$u_z(t)=\begin{cases}u_{\hat{z}}(t), & \text{if } t\in[0,\varsigma_1],\\ u_{\bar{z}}(t), & \text{if } t\in(\varsigma_1,\varsigma_2],\end{cases}$$

and that $z\in W^{1,\infty}(0,\varsigma_2)$ is a solution on $[0,\varsigma_2]$ with $u_z(0)=u_0$. Note that $\mathfrak{f}(u_z(\cdot))\in W^{1,\infty}(0,\varsigma_2)$ by Lemma 3.4.

Repeat these arguments. We find from Lemma 4.1 that $\varsigma_n \geq \min\{(1+2^{-1}+\cdots+n^{-1})d_1^{-1}, T\}$ after the repetition of the n times. The fact that $\sum_{k=1}^{n} k^{-1} \nearrow +\infty$ as $n \to \infty$ makes us finish the repetition finite times.

In this way, if $\boldsymbol{u}_0 = (u_0^1, u_0^2, u_0^3, u_0^4)$ satisfies $(u_0^3, u_0^4) \neq 0$, then we have a solution on the whole interval $[0, T]$. In case of $0 \notin (a, b)$, the proof of Theorem 1 is complete. On the other hand, in case of $a < 0 < b$, we need Step 2 in addition to Step 1.

Step 2. In this step we assume that $\boldsymbol{u}_0 = (u_0^1, u_0^2, 0, 0)$. We may assume $L(0) = 0$ without loss of generality.

Put

$$\lambda_1 = \left[C_2 e^{MT} \int_0^T \max\{\mathfrak{h}(\mathcal{F}(\tau)), 1\} d\tau\right]^{-1},$$
$$\kappa_1 = |\xi|_{L^\infty(0,T)} + M e^{MT} \int_0^T \bar{\mathfrak{f}}(\mathcal{F}(\tau)) d\tau + M e^{MT} \lambda_1^{-1},$$
$$d_1 = \kappa_1 \beta_1^{-1}, \quad \varepsilon_1 = d_1^{-1}(1+\lambda_1\beta_1)^{-1}, \quad \varsigma_1 = \min\{\varepsilon_1, T\},$$

where β_1 is the constant appeared in (L) with $r = 1$. Define an operator $\Psi\colon \mathcal{K}_1 \to C([0, \varsigma_1])$ by (4.1) and (4.2). Note that $L(\mathfrak{f}(\boldsymbol{u}_0))$ vanishes.

Let $z \in \mathcal{K}_1$, and let $0 \leq t_1 < t_2 \leq \varsigma_1$. We claim that $|(\Psi z)(t_1) - (\Psi z)(t_2)| \leq d_1(t_2 - t_1)$. Since $(I - \lambda_1 L^{-1})^{-1}(0) = 0$, $z(0) = 0$ and $\boldsymbol{u}_0 = (u_0^1, u_0^2, 0, 0)$, we see that

$$|(\Psi z)(t_i)| \leq \int_0^{t_i} \left[|z'(\tau)||1 + \lambda_1 \mathfrak{g}(\boldsymbol{u}_z(\tau))| d\tau + \lambda_1 |\mathfrak{f}F(\tau, \boldsymbol{u}_z(\tau), z(\tau))|\right] d\tau$$

by (4.3) and Lemma 3.4. Furthermore, it follows from Lemma 4.1 (i) and (ii) that

$$0 \leq 1 + \lambda_1 \mathfrak{g}(\boldsymbol{u}_z(t)) \leq 1, \quad |\mathfrak{f}F(t, \boldsymbol{u}_z(t), z(t))| \leq \kappa_1 \text{ for } t \in [0, \varsigma_1],$$

and so $|(\Psi z)(t_i)| \leq 1$. Setting $\tau_i = \lambda_1^{-1}[(I - \lambda_1 L^{-1})^{-1} - I](z(t_i) - \lambda_1 \mathfrak{f}(\boldsymbol{u}_z(t_i)))$, we know that $(\Psi z)(t_i) = L(\tau_i)$ and $L(\tau_i) - \lambda_1 \tau_i = z(t_i) - \lambda_1 \mathfrak{f}(\boldsymbol{u}_z(t_i))$. Therefore, we see from (L) that

