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The Hausdorff dimension of the boundary of a tree

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1 Introduction

In this paper we study the Hausdorff dimension of the boundary of a tree with a distance function. A distance function is defined on a tree and makes the boundary a distance space. Therefore the Hausdorff dimension can be defined on the boundary in the same manner as on general distance spaces. Our first aim is the evaluation of the Hausdorff dimension. For this purpose we introduce an additive function defined on the tree, and a function λ defined on the boundary, which is, roughly speaking, the ratio of decrease of the additive function and that of the distance function. The function λ plays an essential role for Theorems 1 and 2. Next we consider the relation between the Hausdorff dimension of a set in the Euclidean space and that of a set in a tree. In fact we can find a set of a tree which has the same Hausdorff dimension as a given bounded set in the Euclidean space (Theorem 3). Using these theorems we have a unified method for calculating the Hausdorff dimension for a set in the Euclidean space.

Let $(X, \mathcal{A}, o)$ be a tree, i.e. a simply connected and locally finite graph, where X is a set of points, $\mathcal{A}$ is a set of arcs and $o \in X$ which is called the root point. For $x, y \in X$ we denote the natural distance by $\rho(x, y)$, which is the least number of arcs joining from x to y if $x \neq y$ and $\rho(x, x) = 0$. We assume that $\#\{y \in X; \rho(x, y) = 1\} \geq 2$ for every $x \in X$. We set $X_n := \{x \in X; \rho(o, x) = n\}$ for $n \geq 0$. Let $D(x)$ be the descendant of $x \in X_n$ and $p(x)$ the parent of x, i.e.

$$D(x) := \bigcup_{k \geq n} \{y \in X_k; \rho(x, y) = k - n\},$$

and $p(x)$ is the point $y \in X_{n-1}$ with $\rho(x, y) = 1$. We set $p^j(x) = p(p^{j-1}(x))$.

Let Ω be the set of all geodesic rays, where a geodesic ray is a sequence of points $(o, x_1, x_2, \dots)$ such that $x_n \in X_n$ and $\rho(x_n, x_{n+1}) = 1$. For $\xi = (x_n)_n \in \Omega$, where $x_0 = o$, we denote $[\xi] := \{x_0, x_1, x_2, \dots\}$. We call Ω the boundary of the tree.

Let $l(x)$ be a positive function defined on X such that $l(x_n)$ strictly decreases to 0 as $n \to \infty$ for any $(x_n)_n \in \Omega$. For $\xi = (x_n)_n, \eta = (y_n)_n \in \Omega$ we define

$$d(\xi, \eta) := \begin{cases} l(x_n) & \text{if } x_0 = y_0, \dots, x_n = y_n, x_{n+1} \neq y_{n+1}, \\ 0 & \text{if } \xi = \eta. \end{cases}$$

Then d is a distance in Ω, and Ω is a compact space. For $x \in X$ let $B(x) := \{\xi \in \Omega; x \in [\xi]\}$. This set can be written as $B(x) = \{\xi \in \Omega; d(\xi, \eta) \leq l(x)\}$ if we take $\eta \in \Omega$ with $x \in [\eta]$. Therefore $B(x)$ can be considered as a closed ball. Note that $B(x)$ is also an open ball since the distance $d(\cdot, \eta)$ is discrete.

For an integer $n \geq 0$ a set $\{x_j\}_j \subset X$ is called an n-covering set of $E \subset \Omega$ if $E \subset \bigcup_j B(x_j)$ and $\{x_j\}_j \subset \bigcup_{k \geq n} X_k$. For $\alpha > 0$ and $E \subset \Omega$ we define

$$\Lambda_\alpha^n(E, l) := \inf \sum_j l(x_j)^\alpha$$

where the infimum extends over all n-covering sets of E, and

$$\Lambda_\alpha(E, l) := \lim_{n \to \infty} \Lambda_\alpha^n(E, l).$$

Λ_α is called the α-dimensional Hausdorff measure. It is well defined since $\Lambda_\alpha^n(E, l)$ increases when $n \to \infty$. Also we define the Hausdorff dimension of E with the distance function l as

$$\dim(E, l) := \inf\{\alpha; \Lambda_\alpha(E, l) = 0\} = \sup\{\alpha; \Lambda_\alpha(E, l) = \infty\}.$$

2 Evaluations

First we remark that $\{l(x)\}$ uniformly decreases to 0.

