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Decidable/Undecidable Properties of Conditional Term Rewriting Systems

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1 Introduction

Conditional term rewriting has a wide range of applications both in equational reasoning and design of declarative programming languages. We address in this paper decidable and undecidable properties of conditional term rewriting systems (CTRSs, for short) with equations in conditions of rules. In particular, we deal with CTRSs of semi-equational, join, and oriented type, in which conditions are interpreted as convertibility, joinability, and reachability, respectively—these types of CTRSs have been widely investigated in literature.

The paper is to show how known decidability results for subclasses of term rewriting systems (TRSs, for short) are extended in conditional setting. More precisely, we examine for classes $\mathcal{C}$ of TRSs having a decidable property ϕ, in which conditional extensions of $\mathcal{C}$ the property ϕ is undecidable, as well as in which extensions of $\mathcal{C}$ the property ϕ remains decidable. The properties considered in this paper are: convertibility, joinability, reachability, termination, and confluence—for these properties, decision problems in term rewriting are rather well-established.

Comparing to sufficient criteria for termination, confluence, etc. of CTRSs, only few decidability/undecidability results are known in conditional term rewriting. This situation is contrasting to that for term rewriting where considerable efforts have been dedicated to the study of decidable/undecidable properties of subclasses of TRSs. For example, it has been shown that all of the properties listed above are decidable for left-linear right-ground TRSs. Furthermore, we believe that this approach gives better understanding of expressiveness of CTRSs subjecting to various restrictions.

One of the best-known undecidability results in conditional term rewriting is that one-step reduction of CTRSs is no longer decidable[3][10]. We will see this fact is

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quite severe in a sense that even in a limited conditional extension of a well-behaved class of TRSs the situation is the same. Several criteria for decidability of one-step reduction, which also imply termination, are known: e.g. decreasing CTRSs[5], deterministic quasi-reductive CTRSs[6]. These criteria are, however, undecidable in general, and tools to detect these criteria have been investigated. More similar to our approach is a result in [3], which says that whether a given term is normal is decidable for orthogonal normal oriented CTRSs with *subterm property.* They proves in fact decidability of innermost one-step reduction for these CTRSs—this property is, however, out of scope of this paper.

Our decidability results are obtained mostly reducing the problem of CTRSs to that of TRSs via a variety of translations. From this point of view, our approach is very similar to [12], where modular aspects of properties of CTRSs are studied via the corresponding "ultra"-properties on TRSs via several kinds of translations.

2 Preliminaries

We assume familiarity with the basic concepts and notations in term rewriting (which can be found in e.g. [2], [11]).

Rules of the form $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ are called conditional rewrite rules; the rule is *right-ground* if r is ground (i.e. has no variables), *left-linear* if l is linear (i.e. every variable occurs at most once); the rule has *right-ground conditions* if $v_1, \ldots, v_n$ are ground, has *left-linear conditions* if $u_1, \ldots, u_k$ are linear, has *ground conditions* if $u_1, \ldots, u_n, v_1, \ldots, v_n$ are ground, and has *at most one condition* if $n \leq 1$. We say $\mathcal{R}$ is right-ground (left-linear) if all rules involved are right-ground (left-linear, respectively); has right-ground (left-linear, ground) conditions if every rule involved has right-ground (left-linear, ground, respectively) conditions. *We assume the number of rules of CTRSs are finite*; thus a CTRS is written as $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ for a finite index set I where c_i stands for the conditions of the rule. We denote by $\mathcal{R}_u$ the underlying TRS $\{l \to r \mid l \to r \Leftarrow c \in \mathcal{R}\}$ of $\mathcal{R}$. *We assume in this paper that CTRSs have no extra-variables,* i.e. every rule $l \to r \Leftarrow c$ in CTRSs satisfies $\mathrm{Var}(r) \cup \mathrm{Var}(c) \subseteq \mathrm{Var}(l)$. Here and hereafter, $\mathrm{Var}(\alpha)$ denotes the set of variables appearing in the expression α. We will denote by $\equiv$ the syntactical equality of entities.

