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Strongly closed subgraphs in a distance-regular graph

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1 Introduction

First we recall the notation and terminologies. All graphs we considered are undirected finite graphs without loops or multiple edges. Let Γ be a connected graph with usual shortest path distance ∂_{Γ}. We identify Γ with the set of vertices. We denote by

$$d_{\Gamma} := \max\{ \partial_{\Gamma}(x,y) \mid x,\, y \in \Gamma \}$$

which is called the *diameter* of Γ. Let

$$\Gamma_j(u) := \{ x \in \Gamma \mid \partial_{\Gamma}(u,x) = j \} \quad \text{and} \quad k_{\Gamma}(u) := |\, \Gamma_1(u) \,|.$$

Γ is called a *regular graph of valency* k if $k_{\Gamma}(u) = k$ for all vertices $u \in \Gamma$.

For two vertices u and x in Γ with $\partial_{\Gamma}(u,x) = j$, let

$$\begin{aligned} C(u,x) &= C_j(u,x) &:=& \ \Gamma_{j-1}(u) \cap \Gamma_1(x), \\ A(u,x) &= A_j(u,x) &:=& \ \Gamma_j(u) \cap \Gamma_1(x) \\ \text{and} \quad B(u,x) &= B_j(u,x) &:=& \ \Gamma_{j+1}(u) \cap \Gamma_1(x). \end{aligned}$$

A connected graph Γ is said to be *distance-regular* if

$$c_j := |\, C_j(u,x) \,|, \quad a_j := |\, A_j(u,x) \,| \quad \text{and} \quad b_j := |\, B_j(u,x) \,|$$

depend only on $j = \partial_{\Gamma}(u,x)$ rather than individual vertices. The numbers c_j, a_j and b_j are called the *intersection numbers* of Γ. It is clear that Γ is a regular graph of valency $k_{\Gamma} = b_0$ if Γ is distance-regular.

The reader is referred to [1],[2] for a general theory of distance-regular graphs.

Let $\Delta \subseteq \Gamma$. We identify Δ with the induced subgraph on it.

A subgraph Δ is called *strongly closed* if $S(x,y) \subseteq \Delta$ for any $x, y \in \Delta$, where

$$S(x,y) := \{y\} \cup C(x,y) \cup A(x,y).$$

It is known that a strongly closed subgraph is distance-regular if it is regular.

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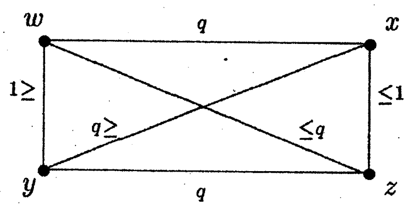
Let q be a positive integer. A quadruple (w,x,y,z) of vertices is called a *root of size* q if $\partial_\Gamma(w,x)=\partial_\Gamma(y,z)=q$, $y\in S(x,w)$ and $z\in S(w,x)$. (See Figure 1.)

A triple (x,y,z) of vertices with $\partial_\Gamma(x,z)=\partial_\Gamma(y,z)=q$ is called a *conron of size* q if there exist three sequences of vertices

$$(x_0,x_1,\ldots,x_m=x),\ (y_0,y_1,\ldots,y_m=y)\quad\text{and}\quad(z_0,z_1,\ldots,z_m=z)$$

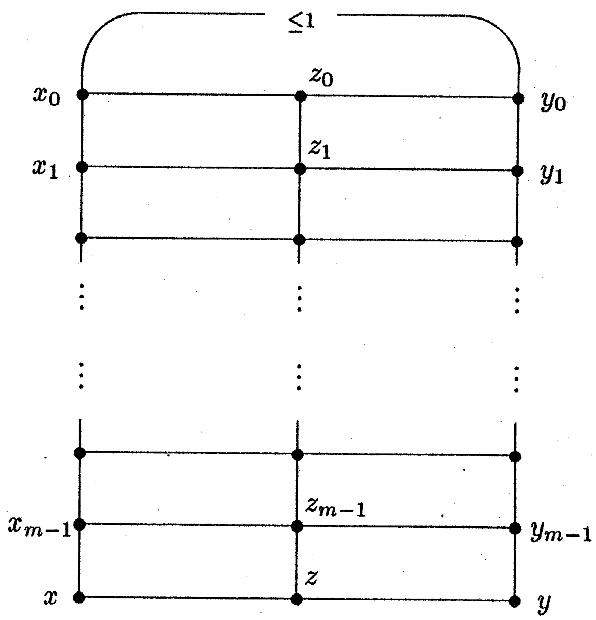
such that $\partial_\Gamma(x_0,y_0)\le 1$, $(x_{i-1},z_{i-1},x_i,z_i)$ and $(y_{i-1},z_{i-1},y_i,z_i)$ are roots of size q for all $1\le i\le m$. (See Figure 2.)

Figure 1.



a root (w,x,y,z) of size q

Figure 2.



a conron (x,y,z) of size q

The conditions $(SS)_q$, $(CR)_q$ and $(SC)_q$ are defined as follow:

$(SS)_q$: $S(x,z)=S(y,z)$ for any triple of vertices (x,y,z) with $\partial_\Gamma(x,z)=\partial_\Gamma(y,z)=q$ and $\partial_\Gamma(x,y)\le 1$.

$(CR)_q$: $S(x,z)=S(y,z)$ for any conron (x,y,z) of size q.