Lemma 1. *For any $\varepsilon > 0$ we have $l(x) < \varepsilon$ except for finitely many $x \in X$.*

Proof. Suppose that there are infinitely many $x \in X$ such that $l(x) > \varepsilon$. Then we can take $x_1 \in X_1$ such that $l(x) > \varepsilon$ for infinitely many $x \in D(x_1)$. Next we take $x_2 \in X_2 \cap D(x_1)$ such that $l(x) > \varepsilon$ for infinitely many $x \in D(x_2)$. Repeat this step, and get a sequence $\{x_n\}_n \in \Omega$ such that $l(x) > \varepsilon$ for infinitely many $x \in D(x_n)$. Since $l(x_n)$ is decreasing, we have $l(x_n) > \varepsilon$. This contradicts our assumption. □

Usually the Hausdorff measure and the Hausdorff dimension are defined in a distance space S as follows: Let $C(t, \delta)$ be a ball centered at $t \in S$ with radius δ. For a set $E \subset S$ we define the Hausdorff measure as

$$\mathcal{H}_\alpha^r(E) := \inf\left\{\sum_j \delta_j^\alpha; E \subset \bigcup_j C(t_j, \delta_j), \delta_j \leq r\right\} \quad \text{for } r > 0,$$

$$\mathcal{H}_\alpha(E) := \lim_{r \to 0} \mathcal{H}_\alpha^r(E);$$

also we define the Hausdorff dimension as

$$\dim E := \inf\{\alpha; \mathcal{H}_\alpha(E) = 0\} = \sup\{\alpha; \mathcal{H}_\alpha(E) = \infty\}.$$

When $S = \Omega$, using Lemma 1, for any $r > 0$ we can find n such that $\mathcal{H}_\alpha^r(E) \le \Lambda_\alpha^n(E, l)$, and vice versa. Therefore our definition for the Hausdorff dimension coincides with the usual one.

Now let $\phi(x)$ be an additive function, i.e.

$$\phi(x) = \sum_{y \in X_{n+1} \cap D(x)} \phi(y) \quad \text{for } x \in X_n.$$

For a nonnegative additive function ϕ and $E \subset \Omega$ we define

$$\Phi(E) := \inf \sum_j \phi(x_j)$$

where the infimum extends over all n-covering sets of E. Since ϕ is additive, Φ is independent of n.

Lemma 2. *Φ is a metric and regular outer measure. Especially, if $\{A_j\}_j$ converges increasingly to A, then $\lim_{j\to\infty} \Phi(A_j) = \Phi(A)$.*

Proof. It is well known that every Borel set is measurable under a metric outer measure (See [3, p. 33, Theorem 19]) and that, if μ is a regular outer measure, then $\lim_{j\to\infty} \mu(A_j) = \mu(A)$ when $\{A_j\}_j$ converges increasingly to A (See [3, p. 17, Theorem 9]). Therefore we have only to prove that Φ is a metric and regular outer measure.

First we shall prove that Φ is a metric outer measure. Let $E, F \subset \Omega$ with $d(E, F) > 0$. We can take n such that $l(x) < d(E, F)/2$ for all $x \in \bigcup_{k\ge n} X_k$. Let $\{x_j\}_j$ be an n-covering set of $E \cup F$. Then we can divide $\{x_j\}_j$ into two disjoint sets $\{y_j\}_j$ and $\{z_j\}_j$ such that $E \subset \bigcup_j B(y_j)$ and $F \subset \bigcup_j B(z_j)$. Therefore

$$\Phi(E) + \Phi(F) \le \sum_j \phi(y_j) + \sum_j \phi(z_j) = \sum_j \phi(x_j).$$

Hence $\Phi(E) + \Phi(F) \le \Phi(E \cup F)$. This means that Φ is a metric outer measure.