For each CTRS $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$, we define TRSs $\mathcal{R}_k$ for $k \in \omega$ inductively as follows:

$$\begin{aligned} \mathcal{R}_0 &= \emptyset \\ \mathcal{R}_{k+1} &= \textstyle\bigcup_{i \in I}\{l_i\theta \to r_i\theta \mid \mathcal{R}_k \vdash u\theta = v\theta \text{ for all } u = v \text{ in } c_i\}. \end{aligned}$$

According to the interpretation of the predicate $\mathcal{R}_k \vdash s = t$, we consider here several types of CTRSs, which are summarized in Table 1. Here, $\xrightarrow{*}_{\mathcal{R}_k}$ ($\xleftrightarrow{*}_{\mathcal{R}_k}$) is the transitive

Table 1:

type	interpretation of $\mathcal{R}_k \vdash s = t$
semi-equational	$s \overset{*}{\leftrightarrow}_{\mathcal{R}_k} t$
join	$s \overset{*}{\rightarrow}_{\mathcal{R}_k} u \overset{*}{\leftarrow}_{\mathcal{R}_k} t$ for some u
oriented	$s \overset{*}{\rightarrow}_{\mathcal{R}_k} t$

reflexive closure of $\rightarrow_{\mathcal{R}_k}$ ($\leftrightarrow_{\mathcal{R}_k}$, respectively). For each types of CTRSs $\mathcal{R}$, its rewrite relation $\rightarrow_{\mathcal{R}}$ is defined as $\bigcup_{k\in\omega} \rightarrow_{\mathcal{R}_k}$.

We say a CTRS $\mathcal{R}$ has decidable one-step reduction if for any terms s, t, whether $s \rightarrow_{\mathcal{R}} t$ holds is decidable. Terms s and t are *convertible* if $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$; terms s and t are *joinable* if $s \overset{*}{\rightarrow}_{\mathcal{R}} u \overset{*}{\leftarrow}_{\mathcal{R}} t$ for some term u; a term t is *reachable from* s if $s \overset{*}{\rightarrow}_{\mathcal{R}} t$. The convertibility (joinability, reachability) is decidable for a CTRS $\mathcal{R}$ if for any terms s and t whether s and t are convertible (resp. whether s and t are joinable, whether t is reachable from s) is decidable.

The following proposition is an immediate consequence of these definitions.

Proposition 2.1 Let $\mathcal{R}$ be a semi-equational (join, oriented) CTRS such that its convertibility (resp. joinability, reachability) is decidable. Then one-step reduction of $\mathcal{R}$ is decidable.

An instance of *Post's Correspondence Problem* (PCP, for short) is like this:

> Let $\}a_1, \ldots, a_m\}$ be an alphabet. Given a finite set P of pairs of non-empty words over this alphabet, namely $P = \{\langle p_i, q_i\rangle \mid 1 \leq i \leq n\}$ where $p_i, q_i \in \{a_1, \ldots, a_m\}^+$, is there a sequence $\delta_1, \ldots, \delta_k \in \{1, \ldots, n\}$ such that $p_{\delta_1} \cdots p_{\delta_k} = q_{\delta_1} \cdots q_{\delta_k}$?

We identify the problem and the set P; and say P is solvable if there exists such a sequence. It is well-known that solvability of P is undecidable even when $m = 2$[17]. Thus, if we find for each P over a fixed alphabet a CTRS $\mathcal{R}$ in the class $\mathcal{C}$ so that $\mathcal{R}$ has the property ϕ if and only if P has a solution (or so that $\mathcal{R}$ has the property ϕ if and only if P has no solutions) then it follows that whether ϕ holds for a CTRS in $\mathcal{C}$ is undecidable. In the succeeding proofs we will use the undecidability of PCP on the fixed alphabets $\{0, 1\}$ to derive our undecidability results. We also provide below unary function symbols $\{0, 1\}$ in our signature (on CTRSs) and a convention that a term $a_1(\cdots(a_l(x))\cdots)$ is denoted by $\overline{w}(x)$ for each word $w = a_1 \cdots a_l \in \{0, 1\}^+$.