$$\begin{aligned}
|(\Psi z)(t_1) - (\Psi z)(t_2)| &\leq (1 + \lambda_1\beta_1)^{-1} |L(\tau_1) - L(\tau_2) - \lambda_1(\tau_1 - \tau_2)| \\
&\leq (1 + \lambda_1\beta_1)^{-1} \int_{t_1}^{t_2} \left[|z'(\tau)||1 + \lambda_1 \mathfrak{g}(\boldsymbol{u}_z(\tau))| + \lambda_1 |\mathfrak{f}F(\tau, \boldsymbol{u}_z(\tau), z(\tau))|\right] d\tau \\
&\leq d_1(t_2 - t_1)
\end{aligned}$$

as claimed.

Hence using Schauder's Fixed Point Theorem, we obtain a solution $\hat{z} \in W^{1,\infty}(0,\varsigma_1)$ on $[0,\varsigma_1]$. If $\varsigma_1 = T$, the proof is complete. Let $\varsigma_1 < T$. If $\boldsymbol{u}_{\hat{z}}(\varsigma_1) = (u^1_{\hat{z}}(\varsigma_1), u^2_{\hat{z}}(\varsigma_1), u^3_{\hat{z}}(\varsigma_1), u^4_{\hat{z}}(\varsigma_1))$ satisfies $(u^3_{\hat{z}}(\varsigma_1), u^4_{\hat{z}}(\varsigma_1)) \neq 0$, then returning to Step 1 we can extend $\hat{z}(t)$ to $[0,T]$. If $\boldsymbol{u}_{\hat{z}}(\varsigma_1) = (u^1_{\hat{z}}(\varsigma_1), u^2_{\hat{z}}(\varsigma_1), 0, 0)$, then choosing $\varsigma_2 = \min\{\varsigma_1 + \varepsilon_1, T\}$ for the above ε_1 and defining

$$\mathcal{K}_2 = \{\zeta \in W^{1,\infty}(\varsigma_1,\varsigma_2) \mid \zeta(\varsigma_1) = \hat{z}(\varsigma_1),\ |\zeta'|_\infty \le d_1\},$$
$$(\Psi\zeta)(t) = (I - \lambda_1 L^{-1})^{-1}(\zeta(t) - \lambda_1 \mathfrak{f}(\boldsymbol{u}_\zeta(t))),\ t \in [\varsigma_1,\varsigma_2] \text{ for } \zeta \in \mathcal{K}_2,$$

we prolong $\hat{z}(t)$ to $[0,\varsigma_2]$. Repeat these arguments.

In this way we gain a solution z on the whole interval $[0,T]$ such that z, $\mathfrak{f}\boldsymbol{u}_z \in W^{1,\infty}(0,T)$. Thus, Theorem 1 has been completely proved. □

5. Proof of the uniqueness theorem

In this section we establish the uniqueness result for (NNS).

Proof of Theorem 2. Let $(z_j, \boldsymbol{u}_j)$, $j = 1, 2$, be weak solutions to (NNS) on $[0,T]$. Recall that $\boldsymbol{u}_j$ is a unique weak solution to the initial-value problem for (SE;z_j) on $[0,T]$ with initial data $\boldsymbol{u}_j(0)$: $\boldsymbol{u}_j \equiv \boldsymbol{u}_{z_j}$. We first show (2.3). Since $z_j(t) = L(\mathfrak{f}(\boldsymbol{u}_j(t)))$, $j = 1, 2$, we see that

$$\beta_{\hat{r}}|z_1(t) - z_2(t)| \le |\mathfrak{f}(\boldsymbol{u}_1(t)) - \mathfrak{f}(\boldsymbol{u}_2(t))|, \quad t \in [0,T], \tag{5.1}$$

by the local Lipschitz continuity of L, cf. (L). Here $\hat{r} \ge \max\{|z_1|_\infty, |z_2|_\infty\}$.