Next we shall show that Φ is regular. Let $A \subset \Omega$. For any positive integer k we can find zero-covering set $\{x_{kj}\}_j$ of A with $\sum_j \phi(x_{kj}) \le \Phi(A) + 1/k$. Let $E = \bigcap_k \bigcup_j B(x_{kj})$. Then E is a Borel set with $A \subset E$. Since $\{x_{kj}\}_j$ is a zero-covering set of E, we have $\Phi(E) \le \sum_j \phi(x_{kj})$. Therefore $\Phi(E) \le \Phi(A)$. □

We introduce a function defined on Ω which plays essential role of our result: Let

$$\lambda(\xi) = \lambda_{\phi,l}(\xi) := \liminf_{n\to\infty} \frac{\log 1/\phi(x_n)}{\log 1/l(x_n)} \quad \text{for } \xi = (x_n)_n \in \Omega.$$

For the convenience we define

$$\frac{\log 1/\phi(x)}{\log 1/l(x)} = \infty \quad \text{if } \phi(x) = 0.$$

Remark that $\lambda(\xi) \ge 0$ since $\phi(x_n) \le \phi(o)$ and $\log 1/l(x_n) \to \infty$ for $\xi = (x_n)_n \in \Omega$.

Theorem 1. *Let ϕ be a nonnegative additive function with $\phi(o) = 1$. Then, for $E \subset \Omega$,*

$$\phi\text{-}\operatorname*{ess\,sup}_{\xi \in E} \lambda_{\phi,l}(\xi) \leq \dim(E, l) \leq \sup_{\xi \in E} \lambda_{\phi,l}(\xi)$$

where ϕ-ess sup means the supremum except a null set of Φ.

Proof. Assume that $\alpha > \dim(E, l)$ and let $F = \{\xi \in E; \lambda(\xi) > \alpha\}$. Also let

$$F_n = \left\{\xi \in F; l(x)^\alpha \geq \phi(x) \text{ for any } x \in [\xi] \cap \left(\bigcup_{k \geq n} X_k\right)\right\}.$$

If $\xi \in F$, then $l(x)^\alpha \geq \phi(x)$ except for finitely many $x \in [\xi]$. Therefore F_n converges increasingly to F.

Now let $\{x_j\}_j$ be an n-covering set of F_n. We may assume that $B(x_j) \cap F_n \neq \emptyset$. If $\xi \in B(x_j) \cap F_n$, then $x_j \in [\xi]$, and thus $l(x_j)^\alpha \geq \phi(x_j)$. Therefore

$$\sum_j l(x_j)^\alpha \geq \sum_j \phi(x_j) \geq \Phi(F_n).$$

Hence $\Lambda_\alpha^n(F_n, l) \geq \Phi(F_n)$. Since $F_n \subset E$, we have

$$\Phi(F_n) \leq \Lambda_\alpha^n(F_n, l) \leq \Lambda_\alpha(F_n, l) \leq \Lambda_\alpha(E, l) = 0.$$

Using Lemma 2, we have $\Phi(F) = 0$. This means that the first inequality holds.

Next assume that $\alpha < \dim(E, l)$ and $\lambda(\xi) < \alpha$ for any $\xi \in E$. Then $l(x)^\alpha \leq \phi(x)$ for infinitely many $x \in [\xi]$. For fixed n and for each $\xi \in E$ we take the closest point $x \in [\xi] \cap \left(\bigcup_{k \geq n} X_k\right)$ to o such that $l(x)^\alpha \leq \phi(x)$. Let Y_n be the set of such points x. Then Y_n is an n-covering set of E and $\{B(x); x \in Y_n\}$ is disjoint. Therefore

$$\Lambda_\alpha^n(E, l) \leq \sum_{x \in Y_n} l(x)^\alpha \leq \sum_{x \in Y_n} \phi(x) \leq \phi(o).$$

Hence $\infty = \Lambda_\alpha(E, l) \leq \phi(o)$, which is a contradiction. This means that the second inequality holds. □

The following lemma is proved by Frostman in the case of the Euclidean spaces ([2, p. 86, Théorème 1]).