3 Undecidability results for right-ground CTRSs

The following fact is well-known.

Theorem 3.1 ([3][10]) One-step reduction is undecidable for CTRSs.

It trivially follows from this theorem that convertibility, joinability, reachability, termination, and confluence are all undecidable for CTRSs. These facts, however, are not surprising since these properties are undecidable also for TRSs.

On the other hand, for the class of (left-linear) right-ground TRSs a number of decidability results has been obtained:

Theorem 3.2 ([4]) Convertibility is decidable for left-linear right-ground TRSs.

Theorem 3.3 ([16]) Joinability and reachability are decidable for right-ground TRSs.

Theorem 3.4 ([9]) Termination is decidable for right-ground TRSs.

Theorem 3.5 ([4][15]) Confluence is decidable for left-linear right-ground TRSs.

We will present in this section sharper undecidability results in conditional term rewriting on convertibility, joinability, reachability, termination, and confluence by clarifying these properties are undecidable for left-linear right-ground CTRSs.

First we show that for whichever type of left-linear right-ground CTRSs one-step reduction, convertibility, joinability, reachability, and termination are undecidable. In fact, we even show a stronger result:

Theorem 3.6 One-step reduction, convertibility, joinability, reachability, and termination are undecidable for left-linear right-ground CTRSs of semi-equational, join, and oriented types having at most one left-linear right-ground condition for each rule.

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \leq i \leq n\}$ with $p_i, q_i \in \{0,1\}^+$. We consider the following CTRS:

$$\mathcal{R}\left\{\begin{array}{lcl}
f(x,y) \to C & \Leftarrow & eq(x,y) = true \\
f(x,y) \to D & \Leftarrow & f(\overline{p_1}(x), \overline{q_1}(y)) = A \\
\cdots\cdots & & \\
f(x,y) \to D & \Leftarrow & f(\overline{p_n}(x), \overline{q_n}(y)) = A \\
E \to F & \Leftarrow & f(\epsilon,\epsilon) = B \\
C \to A & & \\
D \to A & & \\
D \to B & & \\
eq(\epsilon,\epsilon) \to true & & \\
eq(0(x),0(y)) \to true & \Leftarrow & eq(x,y) = true \\
eq(1(x),1(y)) \to true & \Leftarrow & eq(x,y) = true.
\end{array}\right.$$

In CTRSs $\mathcal{R}$ of each types, the one-step reduction $E \to_{\mathcal{R}} F$ occurs if and only if P admits a solution. From this it easily follows that the other properties are also undecidable. ∎

Remark 1 *We note that neither proof in [3] nor that in [10] works for right-ground CTRSs.*

Next we address confluence of left-linear right-ground CTRSs subjecting to the same restriction. Observe that the example above is not effective to show undecidability of confluence.

Theorem 3.7 Confluence is undecidable for left-linear right-ground CTRSs of semi-equational, join, and oriented types having at most one left-linear right-ground condition for each rule.

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \leq i \leq n\}$ with $p_i, q_i \in \{0,1\}^+$. Let

$$\mathcal{R}\left\{\begin{array}{lcl} f(\overline{p_1}(\epsilon), \overline{q_1}(\epsilon)) \to A & & \\ \quad\cdots\cdots & & \\ f(\overline{p_n}(\epsilon), \overline{q_n}(\epsilon)) \to A & & \\ f(\overline{p_1}(x), \overline{q_1}(y)) \to A & \Leftarrow & f(x,y) = A \\ \quad\cdots\cdots & & \\ f(\overline{p_n}(x), \overline{q_n}(y)) \to A & \Leftarrow & f(x,y) = A \\ f(\epsilon, \epsilon) \to \mathit{true} & & \\ f(0(x), 0(y)) \to \mathit{true} & \Leftarrow & f(x,y) = \mathit{true} \\ f(1(x), 1(y)) \to \mathit{true} & \Leftarrow & f(x,y) = \mathit{true}. \end{array}\right.$$

For CTRSs $\mathcal{R}$ of each types, P has no solutions if and only if $\mathit{true} \leftarrow_{\mathcal{R}} t \to_{\mathcal{R}} A$ for no term t if and only if $\mathcal{R}$ is confluent. ∎

4 Decidability results for right-ground CTRSs

In CTRSs presented in the previous section, we admit at most one left-linear right-ground condition per rule. Naturally, one may wonder what happens if conditions are at all ground. We first give an answer to this.

