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Local integral representations of smooth functions and interpolation inequalities

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Let R^n be the n-dimensional Euclidean space. Let $\mathcal{D} = \mathcal{D}(R^n)$ denote the set of $C^\infty-$ functions with compact support , and $2N$ stands for the set of nonnegative even numbers. For a positive integer m, the Riesz kernel $\kappa_m(x)$ of order m on R^n is given by

$$\kappa_m(x) = \frac{1}{\gamma_{m,n}} \begin{cases} |x|^{m-n}, & m-n \notin 2N, \\ (\delta_{m,n} - \log|x|)|x|^{m-n}, & m-n \in 2N \end{cases}$$

with

$$\gamma_{m,n} = \begin{cases} \pi^{n/2} 2^m \Gamma(m/2)/\Gamma((n-m)/2), & m-n \notin 2N, \\ (-1)^{(m-n)/2} 2^{m-1} \pi^{n/2} \Gamma(m/2)((m-n)/2)!, & m-n \in 2N \end{cases}$$

and

$$\delta_{m,n} = \frac{\Gamma'(m/2)}{2\Gamma(m)} + \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{(m-n)/2} - \mathcal{C}) - \log 2$$

where $\mathcal{C}$ is Euler's constant. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

If u is a C^m-function with compact support, then it can be represented by the partial derivatives of m-th order as follows:

$$u(x) = \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy \tag{1}$$

([Re]) where σ_n is the surface area of the unit sphere, and

$$u(x) = \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha \kappa_{2m}(x-y) D^\alpha u(y) dy \tag{2}$$

([Wa]). In this note we give the two kinds of integral representations of C^m-functions, which correspond to (1) and (2). One is based on Taylor's formula and V.I.Burenkov's method [Bu: Theorem 4 in Chap.3], and the other is deduced from the fact that the Riesz kernel κ_{2m} is a fundamental solution for the iterated Laplace operator Δ^m, namely

$$\Delta^m \kappa_{2m} = (-1)^m \delta \tag{3}$$

where δ is the point mass at the origin.

Let $0 < \epsilon_1 < \epsilon_2$. We take a function $\eta \in C^\infty(R^1)$ such that supp $\eta \subset \{\epsilon_1 \leq t \leq \epsilon_2\}$ and

$$\int_0^\infty \eta(t) t^{n-1} dt = \frac{1}{\sigma_n},$$

and set

$$\rho(t) = \int_t^\infty \eta(s) s^{n-1} ds.$$

Moreover we put

$$\omega(x) = \eta(|x|), \tag{4}$$

and

$$\chi(x) = \sigma_n \int_{|x|}^\infty \omega(t\frac{x}{|x|}) t^{n-1} dt. \tag{5}$$

Then $\int \omega(x) dx = 1$ and $\chi(x) = \sigma_n \rho(|x|)$. Since $\rho \in C^\infty([0, \infty))$ and

$$\rho(t) = \begin{cases} \frac{1}{\sigma_n}, & \text{for } 0 \leq t \leq \epsilon_1 \\ 0, & \text{for } t \geq \epsilon_2, \end{cases}$$

we have $\chi \in \mathcal{D}(R^n)$ and

$$\chi(x) = \begin{cases} 1, & \text{for } |x| \leq \epsilon_1 \\ 0, & \text{for } |x| \geq \epsilon_2. \end{cases}$$

PROPOSITION 1. *Let $0 < \epsilon_1 < \epsilon_2$. Then there exist functions $\mu, \chi \in \mathcal{D}$ such that* supp μ, supp $\chi \subset \{|x| \leq \epsilon_2\}, \mu(x) = 0$ *on* $\{|x| \leq \epsilon_1\}, \chi(x) = 1$ *on* $\{|x| \leq \epsilon_1\}$, *and if $u \in C^m(R^n)$, then*

$$u(x) = \int \mu(x-y) u(y) dy + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int \frac{(x-y)^\alpha}{|x-y|^n} \chi(x-y) D^\alpha u(y) dy. \tag{6}$$

PROOF. By Taylor's formula we have

$$u(x) = \sum_{|\gamma|<m} \frac{D^\gamma u(y)}{\gamma!} (x-y)^\gamma \tag{7}$$

$$+ m \sum_{|\alpha|=m} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{m-1} D^\alpha u(y + t(x-y)) dt.$$

We take functions ω and χ defined by (4) and (5). Multiplying (7) by $\omega(x-y)$ and integrating with respect to y, we get

$$\begin{aligned} u(x) &= \sum_{|\gamma|<m} \frac{1}{\gamma!} \int D^\gamma u(y)(x-y)^\gamma \omega(x-y)dy \\ &+ \sum_{|\alpha|=m} \frac{m}{\alpha!} \int (x-y)^\alpha \omega(x-y) \int_0^1 (1-t)^{m-1} D^\alpha u(y+t(x-y))dtdy \\ &= \sum_{|\gamma|<m} \frac{1}{\gamma!} \int D^\gamma u(y)(x-y)^\gamma \omega(x-y)dy \\ &+ \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \left(\int (x-y)^\alpha \omega(x-y) D^\alpha u(y+t(x-y))dy \right) dt \\ &= I_1(x) + I_2(x). \end{aligned}$$