Lemma 3 (Frostman). *Suppose that $\Lambda_\alpha^0(E, l) > 0$. Then there is a nonnegative additive function ϕ with $\phi(o) = 1$ such that $\phi(x) \leq l(x)^\alpha / \Lambda_\alpha^0(E, l)$ for all $x \in X$ and $\phi(x_n) = 0$ for sufficiently large n if $(x_n)_n \in \Omega \setminus \overline{E}$. In other words, $\Phi(\Omega) = 1$, $\Phi(B(x)) \leq l(x)^\alpha / \Lambda_\alpha^0(E, l)$ and Φ is supported in $\overline{E}$.*

Proof. Fix an integer n. We construct nonnegative additive functions ψ_j^n, $j = 0, 1, \dots, n$, as follows: Let $x \in X_n$. If $B(x) \cap E = \emptyset$, then $\psi_j^n(x) = 0$ for $j = 0, \dots, n$. Otherwise $\psi_n^n(x) = l(x)^\alpha$ and

$$\psi_j^n(x) = \min\left\{1, \frac{l(p^{n-j}(x))^\alpha}{\psi_{j+1}^n(p^{n-j}(x))}\right\}\psi_{j+1}^n(x) \quad \text{for } j = 0, \dots, n-1.$$

Let $x \in X_n$ with $B(x) \cap E \neq \emptyset$. First we have $\psi_n^n(x) = l(x)^\alpha$. Second, if $\psi_n^n(p(x)) \leq l(p(x))^\alpha$, then $\psi_{n-1}^n(x) = \psi_n^n(x) = l(x)^\alpha$. Otherwise

$$\psi_{n-1}^n(x') = \frac{l(p(x))^\alpha}{\psi_n^n(p(x))}\psi_n^n(x') \quad \text{for } x' \in X_n \text{ with } p(x') = p(x).$$

Therefore $\psi_{n-1}^n(p(x)) = l(p(x))^\alpha$. After several steps we have

$$\psi_0^n\left(p^j(x)\right) = l\left(p^j(x)\right)^\alpha \quad \text{for some } j = 0, \dots, n. \tag{1}$$

For every $\xi \in E$ we take $x \in [\xi] \cap X_n$ and the largest j satisfying (1). Then we find zero-covering set $\{y_m\}_m$ of E such that $\{B(y_m)\}_m$ is disjoint and $\psi_0^n(y_m) = l(y_m)^\alpha$. Therefore

$$\Lambda_\alpha^0(E, l) \leq \sum_m l(y_m)^\alpha = \sum_m \psi_0^n(y_m) \leq \psi_0^n(o) \tag{2}$$

Let $x \in X_j$ with $j = 0, \dots, n$. Then $\psi_0^n(x) \leq \psi_1^n(x) \leq \dots \leq \psi_j^n(x)$ and

$$\begin{aligned}\psi_j^n(x) &= \sum_{y \in X_n \cap D(x)} \psi_j^n(y) \leq \sum_{y \in X_n \cap D(x)} \frac{l(p^{n-j}(y))^\alpha}{\psi_{j+1}^n(p^{n-j}(y))}\psi_{j+1}^n(y) \\ &= \sum_{y \in X_n \cap D(x)} \frac{l(x)^\alpha}{\psi_{j+1}^n(x)}\psi_{j+1}^n(y) = l(x)^\alpha.\end{aligned}$$

Therefore

$$\psi_0^n(x) \leq l(x)^\alpha \tag{3}$$

By (3), $\{\psi_0^n\}_n$ is bounded. Therefore by taking a subsequence we may assume that $\psi(x) := \lim_{n\to\infty} \psi_0^n(x)$ exists for any $x \in X$. Then we have easily that ψ is a nonnegative additive function. Also we have $\psi(x) \leq l(x)^\alpha$. If $(x_n)_n \in \Omega \setminus \overline{E}$, then $B(x_n) \cap \overline{E} = \emptyset$ for sufficiently large n. Therefore $\psi(x_n) = 0$.