Lemma 4.1 Let ϕ be a property of CTRSs, and C be a class of (unconditional) TRSs such that such that

1. ϕ is decidable for all $\mathcal{R} \in C$,
2. $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{R} \in C$ imply $\mathcal{R}' \in C$,
3. convertibility (joinability, reachability) is decidable for all $\mathcal{R} \in C$.

Then for every CTRS $\mathcal{R}$ of semi-equational (resp. join, and oriented) types having ground conditions, if $\mathcal{R}_u \in C$ then ϕ is decidable for $\mathcal{R}$.

Proof. Let $\mathcal{R}$ be a CTRS of semi-equational type having ground conditions. Let C be a class of TRSs satisfying the conditions above and $\mathcal{R}_u \in C$. We define a sequence of TRSs $\mathcal{S}_0, \mathcal{S}_1, \ldots$ by induction as follows:

$$\begin{array}{lcll} \mathcal{S}_0 & = & \emptyset, & \\ \mathcal{S}_{k+1} & = & \{\, l \to r \mid & l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}, \\ & & & u_i \overset{*}{\leftrightarrow}_{\mathcal{S}_k} v_i \text{ for all } i = 1, \ldots n \,\}. \end{array}$$

Since $\mathcal{S}_k \subseteq \mathcal{R}_u$, we know $\mathcal{S}_k \in C$ and hence $u_i \overset{*}{\leftrightarrow}_{\mathcal{S}_k} v_i$ is decidable by our assumption. Thus, $\mathcal{S}_0, \mathcal{S}_1, \ldots$ are effectively defined by induction on k. Moreover, by definition, $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{R}_u$ holds. Suppose $\mathcal{R}$ has l rules. Then, there are at most $l+1$ different TRSs in this sequence. It should be clear also from definition that $\mathcal{S}_i = \mathcal{S}_{i+1}$ implies $\mathcal{S}_i = \mathcal{S}_j$ for all $j \geq i$. Hence, it follows that $\mathcal{S}_l = \mathcal{S}_{l+1} = \cdots$.

By induction on k, one easily verifies that $\to_{\mathcal{R}_k}$ equals $\to_{\mathcal{S}_k}$ for each k, and thus we conclude that $\to_{\mathcal{R}} = \bigcup_{k\in\omega} \to_{\mathcal{R}_k} = \bigcup_{k\in\omega} \to_{\mathcal{S}_k} = \to_{\mathcal{S}_l}$. Since $\mathcal{S}_l \subseteq \mathcal{R}_u$, we know $\mathcal{S}_l \in C$ and thus ϕ is decidable for $\mathcal{S}_l$ by our assumption. Therefore, ϕ is decidable for $\mathcal{R}$. ∎

Using this lemma and Theorems 3.2–3.5, it follows immediately,

Theorem 4.2 1. Convertibility and confluence are decidable for left-linear right-ground CTRSs of semi-equational, join, and oriented types having ground conditions.

2. Joinability, reachability, and termination are decidable for right-ground CTRSs of join and oriented types having ground conditions.

3. Joinability, reachability, and termination are decidable for left-linear right-ground CTRSs of semi-equational type having ground conditions.

It is very likely that we allowed a defined symbol f to occur in conditions of rules is a key for the undecidability results shown in the previous section. We next study decidable/undecidable properties of (left-linear) right-ground CTRSs containing no defined symbols in conditions of the rules.