Put $\boldsymbol{v}_j(t) = S(-z_j(t))\boldsymbol{u}_j(t)$. Then $\boldsymbol{v}_j$ is a solution to (ODE;z_j) on $[0,T]$ with $\boldsymbol{v}_j(0) = S(-z_j(0))\boldsymbol{u}_j(0)$. We claim that

$$|\mathfrak{f}(\boldsymbol{u}_1(t)) - \mathfrak{f}(\boldsymbol{u}_2(t))| \le \|\mathfrak{f}\| e^{\omega\hat{r}} \|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|, \quad t \in [0,T]. \tag{5.2}$$

Indeed, we suppose that $z_1(t) < z_2(t)$ at t, then we see $\mathfrak{f}(\boldsymbol{u}_1(t)) > \mathfrak{f}(\boldsymbol{u}_2(t))$ at t, since L is strictly decreasing. In addition, $\sigma \mapsto \mathfrak{f}(S(\sigma)\boldsymbol{v}_1(t))$ is nondecreasing by (3.1). Hence it follows that

$$\begin{aligned}|\mathfrak{f}(\boldsymbol{u}_1(t)) - \mathfrak{f}(\boldsymbol{u}_2(t))| &= \mathfrak{f}(\boldsymbol{u}_1(t)) - \mathfrak{f}(\boldsymbol{u}_2(t)) \\ &= \mathfrak{f}(S(z_1(t))\boldsymbol{v}_1(t)) - \mathfrak{f}(S(z_2(t))\boldsymbol{v}_1(t)) \\ &\qquad\qquad + \mathfrak{f}(S(z_2(t))\boldsymbol{v}_1(t)) - \mathfrak{f}(S(z_2(t))\boldsymbol{v}_2(t)) \\ &\le \|\mathfrak{f}\| e^{\omega\hat{r}} \|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\| \qquad \text{at } t\end{aligned}$$

as claimed.

Next, claim that

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\| \le C\Big(\|\boldsymbol{v}_1(0) - \boldsymbol{v}_2(0)\| + C\int_0^t |z_1(\tau) - z_2(\tau)| d\tau\Big), \quad t \in [0, T], \tag{5.3}$$

where C depends on $\hat{r} \ge \max\{|z_1|_\infty, |z_2|_\infty\}$. Definition of F and condition (F2) provide with the local Lipschitz continuity of $\sigma \mapsto S(-\sigma)F(t, S(\sigma)\boldsymbol{u}, \sigma)$: For each $r > 0$ there is a constant $C(r)$ such that

$$\|S(-\sigma_1)F(t, S(\sigma_1)\boldsymbol{u}, \sigma_1) - S(-\sigma_2)F(t, S(\sigma_2)\boldsymbol{u}, \sigma_2)\| \le C(r)|\sigma_1 - \sigma_2|$$

for $t \in [0, T]$, $\boldsymbol{u} \in D$ and σ_1, $\sigma_2 \in [-r, r]$. Using the local Lipschitz continuity of $\sigma \mapsto S(-\sigma)F(t, S(\sigma)\boldsymbol{u}, \sigma)$ combined with the Lipschitz continuity of $\boldsymbol{u} \mapsto F(t, \boldsymbol{u}, \sigma)$, we have

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\| \le \|\boldsymbol{v}_1(0) - \boldsymbol{v}_2(0)\| + C\int_0^t |z_1(\tau) - z_2(\tau)| d\tau + C\int_0^t \|\boldsymbol{v}_1(\tau) - \boldsymbol{v}_2(\tau)\| d\tau.$$

By Gronwall's Lemma we get (5.3).

Therefore, it follows from (5.1)–(5.3) that

$$|z_1(t) - z_2(t)| \le C\Big(\|\boldsymbol{v}_1(0) - \boldsymbol{v}_2(0)\| + C\int_0^t |z_1(\tau) - z_2(\tau)| d\tau\Big), \quad t \in [0, T],$$

and then apply Gronwall's Lemma to obtain (2.3).