Let $\phi(x) = \psi(x)/\psi(o)$. Since $\psi(o) \geq \Lambda_\alpha^0(E, l)$ by (2), we have the result. □

Lemma 4. *$\Lambda_\alpha^0(E, l) = 0$ implies that $\Lambda_\alpha(E, l) = 0$.*

Proof. For any integer n we set $r = \min\left\{l(x); x \in \bigcup_{k<n} X_k\right\}$ and let $\varepsilon < r^\alpha$. Since $\Lambda_\alpha^0(E, l) = 0$, we can find a zero-covering set $\{x_j\}_j$ of E such that $\sum_j l(x_j)^\alpha < \varepsilon$. Then x_j can not be in $\bigcup_{k<n} X_k$. This means that $\{x_j\}_j$ is an n-covering set of E. Therefore $\Lambda_\alpha^n(E, l) < \varepsilon$. Hence we have the result. □

Theorem 2. *Let E be a compact subset of Ω. Then*

$$\dim(E,l) = \sup_{\phi}\left(\phi\text{-}\operatorname*{ess\,sup}_{\xi\in E}\lambda_{\phi,l}(\xi)\right) = \inf_{\phi}\left(\sup_{\xi\in E}\lambda_{\phi,l}(\xi)\right)$$

where $\sup_\phi$ or $\inf_\phi$ extends over all nonnegative additive functions ϕ with $\phi(o)=1$.

Proof. Using Theorem 1 we have only to prove that

$$\dim(E,l) \le \sup_{\phi}\left(\phi\text{-}\operatorname*{ess\,sup}_{\xi\in E}\lambda_{\phi,l}(\xi)\right),$$

$$\dim(E,l) \ge \inf_{\phi}\left(\sup_{\xi\in E}\lambda_{\phi,l}(\xi)\right).$$

Let $\alpha < \dim(E,l)$. Then $\Lambda_\alpha(E,l) = \infty$. By Lemmas 4 and 3 we can find a nonnegative additive function ϕ such that $\phi(o) = 1$ and $\phi(x) \le l(x)^\alpha / \Lambda_\alpha^0(E,l)$. Then $\lambda(\xi) \ge \alpha$ for any $\xi \in E$. Therefore $\phi\text{-}\operatorname{ess\,sup}_{\xi\in E}\lambda(\xi) \ge \alpha$. Hence we have the first inequality.

Next we shall prove the second inequality. Assume that $\alpha > \dim(E,l)$. Then $\Lambda_\alpha(E,l) = 0$. Let $Z = \{x \in X; B(x)\cap E = \emptyset\}$ and let $\phi(x) = 0$ for $x \in Z$. Let $\phi(o) = 1$. Since $\Lambda_\alpha^1(E,l) = 0$, we can find a one-covering set Y_1 of E such that $\sum_{y\in Y_1} l(y)^\alpha \le l(o)^\alpha$. We may assume that $\{B(y); y \in Y_1\}$ is disjoint and $Y_1 \cap Z = \emptyset$. Let

$$\phi(y) = \frac{l(y)^\alpha}{\sum_{z\in Y_1} l(z)^\alpha} \quad \text{for } y \in Y_1.$$