The set D of defined symbols is defined like this: $D = \{\mathrm{root}(l) \mid l \to r \Leftarrow c \in \mathcal{R}\}$. Here $\mathrm{root}(l)$ is the function symbol occurring at the root position of l. A rule $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ is said to have *constructor conditions* if $u_1, \ldots, u_n, v_1, \ldots, v_n$ involve no defined symbols.

Theorem 4.3 1. Convertibility and confluence are decidable for left-linear right-ground CTRSs of join and oriented types having at most one left-linear right-ground constructor condition for each rule.

2. Joinability, reachability, and termination are decidable for right-ground CTRSs of join and oriented types having at most one left-linear right-ground constructor condition for each rule.

Proof. Suppose that $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ is a left-linear right-ground CTRS of oriented type having at most one left-linear right-ground constructor condition for each rule.

Let $l_i \to r_i \Leftarrow c_i \in \mathcal{R}$ and suppose that c_i equals $u_1 = v_1$ By our assumption, every variable occurs in u_1 at most once. Thus, since u_1 is a constructor term, one can assume without loss of generality that there exists a unique ground substitution σ_i such that $\mathrm{dom}(\sigma_i) = \mathrm{Var}(c_i)$ and $u_1\sigma_i \equiv v_1$. Let $\mathcal{S}$ be a TRS $\{l_i\sigma_i \to r_i \mid i \in I\}$. Then, it is not hard to show that the reduction relation of $\mathcal{S}$ and $\mathcal{R}$ coincide, i.e. $\xrightarrow{*}_{\mathcal{S}} = \xrightarrow{*}_{\mathcal{R}}$. Since $\mathcal{S}$ is a left-linear right-ground TRS, for any terms s, t whether $s \overset{*}{\leftrightarrow}_{\mathcal{S}} t$ holds is decidable by Theorem 3.2. Also, confluence of $\mathcal{S}$ is decidable by Theorem 3.5. For join type, it suffices to note that right-hand sides of conditions are $\mathcal{R}_u$-normal form.

The other statements are proved similarly using Theorems 3.2–3.5. ∎

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6 and with that in the proof of Theorem 3.7 which breaks the additional requirement that conditions are constructor terms.

5 Reachability of growing CTRSs

A TRS $\mathcal{R}$ is growing if for any $l \to r \in \mathcal{R}$ and for any position p, $l/p \in \mathrm{Var}(r)$ implies $|p| \leq 1$. The following theorem is known.

Theorem 5.1 ([13]) Reachability is decidable for left-linear growing TRSs.

In this section, we present an extension of this result for conditional term rewriting.

Definition 5.2 For a conditional rewrite rule $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ and $i \in I$, let

$$\begin{array}{l} \mathcal{O}_i(l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n) \\ = \left\{ \begin{array}{ll} \{\, l \to \nabla_i(u_1, \ldots, u_n, r),\ \nabla_i(v_1, \ldots, v_n, z) \to z \,\} & \text{if } n \geq 1, \\ \{\, l \to r \,\} & \text{if } n = 0, \end{array} \right. \end{array}$$

where z is a new variable and ∇_i is a new function symbol with the appropriate arity. Suppose $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$. Then we define

$$\mathcal{O}(\mathcal{R}) = \bigcup_{i \in I} \mathcal{O}_i(l_i \to r_i \Leftarrow c_i).$$

Symbols in $\{\nabla_i \mid i \in I\}$ are called ∇-*symbols.* The set of terms containing no ∇-symbols is denoted by $\mathcal{T}$.

From now on until Theorem 5.7, we assume that $\mathcal{R}$ is a left-linear CTRS of oriented type having right-ground conditions.