$(SC)_q$: For any given pair of vertices at distance q, there exists a strongly closed subgraph of the diameter q containing them.

The following are our main results.

Theorem 1 ([4, 6]) *Let Γ be a distance-regular graph with $a_1 > 0$. Let q be an integer with $b_{q-1} > b_q$. The following two conditions hold if and only if $(SC)_q$ holds.*
(i) $(SS)_i$ *holds for all* $1 \leq i < q$,
(ii) $(CR)_q$ *holds.*

Theorem 2 ([3]) *Let Γ be a distance-regular graph with $r = r(\Gamma) := \max\{i \mid (c_i, b_i) = (c_1, b_1)\}$. Then $(CR)_{r+1}$ holds if and only if $(SC)_{r+1}$ holds.*

2 The Proof of Main Result

First we show the following relations among the conditions $(SS)_q$, $(CR)_q$ and $(SC)_q$.

Proposition 3 *(1) If $(SC)_q$ holds, then $(CR)_q$ holds.*
(2) If $(SC)_q$ holds, then $(SS)_h$ holds for all $h \leq q$.

Proof. (1) Let (x, y, z) be a conron with sequences

$$(x_0, x_1, \ldots, x_m = x),\ (y_0, y_1, \ldots, y_m = y) \quad \text{and} \quad (z_0, z_1, \ldots, z_m = z)$$

as in Figure 2. Let Δ be a strongly closed subgraph of the diameter q containing x and z. Then z_{m-1}, $x_{m-1} \in S(x,z) \cup S(z,x) \subseteq \Delta$. Inductively, we have

$$z_{m-i},\ x_{m-i} \in S(x_{m-i+1}, z_{m-i+1}) \cup S(z_{m-i+1}, x_{m-i+1}) \subseteq \Delta \quad \text{for all} \quad 1 \leq i \leq m.$$

Whence $y_0 \in S(z_0, x_0) \subseteq \Delta$ and $y_i \in S(z_{i-1}, y_{i-1}) \subseteq \Delta$ for all $1 \leq i \leq m$. Therefore we have $y \in \Delta$ and

$$S(x,z) = \{z\} \cup \Delta_1(z) = S(y,z).$$

This proves our assertion.
(2) Let (x, y, z) be any triple of vertices in Γ with $\partial_\Gamma(x,z) = \partial_\Gamma(y,z) = h \leq q$ and $\partial_\Gamma(x,y) \leq 1$. Suppose there exists $w \in S(y,z) - S(x,z)$. Then $\partial_\Gamma(x,y) = 1, \partial_\Gamma(x,w) = h+1$ and hence $\partial_\Gamma(y,w) = h$. Let $w_h := w$ and take $w_{i+1} \in B(x, w_i) \subseteq B(y, w_i)$ for $i = h, h+1, \ldots, q$. Then $\partial_\Gamma(x, w_q) = q+1$ and $\partial_\Gamma(y, w_q) = q$. Since $(SC)_q$ holds, there exists Δ a strongly closed subgraph of diameter q containing y and w_q. Then $w_{j-1} \in S(y, w_j) \subseteq \Delta$ for all $j = q, \ldots, h$. Thus $z \in S(y, w_h) \subseteq \Delta$ and $x \in S(z,y) \subseteq \Delta$. This contradicts $q = d_\Delta \geq \partial_\Gamma(x, w_q) = q+1$. The desired result is proved. □

Let Γ be a distance-regular graph with $b_{q-1} > b_q$. Assume that the conditions (i)(ii) in Theorem 1 holds. We sketch the construction of strongly closed subgraphs.

Fix a pair of vertices (u,v) in Γ at distance q. For any $x,\ y \in \Gamma_q(u)$, we define the relation $x \approx y$ iff (x,y,u) is a conron. Then this is an equivalence relation on $\Gamma_q(u)$. Set Ψ be the equivalence class containing v under this equivalence relation $\approx$. Define $\Delta(u,v) := P(u,\Psi)$ the subgraph induced on all vertices lying on shortest paths between u and vertices in Ψ.

Them $\Delta(u,v)$ becomes a strongly closed subgraph of diameter q. In particular, it is distance-regular. (See [4, 6]). □

3 Applications

Theorem 4 ([6]) *Let Γ be a regular thick near polygon with $r = r(\Gamma)$. If $2r+1 \le d_\Gamma$, then $b_{q-1} > b_q$ and $(SC)_q$ holds for all q with $r+1 \le q \le d-r$. In particular, $r \in \{1,2,3,5\}$.*

Theorem 5 ([5]) *Let Γ be a distance-regular graph with $(c_1,b_1) = \cdots = (c_r,b_r) \neq (c_{r+1},b_{r+1}) = \cdots = (c_{2r},b_{2r})$ where $c_{r+1} \ge 2$. Then one of the following holds:*
(i) $r \le 2$,
(ii) $a_1 = a_{r+1} = 0$, $c_{r+1} = 2$ *and* $r \equiv 0 \pmod 2$.

Sketch of the Proof of Theorem 5. Assume $r \ge 3$.
(1) Show $(CR)_{r+1}$ holds and r is even.
(2) $(SC)_{r+1}$ holds from Theorem 1. A strongly closed subgraph Δ is a genelarized polygon.
(3) Δ is an ordinary polygon. Hence (ii) holds. □

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