It is well defined since $\sum_{y\in Y_1}\phi(y) = 1$. We have

$$\phi(y) \ge \frac{l(y)^\alpha}{l(o)^\alpha} \quad \text{for } y \in Y_1.$$

Next let $x \in Y_1$ and $n = \rho(o,x)$. Since $\Lambda_\alpha^{n+1}(E\cap B(x), l) = 0$, we can find an $(n+1)$-covering set $Y_2(x)$ of $E \cap B(x)$ such that $\sum_{y\in Y_2(x)} l(y)^\alpha \le l(x)^\alpha$. We may assume that $\{B(y); y\in Y_2(x)\}$ is disjoint and $Y_2(x)\cap Z = \emptyset$. Let

$$\phi(y) = \frac{l(y)^\alpha}{\sum_{z\in Y_2(x)} l(z)^\alpha}\phi(x) \quad \text{for } y \in Y_2(x).$$

It is well defined since $\sum_{y\in Y_2(x)}\phi(y) = \phi(x)$. We have

$$\phi(y) \ge \frac{l(y)^\alpha}{l(x)^\alpha}\phi(x) \ge \frac{l(y)^\alpha}{l(o)^\alpha} \quad \text{for } y \in Y_2(x).$$

Let $Y_2 = \bigcup_{x\in Y_1} Y_2(x)$. Then Y_2 is an n_2-covering set of E for some n_2 and $Y_2 \cap Y_1 = \emptyset$. Similarly, for every m, there is an n_m-covering set Y_m of E for some n_m such that

$$\phi(y) \ge \frac{l(y)^\alpha}{l(o)^\alpha} \quad \text{for } y \in Y_m$$

and $Y_m \cap Y_{m'} = \emptyset$ if $m \neq m'$.

Let $\xi \in E$. Then we can find $x_m \in [\xi] \cap Y_m$ for each m. Therefore

$$\frac{\log 1/\phi(x_m)}{\log 1/l(x_m)} \leq \frac{\alpha \log l(o) + \alpha \log 1/l(x_m)}{\log 1/l(x_m)} \to \alpha.$$

This means $\sup_{\xi \in E} \lambda(\xi) \leq \alpha$. Hence we have the result. □

3 Comparison principle

We shall discuss the relation between the usual Hausdorff dimension of a set in $\mathbb{R}^N$ and that of a set of a tree. The definitions of the Hausdorff measure and the Hausdorff dimension are mentioned at the begining of the previous section.

Theorem 3. *Let K be a bounded set in $\mathbb{R}^N$. Then there exist a tree $(X, \mathcal{A}, o)$, a distance function l and $E \subset \Omega$ such that $\Lambda_\alpha(E, l)$ is comparable with $\mathcal{H}_\alpha(K)$ for each $\alpha > 0$, where the comparison constants depend only on N. Especially*

$$\dim K = \dim(E, l).$$

Proof. Take a cube Q_0 with $K \subset Q_0$. Let $\mathcal{Q}_0 = \{Q_0\}$. We divide dyadically Q_0 into 2^N mutually disjoint cubes. We denote $\mathcal{Q}_1$ the collection of such 2^N cubes. Next we divide dyadically each cube of $\mathcal{Q}_1$ into 2^N mutually disjoint cubes and we denote $\mathcal{Q}_2$ the collection of such 2^{2N} cubes. Similarly we get $\mathcal{Q}_n$. For every $Q \in \mathcal{Q}_n$ we can find the unique cube $q(Q) \in \mathcal{Q}_{n-1}$ with $Q \subset q(Q)$.

Next we take a homogeneous tree such that $\#(X_n \cap D(x)) = 2^N$ for every $x \in X_{n-1}$. Let f be a bijective mapping from $\bigcup_{n \geq 0} \mathcal{Q}_n$ to $\bigcup_{n \geq 0} X_n$ such that $f(Q) \in X_n$ and $f(q(Q)) = p(f(Q))$ for $Q \in \mathcal{Q}_n$. Also let $l(f(Q)) = \operatorname{diam} Q$.