Definition 5.3 Let $s \equiv C[g\sigma]_p \to_{\mathcal{O}(\mathcal{R})} C[h\sigma]_p \equiv t$ with $g \to h \in \mathcal{O}(\mathcal{R})$. Let $q \in \mathrm{Pos}(t)$. The antecedent of q is a position in s defined as follows:

1. If $p \not< q$ then $q \in \mathrm{Pos}(s)$, and the antecedent of q is q.
2. Otherwise, i.e. $p < q$.
   (a) Suppose that there exists $p_x \in \mathrm{Pos}(h)$ such that $j/p_x \equiv x \in \mathcal{V}$ and $p.p_x \leq q$. Then, by linearity of g, there exists a unique position $p'_x \in \mathrm{Pos}(g)$ such that $g/p'_x \equiv x$. The antecedent of q is the position $p.p'_x.(q \backslash p_x)$.
   (b) Otherwise, the antecedent of q is undefined.

Definition 5.4 1. A subterm u of t is called a ∇-subterm of t if $\mathrm{root}(u)$ is a ∇-symbol.

2. Let $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$ with $s \in \mathcal{T}$. We now define $\widetilde{v}$ for each subterm v of t, and an origin of u for each ∇-subterm u of t, by induction on the length of $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$.
   (a) Base step. Put $\widetilde{v} \equiv v$ for each subterm v of s. There is no ∇-subterm of s by our assumption.
   (b) Induction step. Suppose $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} w \to_{\mathcal{O}(\mathcal{R})} t$, $w \equiv C[g\sigma]_p$, and $t \equiv C[h\sigma]_p$ with $g \to h \in \mathcal{O}(\mathcal{R})$. Let $u \equiv t/q$ be a ∇-subterm of t.
      i. If $p \not\leq q$ then the origin of u is that of w/q.
      ii. If $p = q$ then the origin of u is $\widetilde{w/p}$.
      iii. Otherwise, i.e. $p < q$. Then, by the definition of $\mathcal{O}(\mathcal{R})$, there exists a position $p_x \in \mathrm{Pos}(h)$ such that $h/p_x \in \mathcal{V}$ and $p.p_x \leq q$. Thus, there exists a unique antecedent $q' \in \mathrm{Pos}(w)$ of q. The origin of u is that of w/q'.

   For each subterm v of t, $\widetilde{v}$ results from v by replacing all its maximal ∇-subterms by their respective origins.

Lemma 5.5 Let $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$ with $s, t \in \mathcal{T}$. Then, $s \xrightarrow{*}_{\mathcal{R}} t$.

Proof. Let $s \in \mathcal{T}$. By induction on the lenght of the reduction, one can prove that $\tilde{t}' \xrightarrow{*}_{\mathcal{R}} \tilde{t}$ for any s', t', t such that $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} s' \trianglerighteq t' \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$. Here $s' \trianglerighteq t'$ denotes that t' is a subterm of s'. Then it follows that $\tilde{s} \xrightarrow{*}_{\mathcal{R}} \tilde{t}$ for any $s \in \mathcal{T}$ and t such that $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$, and since $t \equiv \tilde{t}$ when $t \in \mathcal{T}$, the statement of the lemma follows immediately. ∎

Remark 2 *In the lemma above, our assumption that conditions of rewrite rules in $\mathcal{R}$ are right-ground can not be dropped. To see this, let*

$$\mathcal{R}\begin{cases} a(c) \to b(d) & \\ A(x) \to B & \Leftarrow \quad a(x) = b(x). \end{cases}$$

Then we have $\mathcal{O}(\mathcal{R}) = \{a(c) \to b(d),\ A(x) \to \nabla_2(a(x), B),\ \nabla_2(b(x), z) \to z\}$, and thus $A(c) \to_{\mathcal{O}(\mathcal{R})} \nabla_2(a(c), B) \to_{\mathcal{O}(\mathcal{R})} \nabla_2(b(d), B) \to_{\mathcal{O}(\mathcal{R})} B$ holds. But since there is no term t satisfying $a(t) \xrightarrow{*}_{\mathcal{R}} b(t)$, we have $A(t) \xrightarrow{*}_{\mathcal{R}} B$ for no term t.

Remark 3 *Our translation is very similar to that appeared in [3] although they assume also orthogonality. Also, similar translations and preservation results have been appeared in [7], [18], [12]. But neither of them is effective for our theorem below.*

We define growingness of a CTRS as follows.