It remains to show that (2.3) implies the uniqueness. Assume $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0)$. Then it is obvious that $z_1 \equiv z_2$ by (2.3). Noting that a weak solution to (SE;z) is at most one for $z \in C([0, T])$, we conclude $\boldsymbol{u}_1 \equiv \boldsymbol{u}_{z_1} \equiv \boldsymbol{u}_{z_2} \equiv \boldsymbol{u}_2$. $\square$

We conclude with the final remarks.

Remark. We can show that the unknown $u(t,x)$ is compactly supported in x under the additional assumptions similar to [12, 14]. We can also discuss continuous dependence of $u(t,x)$ on initial data in a way similar to [16].

References

1. G. Allain and P. Colli, *A mathematical study of a muscle contraction model in which the fibre is a continuum of elements*, Adv. Applied Math. **9** (1988), 104–126.

2. P. Colli, *A mathematical model of heterogeneous behavior of single muscle fibres*, J. Math. Biol. **24** (1986), 103–118.

3. ———, *On a nonlinear and nonlocal evolution equation related to muscle contraction*, Nonlinear Anal. TMA **13** (1989), 1149–1162.

4. P. Colli, V. Comincioli, G. Naldi and A. Torelli, *A mathematical study of the plasticity effects in muscle contraction*, Appl. Math. Optim. **22** (1990), 1–26.

5. P. Colli and M. Grasselli, *Mathematical study of a nonlinear transport-diffusion problem related to muscle contraction*, Differential Integral Equations **3** (1990), 837–849.

6. ———, *Parabolic perturbation of a nonlinear hyperbolic problem arising in physiology*, J. Differential Equations **101** (1993), 178–212.

7. V. Comincioli and A. Torelli, *Mathematical aspects of the cross-bridge mechanism in muscle contraction*, Nolinear Anal. TMA **7** (1983), 661–683.

8. ———, *A mathematical model of contracting muscle with viscoelastic elements*, SIAM J. Math. Anal. **19** (1988), 593–612.

9. ———, *A mathematical study of a continuum-state cross-bridge model of muscle contraction*, Math. Methods Appl. Sci. **10** (1988), 289–302.

10. V. Comincioli, A. Torelli, C. Poggesi and C. Reggiani, *A four-state cross bridge model for muscle contraction. Mathematical study and validation*, J. Math. Biol. **20** (1984), 277–304.

11. L. Gastaldi and F. Tomarelli, *A nonlinear and nonlocal evolution equation describing the muscle contraction*, Nonlinear Anal. TMA **11** (1987), 163–182.

12. N. Kato and T. Yamaguchi, *Nonlinear nonlocal differential equations related to muscle contraction mechanism*, Nonlinear Anal. TMA **30** (1997), 3909–3915.

13. ———, *Nonlinear nonlocal transport-diffusion equations arising in physiology*, to appear in Nihonkai Math. J.

14. ———, *Nonlinear nonlocal hyperbolic equations arising in physiology*, to appear in Adv. Math. Sci. Appl.

15. T. Matsumoto, *Time-dependent nonlinear perturbations of analytic semigroups in Banach spaces*, Adv. Math. Sci. Appl. **7** (1997), 119–163.

16. T. Matsumoto, S. Oharu and T. Yamaguchi, *A nonlinear nonlocal transport system related to the cross-bridge mechanism of muscle contraction*, preprint.

17. N. H. Pavel, *Semilinear equations with dissipative time-dependent domain perturbations*, Israel J. Math. **46** (1983), 103–122.

18. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

19. A. Torelli, *A non linear hyperbolic equation related to the dynamics of cardiac muscle*, Portugal. Math. **41** (1982), 171–188.

20. T. Yamaguchi, *Nonlocal nonlinear systems of transport equations in weighted L^1 spaces: an operator theoretic approach*, preprint.