Let $t \in K$. We can find $Q_n(t) \in \mathcal{Q}_n$ for each n such that $t \in Q_n(t)$ and $Q_{n-1}(t) = q(Q_n(t))$. We set $E = \{(f(Q_n(t)))_n \in \Omega; t \in K\}$.

Now let $\{x_m\}_m$ be an n-covering set of E. Let $t \in K$. Since $(f(Q_j(t)))_j \in E$, we can find an m with $(f(Q_j(t)))_j \in B(x_m)$. Then there is a j such that $f(Q_j(t)) = x_m$. Therefore $t \in Q_j(t) = f^{-1}(x_m)$. Hence $K \subset \bigcup_m f^{-1}(x_m)$. Let C_m be a ball with radius $\operatorname{diam} f^{-1}(x_m)$ such that $f^{-1}(x_m) \subset C_m$. Then $\{C_m\}_m$ be a covering of K. Since $\operatorname{diam} f^{-1}(x_m) = l(x_m)$, we have $\mathcal{H}_\alpha^r(K) \leq \sum_m l(x_m)^\alpha$ where $r = \max_m l(x_m)$. Therefore $\mathcal{H}_\alpha^r(K) \leq \Lambda_\alpha^n(E, l)$. Since $r \to 0$ when $n \to \infty$, we have $\mathcal{H}_\alpha(K) \leq \Lambda_\alpha(E, l)$.

Let $C = C(t, \delta)$, the ball centered at $t \in \mathbb{R}^N$ with radius δ, and

$$J(C) = \left\{ Q \in \bigcup_{n \geq 1} \mathcal{Q}_n; \operatorname{diam} Q \leq \delta < \operatorname{diam} q(Q), Q \cap C \neq \emptyset \right\}.$$

Remark that the number of $J(C)$ is less than a constant c which is independent of C. Now let $\{C_m\}_m$ be a covering of K where C_m is a ball with radius δ_m and $\delta_m \leq r$. For

$\xi = (x_n)_n \in E$ we can find $t \in K$ such that $x_n = f(Q_n(t))$. We know that $t \in C_m$ for some m and $\operatorname{diam} Q_n(t) \leq \delta_m < \operatorname{diam} q(Q_n(t))$ for some n. Therefore we have $Q_n(t) \in J(C_m)$ for some m and some n. Note that $\xi \in B(x_n) = B(f(Q_n(t)))$. Let n_0 be the smallest number satisfying $Q_{n_0}(t) \in \bigcup_m J(C_m)$ for some $t \in K$. Then $\{f(Q); Q \in \bigcup_m J(C_m)\}$ is an n_0-covering set of E. Since $l(f(Q)) = \operatorname{diam} Q \leq \delta_m$ for $Q \in J(C_m)$,

$$\Lambda_\alpha^{n_0}(E, l) \leq \sum_m \sum_{Q \in J(C_m)} l(f(Q))^\alpha \leq \sum_m c\delta_m^\alpha.$$

Hence $\Lambda_\alpha^{n_0}(E, l) \leq c\mathcal{H}_\alpha^r(K)$. Since $n_0 \to \infty$ when $r \to 0$, we have $\Lambda_\alpha(E, l) \leq c\mathcal{H}_\alpha(K)$. Hence we have the result. □

We shall give some examples. Using Our theorems, we get the Hausdorff dimension by simple calculation for some sets in $\mathbb{R}^N$.

Example 1 (The 1/3-Cantor set). $\dim K = \log 2/\log 3$ if K is the 1/3-Cantor set.