Definition 5.6 A CTRS $\mathcal{R}$ are said to be *growing* if for any $l \to r \Leftarrow c \in \mathcal{R}$ and for any position $p \in \mathrm{Pos}(l)$, $l/p \in \mathrm{Var}(r) \cup \mathrm{Var}(c)$ implies $|p| \leq 1$.

Using Lemma 5.5, one easily shows

Theorem 5.7 Reachability is decidable for left-linear growing CTRSs of oriented type having right-ground conditions.

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6, which breaks growingness condition.

6 Termination of right-ground CTRSs

A proof analogous to that of decidability of termination of (right-)ground TRSs[9] is effective to establish a relation between decidability of termination and that of one-step reduction for right-ground CTRSs.[1]

The next lemma is proved in the same way as the one for TRSs.

[1]Original lemmas are presented under an additional (auxiliary) assumption that left-hand sides of rewrite rules are also ground.

Lemma 6.1 Let $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ be a right-ground CTRS. If $\mathcal{R}$ is non-terminating then there exists $i \in I$ such that r_i is non-terminating.

The proof of the following lemma[2] needs an alternation, for the original way in [9] (which is adopted also in [11] and [2]) to use induction on the number of rewrite rules does not work.

Lemma 6.2 Let $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ be a right-ground CTRS. If $\mathcal{R}$ is non-terminating then there exists $i \in I$ such that $r_i \xrightarrow{+} C[r_i]$.

Proof. Use the minimal infinite reduction sequence argument; see e.g. [14], [8]. ∎

Theorem 6.3 Let $\mathcal{R}$ be a right-ground CTRS. If one-step reduction of $\mathcal{R}$ is decidable then termination is a decidable property of $\mathcal{R}$.

Theorem 6.4 For any right-ground CTRS $\mathcal{R}$ if $\mathcal{R}$ is non-terminating then there exists a natural number k such that $\mathcal{R}_k$ is non-terminating.

Remark 4 *The theorem above contrasts to non-right-ground case. Let*

$$\mathcal{R}\begin{cases} g(x) \to g(f(x)) & \Leftarrow \quad h(x) = b \\ h(a) \to b & \\ h(f(x)) \to b & \Leftarrow \quad h(x) = b. \end{cases}$$

Then, for CTRSs $\mathcal{R}$ of each types,

$$\begin{array}{lcl} \mathcal{R}_0 & = & \emptyset \\ \mathcal{R}_1 & = & \{h(a) \to b\} \\ \mathcal{R}_2 & = & \mathcal{R}_1 \cup \{g(a) \to g(f(a)), h(f(a)) \to b\} \\ \mathcal{R}_3 & = & \mathcal{R}_2 \cup \{g(f(a)) \to g(f(f(a))), h(f(f(a))) \to b\} \\ \mathcal{R}_4 & = & \mathcal{R}_3 \cup \{g(f(f(a))) \to g(f(f(f(a)))), h(f(f(f(a)))) \to b\} \\ & & \cdots\cdots \end{array}$$

are all terminating, while $\mathcal{R}$ is non-terminating as:

$$g(a) \to_{\mathcal{R}} g(f(a)) \to_{\mathcal{R}} g(f(f(a))) \to_{\mathcal{R}} \cdots.$$

By combining Proposition 2.1 and Theorems 5.7 and 6.3 we obtain

Corollary 6.5 Termination is decidable for left-linear right-ground growing CTRSs of oriented type having right-ground conditions.

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6, which breaks growingness condition.

[2]The original lemma shows $l_i \xrightarrow{+} C[l_i]$ instead of $r_i \xrightarrow{+} C[r_i]$.

7 Conclusion

In this paper we studied decidability/undecidability of convertibility, joinability, reachability, termination, and confluence for subclasses of CTRSs of semi-equational, join, and oriented types—in particular, those that are related to known decidability/undecidability results for (C)TRSs.

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