By integration by part we have

$$I_1(x) = \int \mu(x-y)u(y)dy$$

where

$$\mu(x) = \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma (x^\gamma \omega(x)).$$

Since $\omega \in \mathcal{D}$ and supp $\omega \subset \{\epsilon_1 \le |x| \le \epsilon_2\}$ by (4), μ also has the same properties. Further by the change of variables $y + t(x-y) = z$, we obtain

$$\begin{aligned} I_2(x) &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \left(\int \frac{(x-z)^\alpha}{(1-t)^m} \omega(\frac{x-z}{1-t}) D^\alpha u(z) \frac{dz}{(1-t)^n} \right) dt \\ &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int D^\alpha u(z)(x-z)^\alpha \left(\int_0^1 \omega(\frac{x-z}{1-t}) \frac{dt}{(1-t)^{n+1}} \right) dz \end{aligned}$$

because of $x-y = (x-z)/(1-t)$. Moreover by the change of variable $|x-z|/(1-t) = s$, we get

$$\begin{aligned} I_2(x) &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int D^\alpha u(z)(x-z)^\alpha \left(\int_{|x-z|}^\infty \omega(s\frac{x-z}{|x-z|}) \frac{s^{n-1}}{|x-z|^n} ds \right) dz \\ &= \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int D^\alpha u(z) \frac{(x-z)^\alpha}{|x-z|^n} \chi(x-z)dz \end{aligned}$$

because of $(x-z)/(1-t) = s(x-z)/|x-z|$. Thus we obtain (6).

PROPOSITION 2. *Let $0 < \epsilon_1 < \epsilon_2$. Then there exist functions $\zeta, \xi \in \mathcal{D}$ such that* supp ζ, supp $\xi \subset \{|x| \le \epsilon_2\}, \zeta(x) = 0$ *on* $\{|x| \le \epsilon_1\}, \xi(x) = 1$ *on* $\{|x| \le \epsilon_1\}$, *and if* $u \in C^m(R^n)$, *then*

$$(8) \qquad u(x) = \int \zeta(x-y)u(y)dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha(\xi\kappa_{2m})(x-y)D^\alpha u(y)dy.$$

PROOF. First we assume that $u \in \mathcal{D}$. Since $u(x-y)$ belongs to $\mathcal{D}$ as a function of y, the formula (3) gives

$$\begin{aligned} u(x) &= \langle \delta(y), u(x-y)\rangle = \langle (-1)^m \Delta^m \kappa_{2m}(y), u(x-y)\rangle \\ &= \langle (-1)^m \kappa_{2m}(y), (\Delta^m u)(x-y)\rangle = \int (-1)^m \kappa_{2m}(y)\Delta^m u(x-y)dy \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the pairing between distributions and test functions. We take a function $\xi \in \mathcal{D}$ such that

$$(9) \qquad \xi(x) = \begin{cases} 1, & |x| \leq \epsilon_1. \\ 0, & |x| \geq \epsilon_2. \end{cases}$$

If we set $\zeta(x) = (-1)^m \Delta^m((1-\xi)\kappa_{2m})(x)$, then by integration by part we have

$$\begin{aligned} u(x) &= \int (1-\xi(y))(-1)^m \kappa_{2m}(y)\Delta^m u(x-y)dy + \int \xi(y)(-1)^m \kappa_{2m}(y)\Delta^m u(x-y)dy \\ &= \int \zeta(y)u(x-y)dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha(\xi\kappa_{2m})(y)D^\alpha u(x-y)dy \\ &= \int \zeta(x-y)u(y)dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha(\xi\kappa_{2m})(x-y)D^\alpha u(y)dy. \end{aligned}$$

By (3) and (9) we see that $\zeta(x) = 0$ for $|x| \leq \epsilon_1$ and $|x| \geq \epsilon_2$, and hence $\zeta \in \mathcal{D}$. Therefore we obtain the proposition for $u \in \mathcal{D}$. In case $u \in C^m(R^n)$, the proposition is obtained by approximating u by a sequence $\{u_j\} \subset \mathcal{D}$ such that $D^\alpha u_j$ converges to $D^\alpha u$ locally uniformly as $j \to \infty$ for $|\alpha| \leq m$. This completes the proof.

By taking differentiation under the integral sign in (6) and (8), we obtain the following corollary.