Proof. Let Q_0 be the closed interval $[0, 1]$. We divide Q_0 into three intervals $[0, 1/3]$, $(1/3, 2/3)$ and $[2/3, 1]$. We denote $\mathcal{Q}_1$ the collection of such three intervals. Next we divide each interval of $\mathcal{Q}_1$ into mutually disjoint three intervals and denote $\mathcal{Q}_2$ the collection of such 3^2 intervals. Similarly we get $\mathcal{Q}_n$. Figure 1 shows the intervals of $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Next we take a homogeneous tree such that $\#(X_n \cap D(x)) = 3$ for $x \in X_{n-1}$. Also let $l(x) = 3^{-n}$ for $x \in X_n$. Take $Y_n \subset X_n$ such that $Y_0 = \{o\}$ and $\#(Y_n \cap D(x)) = 2$ for $x \in Y_{n-1}$. In Figure 1 the hatched intervals correspond to Y_0, Y_1 or Y_2. Let $E = \{(x_n)_n \in \Omega; x_n \in Y_n \text{ for all } n\}$. Then we can prove $\dim K = \dim(E, l)$ similarly to the proof of Theorem 3.

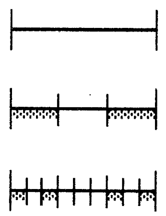

図 1: The Cantor set

For $x \in X_n$ let $\phi(x) = 2^{-n}$ if $B(x) \cap E \neq \emptyset$ and $\phi(x) = 0$ otherwise. Then ϕ is a nonnegative additive function with $\phi(o) = 1$. For $\xi = (x_n)_n \in E$

$$\frac{\log 1/\phi(x_n)}{\log 1/l(x_n)} = \frac{\log 2}{\log 3}.$$

Therefore $\lambda(\xi) = \log 2/\log 3$. Hence Theorem 1 implies the result. □

Example 2 (The Sierpiński gasket). $\dim K = \log 3/\log 2$ if K is the Sierpiński gasket.

Proof. Let Q_0 be the closed triangle. We divide Q_0 into four disjoint triangles by connecting midpoints of edges where the center one is open triangle and some vertexes are removed from other three. We denote $\mathcal{Q}_1$ the collection of such four triangles. Next we divide each triangle of $\mathcal{Q}_1$ into four disjoint triangles and denote $\mathcal{Q}_2$ the collection of such 4^2 triangles. Similarly we get $\mathcal{Q}_n$. Figure 2 shows the triangles of $\mathcal{Q}_2$. Next we take a homogeneous tree such that $\#(X_n \cap D(x)) = 4$ for $x \in X_{n-1}$. Also let $l(x) = 2^{-n}$ for $x \in X_n$. Take a set E corresponding to K similarly to Example 1. (In Figure 2 the hatched triangles correspond to Y_2.)

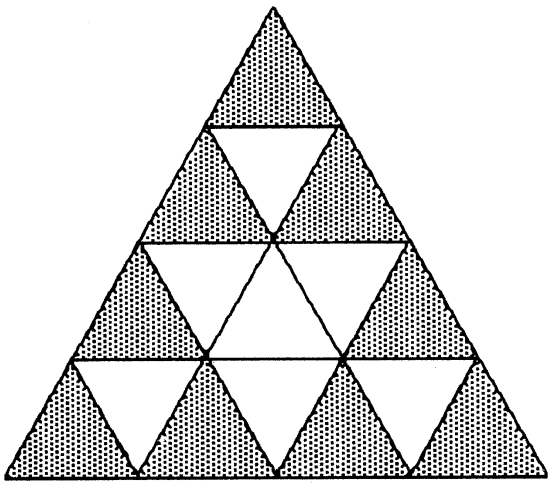

図 2: The Sierpiński gasket

For $x \in X_n$ let $\phi(x) = 3^{-n}$ if $B(x) \cap E \neq \emptyset$ and $\phi(x) = 0$ otherwise. Then ϕ is a nonnegative additive function with $\phi(o) = 1$. We have $\lambda(\xi) = \log 3/\log 2$ for $\xi \in E$. Therefore Theorem 1 implies the result. □

参考文献

[1] K. J. Falconer. *The geometry of fractal sets.* Cambridge Univ. Press, 1985.

[2] O. Frostman. Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddel. Lunds Univ. Mat. Sem.*, 3:1–118, 1935.

[3] C. A. Rogers. *Hausdorff Measures.* Cambridge University Press, 1970,1998.