COROLLARY 3. *Let* $0 < \epsilon_1 < \epsilon_2$ *Then there exist functions* $\mu, \chi, \zeta, \xi \in \mathcal{D}$ *such that* supp μ, supp χ, supp ζ, supp $\xi \subset \{|x| \leq \epsilon_2\}, \mu(x) = \zeta(x) = 0$ *on* $\{|x| \leq \epsilon_1\}, \chi(x) = \xi(x) = 1$ *on* $\{|x| \leq \epsilon_1\}$, *and if* $u \in C^m(R^n)$, *then for* $|\gamma| \leq m-1$

$$D^\gamma u(x) = \int D^\gamma \mu(x-y)u(y)dy + \sum_{|\alpha|=m} \frac{m}{\alpha!\sigma_n} \int D^\gamma(\chi_\alpha)(x-y)D^\alpha u(y)dy$$

where $\chi_\alpha(x) = x^\alpha \chi(x)/|x|^n$, *and*

$$D^\gamma u(x) = \int D^\gamma \zeta(x-y)u(y)dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^{\gamma+\alpha}(\xi \kappa_{2m})(x-y)D^\alpha u(y)dy.$$

As an application of local integral representations, we establish interpolation inequalities.

PROPOSITION 4. *Let* $r \geq 0, \epsilon > 0$ *and* $1 < p < \infty$. *Then for* $u \in C^m(R^n)$ *and* $|\gamma| < m - (n/p)$,

$$\max_{|x|\leq r} |D^\gamma u(x)| \leq C_\epsilon^1 \left(\int_{|y|\leq r+\epsilon} |u(y)|dy + \sum_{|\alpha|=m} \left(\int_{|y|\leq r+\epsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \right).$$

where C_ϵ^1 *is independent of* r.

PROOF. Let $|x| \leq r$ and $|\gamma| < m - (n/p)$. By applying Corollary 3 for $\epsilon_1 = \epsilon/2, \epsilon_2 = \epsilon$ and Hölder's inequality, we have

$$\begin{aligned}
|D^\gamma u(x)| &= \int_{|x-y|\leq\epsilon} |D^\gamma \mu(x-y)u(y)|dy + \sum_{|\alpha|=m} \frac{m}{\alpha!\sigma_n} \int_{|x-y|\leq\epsilon} |D^\gamma(\chi_\alpha)(x-y)D^\alpha u(y)|dy \\
&\leq \max_{|y|\leq\epsilon} |D^\gamma \mu(y)| \int_{|y|\leq r+\epsilon} |u(y)|dy \\
&\quad + \sum_{|\alpha|=m} \frac{m}{\alpha!\sigma_n} \left(\int_{|x-y|\leq\epsilon} |D^\gamma \chi_\alpha(x-y)|^{p'} dy \right)^{1/p'} \left(\int_{|y|\leq r+\epsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \\
&\leq C_\epsilon^1 \left(\int_{|y|\leq r+\epsilon} |u(y)|dy + \sum_{|\alpha|=m} \left(\int_{|y|\leq r+\epsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \right)
\end{aligned}$$

where $(1/p) + (1/p') = 1$ and

$$C_\epsilon^1 = \max_{|\gamma|<m-(n/p)} \left(\max_{|y|\leq\epsilon} |D^\gamma u(y)| + \max_{|\alpha|=m} \frac{m}{\alpha!\sigma_n} \left(\int_{|y|\leq\epsilon} |D^\gamma \chi_\alpha(y)|^{p'} dy \right)^{1/p'} \right) < \infty.$$

PROPOSITION 5. *Let* $r \geq 0$ *and* $\epsilon > 0$. *Then for* $u \in C^m(R^n)$ *and* $|\gamma| \leq m-1$,

$$\max_{|x|\leq r} |D^\gamma u(x)| \leq C_\epsilon^2 \left(\max_{|y|\leq r+\epsilon} |u(y)| + \sum_{|\alpha|=m} \max_{|y|\leq r+\epsilon} |D^\alpha u(y)| \right)$$

where C_ϵ^2 *is independent of* r.

PROOF. Let $|x| \leq r$ and $|\gamma| \leq m-1$. By applying Corollary 3 for $\epsilon_1 = \epsilon/2$ and $\epsilon_2 = \epsilon$, we have

$$\begin{aligned}
|D^\gamma u(x)| &\leq \int_{|x-y|\leq\epsilon} |D^\gamma\zeta(x-y)u(y)|dy + \sum_{|\alpha|=m} \frac{m!}{\alpha!}\int_{|x-y|\leq\epsilon} |D^{\alpha+\gamma}(\xi\kappa_{2m})(x-y)D^\alpha u(y)|dy \\
&\leq \max_{|y|\leq r+\epsilon} |u(y)| \int |D^\gamma\zeta(y)|dy \\
&\qquad + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \max_{|y|\leq r+\epsilon} |D^\alpha u(y)| \int |D^{\alpha+\gamma}(\xi\kappa_{2m})(y)|dy \\
&\leq C_\epsilon^2 \left(\max_{|y|\leq r+\epsilon} |u(y)| + \sum_{|\alpha|=m} \max_{|y|\leq r+\epsilon} |D^\alpha u(y)| \right)
\end{aligned}$$

where

$$C_\epsilon^2 = \max_{|\gamma|\leq m-1} \left(\int |D^\gamma\zeta(y)|dy + \max_{|\alpha|=m} \frac{m!}{\alpha!} \int |D^{\alpha+\gamma}(\xi\kappa_{2m})(y)|dy \right) < \infty.$$

For $1 \leq p < \infty$ and a positive integer m, we set

$$|u|_{m,p} = \sum_{|\alpha|=m} ||D^\alpha u||_p$$

where

$$||u||_p = (\int |u(x)|^p dx)^{1/p}.$$

PROPOSITION 6. *Let j, m be positive integers and $j < m$. Then for $u \in C^m$*

$$|u|_{j,p} \leq C_1||u||_p + C_2|u|_{m,p}$$

where

$$C_1 = \sum_{|\gamma|=j} ||D^\gamma\mu||_1, \quad C_2 = \max_{|\alpha|=m} \frac{m}{\alpha!\sigma_n} \sum_{|\gamma|=j} ||D^\gamma\chi_\alpha||_1.$$

PROOF. Let $|\gamma| = j$. By Corollary 3 and Young's inequality we have

$$||D^\gamma u||_p \leq ||D^\gamma\mu||_1||u||_p + \sum_{|\alpha|=m} \frac{m}{\alpha!\sigma_n}||D^\gamma\chi_\alpha||_1||D^\alpha u||_p.$$

Hence the proposition holds.

From an elementary calculation we have

Lemma 7. *Let $a, b, \theta, \tau > 0$. The function $\varphi(x) = ax^{\theta} + bx^{-\tau}(x > 0)$ attains the minimum $((\tau/\theta)^{\theta/(\theta+\tau)} + (\theta/\tau)^{\tau/(\theta+\tau)})a^{\tau/(\theta+\tau)}b^{\theta/(\theta+\tau)}$ at $x = (b/a)^{1/(\theta+\tau)}(\tau/\theta)^{1/(\theta+\tau)}$.*

For a function u and a positive number ϵ, we set

$$u_{(\epsilon)}(x) = u(\epsilon x).$$

We easily see

Lemma 8. *For a multi-index α, $(D^{\alpha}u_{(\epsilon)})(x) = \epsilon^{|\alpha|}(D^{\alpha}u)(\epsilon x)$.*

Proposition 9. *Let j, m be positive integers and $j < m$. The following three inequalities are equivalent:*

(i) $$|u|_{j,p} \leq C_1||u||_p + C_2|u|_{m,p} \qquad \text{for any } u \in C^m,$$

(ii) $$|u|_{j,p} \leq C_1\epsilon^j||u||_p + C_2\epsilon^{-m+j}|u|_{m,p} \quad \text{for any } \epsilon > 0, u \in C^m,$$

(iii) $$|u|_{j,p} \leq C_1^{1-(j/m)}C_2^{j/m}((\frac{m-j}{j})^{j/m}+(\frac{j}{m-j})^{1-(j/m)})||u||_p^{1-(j/m)}|u|_{m,p}^{j/m} \quad \text{for any } u \in C^m.$$

Proof. (ii) $\Longrightarrow$ (i) It suffices to put $\epsilon = 1$.
(i) $\Longrightarrow$ (ii) Let $u \in C^m$ and $\epsilon > 0$. Since $u_{(\epsilon)} \in C^m$, by the assumption (i), we have

$$|u_{(\epsilon)}|_{j,p} \leq C_1||u_{(\epsilon)}||_p + C_2|u_{(\epsilon)}|_{m,p}.$$

The inequality (ii) follows from the equality

$$|u_{(\epsilon)}|_{k,p} = \epsilon^{k-(n/p)}|u|_{k,p} \qquad (k = 0, 1, \cdots, m),$$

which is obtained by Lemma 8.
(ii) $\Longleftrightarrow$ (iii) This follows from Lemma 7.

Thus by Propositions 6 and 9 we obtain

Corollary 10. ([Ad: Theorem 4.17]) *Let j, m be positive integers and $j < m$. Then for $u \in C^m$*

$$|u|_{j,p} \leq C_1^{1-(j/m)}C_2^{j/m}((\frac{m-j}{j})^{j/m} + (\frac{j}{m-j})^{1-(j/m)})||u||_p^{1-(j/m)}|u|_{m,p}^{j/m}$$

where the constants C_1, C_2 are the same as in Proposition 6.

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